Nonlinear-σ-model Padé calculation of πN phase shifts*

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A calculation of the πN scattering amplitude, based on the nonlinear σ model, through the one-loop approximation of perturbation theory, is given. Unitary partial-wave amplitudes are obtained by constructing the first diagonal Padé approximant for the perturbation series. The s- and p-wave phase shifts are computed and compared with experimental phase shifts.

I. INTRODUCTION

The SU(2) × SU(2) current algebra (CA) and the partially conserved axial-vector current (PCAC) have been important conceptual developments[1] in low-energy hadron physics, but they have led to only a handful of numbers to be confronted with experiment. Typically, CA and PCAC imply Ward identities which can be converted into low-energy theorems (off-shell). Results obtained include $\pi\pi$ s-wave scattering lengths, πN s-wave scattering lengths, the Goldberger-Treiman relation, and hyperon nonleptonic s-wave decay constants. The work described in this paper is part of a continuing program[2] to go beyond these results, in particular, to compute $\pi\pi$ and πN scattering amplitudes above threshold, up into the resonance retion.

We adopt the point of view that the real world is close to the Goldstone-Nambu mode of chiral SU(2) × SU(2) invariance. This point of view can be realized in a specific and detailed way in different variants of the so-called σ models,[1] and we will use these as a crutch to do the calculations. This does not necessarily imply a belief that there is anything fundamental about the σ model Lagrangians; the whole calculation could be described as a particular approximate S-matrix calculation, in which one starts with a small number of parameters as input and uses analyticity and approximate unitarity and crossing (in some, one hopes, optimal way) to compute an approximation to the S matrix valid in some range of energy. Given such a specific framework, the present approach to strong-interaction calculations (two-body channels, low to intermediate energies) is to generate a perturbation series for the scattering amplitude and use it to construct Padé approximants. Recall that the sequence of Padé approximants is an algorithm for the summation of divergent series which has been applied in recent years to a number of problems of statistical mechanics and strong-interaction dynamics.[3] There does not exist any proof of the convergence of the sequence of Padé approximants in a real field theory, but it has been proved that the diagonal approximants converge to the scattering amplitude in potential theory. This is a strong indication that at least the bound-state (resonance) problem (poles in the scattering amplitude as functions of coupling constant g) is handled by the Padé algorithm. Some indication of the power of the Padé algorithm to sum series which fail to converge for any value of g has been obtained from consideration of the anharmonic oscillator. It is known that this Hamiltonian possesses eigenvalues $E_n(g)$ for which $g=0$ is a branch point, i.e., the perturbation series for the energy levels must diverge. Nevertheless, Loeffel *et al.*[4] proved that the diagonal Padé approximants formed from the coefficients of the divergent perturbation expansion do converge to the correct $E_n(g)$. An important feature of the diagonal Padé approximants, applied to partial-wave amplitudes, is that they satisfy elastic unitarity exactly. Thus, the present approach may be viewed as a scheme for the (elastic) unitarization of the current-algebra and PCAC threshold results. All such schemes necessarily violate crossing symmetry. However, in the $\pi\pi$ problem one has a quantitative measure of this violation in the form of the Roskies relations,[5] and the nonlinear-σ-model Padé calculation was found[2] to satisfy the constraints of crossing to a high degree of accuracy, an improvement over previous unitarization schemes which have been tested in this regard.

Having decided to try to compute the scattering amplitude from the information contained in the formal perturbation series by way of the Padé algorithm, we are faced with the task of defining and computing the ("almost") chiral-invariant perturbation series. The problems involved depend on whether we are dealing with the linear σ model (LσM) or the nonlinear σ model (NLσM). In the LσM the problem is the appearance in the Lagrangian of the σ field. This is a problem because of the questionable existence of the σ particle. At best it is a very broad resonance (ϵ) in the $I=0$ s-wave $\pi\pi$ scattering; then one has the problems associated with unstable particles in a Lagrangian and in perturbation theory. Thus, one would like the σ to appear, if at all, as output rather than input of a calculation. Elimination of the σ field from the LσM leads to the NLσM. The problem with constructing the perturbation series for the NLσM is that it appears to be unrenormalizable. However, because of the relation between the NLσM and the LσM, the perturbation series for the NLσM can be defined. The way to do this was first suggested by Bessis and Zinn-Justin.[6] Essentially the LσM is used as a Lorentz-invariant, chiral-invariant, regularization of the NLσM.

The procedure is to eliminate the σ particle from the LσM as follows. Compute all Feynman diagrams (to any given order) in the renormalized perturbation expansion of the LσM. Then expand the resulting invariant matrix element in powers of m_σ and $\ln m_\sigma$ and drop all terms which would vanish if the limit $m_\sigma \to \infty$ were taken. This procedure eliminates all poles and thresholds associated with the σ particle, and leaves terms which are independent of m_σ and also terms which depend on m_σ (polynomial in m_σ and $\ln m_\sigma$, with coefficients which depend on the kinematic variables of the amplitude[2,6]). m_σ thus remains as an arbitrary parameter, but does not have an interpretation as the mass of any particle. Bessis and Zinn-Justin gave an elegant heuristic proof, based on the Feynman path-integral formulation of the generating functionals for the Green's functions of the LσM and of the NLσM, that the $m_\sigma \to \infty$ limit of the LσM produces the NLσM. The proof is formal because the limit does not exist for the perturbation expansion beyond the tree approximation; but the m_σ-truncation procedure described above is well defined because of the well-studied renormalizability of the LσM, and hence does provide the perturbation series, to any order, to be used for the construction of the Padé approximants, and all resonances (including the σ, if it exists) are to appear as output. The input parameters are the physical masses and coupling constants of the pion and the nucleon, and the Lorentz-invariant, chiral-invariant NLσM cutoff parameter, m_σ.

At the one-loop order (which is all that is required for the construction of the [1, 1] Padé approximant), the equivalence of the m_σ-truncated LσM and the NLσM is particularly transparent. Think of constructing the one-loop NLσM amplitude for some scattering process by iteration of the unitarity equation (in all channels) combined with dispersion relations.

$$\operatorname{Im} M^{(1)}_{\mathrm{NL}\sigma\mathrm{M}} = \int_\Phi |M^{(0)}_{\mathrm{NL}\sigma\mathrm{M}}|^2 ,$$

where $M^{(0)}_{\mathrm{NL}\sigma\mathrm{M}}$ is the tree approximation (or Born term) of the NLσM invariant matrix element for the process in question. But it is well known that in the tree approximation

$$M^{(0)}_{\mathrm{NL}\sigma\mathrm{M}} = \lim_{\substack{m_\sigma \to \infty \\ (\lambda \to \infty)}} M^{(0)}_{\mathrm{L}\sigma\mathrm{M}} .$$

(The meson coupling λ must also go to infinity, $\lambda \alpha m_\sigma / f_\pi$). Because the phase-space integration is compact, the limit $m_\sigma \to \infty$ and the phase space integration can be interchanged, so

$$\operatorname{Im} M^{(1)}_{\mathrm{NL}\sigma\mathrm{M}} = \lim_{m_\sigma \to \infty} \operatorname{Im} M^{(1)}_{\mathrm{L}\sigma\mathrm{M}} .$$

Analyticity implies that $M^{(1)}$ is determined by $\operatorname{Im} M^{(1)}$ up to a polynomial (in s and t). The dispersion relation for $\operatorname{Re} M^{(1)}_{\mathrm{NL}\sigma\mathrm{M}}$ requires subtractions (the limit $m_\sigma \to \infty$ and the noncompact dispersion integration are not interchangeable), so we see that the definition of $M^{(1)}_{\mathrm{NL}\sigma\mathrm{M}}$ by the m_σ truncation of $M^{(1)}_{\mathrm{L}\sigma\mathrm{M}}$, just serves to fix all of the subtraction constants in the polynomial for $M^{(1)}_{\mathrm{NL}\sigma\mathrm{M}}$ in terms of one arbitrary parameter, m_σ.

The whole procedure has been carried through the one-loop order and the computation of the phase shifts from the [1, 1] Padé approximant for the $\pi\pi \to \pi\pi$ process, including pions and nucleons in the intermediate state, and the work is reported in considerable detail in Ref. 2. The results, as described there, were sufficiently encouraging so that we have gone on to the $\pi N \to \pi N$ calculation reported in this paper. For the πN calculation we have chosen to make use of the considerable simplification which can be obtained in the one-loop order if one works in the chiral-invariant limit. Then one can bypass the whole procedure of first computing all of the LσM one-loop Feynman integrals and then computing their asymptotic (in m_σ) expansions. We can compute directly with the NLσM, because, as we have just described, the one-loop amplitude is determined via unitarity and analyticity by the tree diagrams, up to a subtraction polynomial; and in the chiral-invariant limit there is an exact low-energy theorem (LET) which determines all but one of the required subtraction constants--a result equivalent to the m_σ truncation of the LσM one-loop amplitude. We have decided to proceed this way for a couple of reasons. First, we believe that the real world is close to the Nambu-Goldstone mode realization of SU(2) × SU(2) invariance, as evidenced by the success of the Goldberger-Treiman relation, the Adler-Weisberger-Weinberg LET's etc. Thus we expect this approximation to be entirely appropriate in view of the other, less controllable uncertainties introduced by the restriction to the [1, 1] Padé approximant, the neglect of kaons and other heavier strange particles, and particularly, the neglect

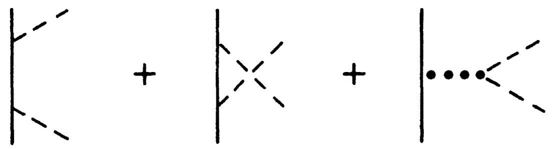

FIG. 1. LσM tree diagrams for $\pi N \to \pi N$.

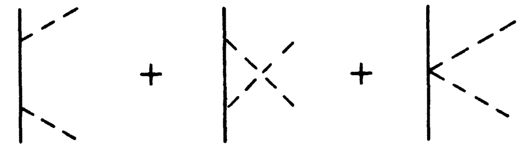

FIG. 2. NLσM tree diagrams for $\pi N \to \pi N$.

of inelastic processes, which are known to be important above 1500 MeV. Another reason for seeking simplification of the calculation is the existence in the literature of two previous related calculations for the πN problem,[7] which should give similar results for appropriately chosen values of the parameters, but rather are in strong disagreement with each other. By doing the calculation in the chiral limit we greatly simplify it, and hence increase our control over it — as discussed in detail below.

II. THE ONE-LOOP NLσM AMPLITUDE

The manifestly chiral-invariant linear σ-model Lagrangian, for pions, Σ's and nucleons, is

$$\mathcal{L}_{\mathrm{L\sigma M}} = \tfrac{1}{2}[(\partial\vec{\phi})^2 + (\partial\chi)^2] - \tfrac{1}{2}\mu_0{}^2(\vec{\phi}^2+\chi^2) + \bar{\psi}i\gamma\cdot\partial\psi - \tfrac{1}{4}\lambda_0{}^2(\vec{\phi}^2+\chi^2)^2 - g_0\bar{\psi}(\chi - i\gamma_5\vec{\tau}\cdot\vec{\phi})\psi. \quad (2.1)$$

The Goldstone-Nambu mode of chiral SU(2) $\times$SU(2) invariance is realized by way of a nonzero vacuum expectation value of the σ field

$$\langle\chi\rangle = v_0. \quad (2.2)$$

Rewrite the Lagrangian in terms of the translated field

$$\hat{\chi} = \chi - v_0, \quad (2.3)$$

$$\mathcal{L}_{L\sigma M} = \tfrac{1}{2}(\partial\vec{\phi})^2 + \tfrac{1}{2}[(\partial\hat{\chi})^2 - M_1{}^2\hat{\chi}^2] + \bar{\psi}(i\gamma\cdot\partial - m_1)\psi - \lambda_0{}^2 v_0\hat{\chi}(\vec{\phi}^2+\hat{\chi}^2) - \tfrac{1}{4}\lambda_0{}^2(\vec{\phi}^2+\hat{\chi}^2)^2 + ig_0\bar{\psi}\gamma_5\vec{\tau}\cdot\vec{\phi}\psi - g_0\bar{\psi}\psi\hat{\chi}\ , \quad (2.1')$$

with

$$\mu_1{}^2 = \mu_0^2 + \lambda_0{}^2 v_0{}^2 = 0,$$
$$M_1{}^2 = 2\lambda_0{}^2 v_0{}^2, \quad m_1 = g_0 v_0. \quad (2.4)$$

The tree diagrams (Born term) for the πN scattering amplitude following from the Lagrangian (2.1′) are shown in Fig. 1. The corresponding invariant matrix element is

$$M_{ab}(p,q,p',q') \underset{\mathrm{L\sigma M}}{=} \bar{u}(p')\left[G^2\gamma_5\tau_b \frac{1}{\gamma(p+q)-m}\gamma_5\tau_a + G^2\gamma_5\tau_a\frac{1}{\gamma(p-q')-m}\gamma_5\tau_b - \frac{2G\lambda^2 v}{t-M^2}\delta_{ab}\right]u(p). \quad (2.5)$$

Eliminate the σ particle by taking $M\to\infty$, and also $\lambda\to\infty$ to maintain the chiral constraint $M^2 = 2\lambda^2 m^2/G^2$ [see Eq. (2.4)]. One obtains the Born term of the NLσM,

$$M_{ab}(p,q,p',q') \underset{\mathrm{NL\sigma M}}{=} G^2\bar{u}(p')\left[\gamma_5\tau_b\frac{1}{\gamma(p+q)-m}\gamma_5\tau_a + \gamma_5\tau_a\frac{1}{\gamma(p-q')-m}\gamma_5\tau_b + \frac{1}{m}\delta_{ab}\right]u(p). \quad (2.6)$$

The corresponding NLσM tree diagrams are shown in Fig. 2. In terms of the standard decomposition

$$M_{ab}(p,q,p',q') = \bar{u}(p')\{\delta_{ab}[A^{(+)}(s,t,u) + \tfrac{1}{2}\gamma(q+q')B^{(+)}(s,t,u)] + \tfrac{1}{2}[\tau_b,\tau_a][A^{(-)}(s,t,u) + \tfrac{1}{2}\gamma(q+q')B^{(-)}(s,t,u)]\}u(p), \quad (2.7)$$

with

$$M^{(+)} = \tfrac{1}{3}(M^{(1)} + 2M^{(3)}), \quad M^{(-)} = \tfrac{1}{3}(M^{(1)} - M^{(3)}), \quad (2.8)$$

we have

$$A_{\mathrm{tree}}^{(+)} = \frac{G^2}{m}, \quad A_{\mathrm{tree}}^{(-)} = 0$$
$$B_{\mathrm{tree}}^{(+)} = G^2\left(\frac{1}{m^2-s} - \frac{1}{m^2-u}\right), \quad (2.9)$$
$$B_{\mathrm{tree}}^{(-)} = G^2\left(\frac{1}{m^2-s} - \frac{1}{m^2-u}\right).$$

In principle, as discussed in the Introduction, we should next work out the renormalized perturbation series based on the Lagrangian (2.1′), compute all the renormalized one-loop diagrams, and then make the double expansion in M and $\ln M$ of all the resulting functions, throwing away all terms which vanish in the limit $M\to\infty$. The result would be, by definition, the invariant matrix element for πN scattering of the NLσM, computed through second order (one-loop). This is a lengthy procedure, first because there are a large number of LσM one-loop diagrams, and second, because there is considerable cancellation of singular terms in the limit $M \to \infty$ one has to compute several terms in the asymptotic expansions of each of the LσM Feynman integrals. However, as we have already pointed out, the imaginary parts of the second-order amplitude (discontinuities in s,t,u) are determined, via iteration of the unitarity equations, by the simple known first-order amplitudes. Then analyticity (in general, or of the perturbation-theory Feynman integrals in particular) implies that the entire second-order amplitude is determined up to a polynomial in the variables s,t,u, of order determined by the number of subtractions necessary in the dispersion relations used to construct the real part of the amplitude from its imaginary part. After computing the s-, t-, and u-channel discontinuities (see below), and observing the large s,t,u behavior of the functions, we determine

$$\begin{aligned} &A^{(+)}_{1\text{-loop}} \sim a + bt + \cdots \,, \\ &A^{(-)}_{1\text{-loop}} \sim a' + \cdots \,, \\ &B^{(+)}_{1\text{-loop}} \sim b_P\left(\frac{1}{m^2-s} - \frac{1}{m^2-u}\right) + c' + \cdots \,, \\ &B^{(-)}_{1\text{-loop}} \sim b_P\left(\frac{1}{m^2-s} + \frac{1}{m^2-u}\right) + c + \cdots \,, \end{aligned} \tag{2.10}$$

where the ellipses indicate known functions (convergent integrals). Crossing symmetry implies $a' = c' = 0$. Coupling-constant renormalization—the condition that the conventional πN coupling constant is determined by the residue of the amplitude at the nucleon pole—provides one condition to determine b_P. The low-energy theorem (LET) following from chiral invariance is

$$A^{(+)}(\omega, t=0) + \omega B^{(+)}(\omega, t=0) = O(\omega^2), \tag{2.11a}$$

$$A^{(-)}(\omega, t=0) + \omega B^{(-)}(\omega, t=0) = (1 - g_A^2)\frac{\omega}{2f_\pi^2} + O(\omega^2), \tag{2.11b}$$

where

$$\omega = \frac{p \cdot q}{m} \underset{\mu=0}{=} |\vec{q}| + O(|\vec{q}|^2),$$

where $|\vec{q}|$ is the c.m. three momentum. Equations (2.11a) and (2.11b) provide two conditions which determine the constants a, c so that only one constant, b, in (2.10) remains undetermined—a result equivalent to the m_σ truncation of $M^{1\text{-loop}}_{\text{L}\sigma\text{M}}$ which gives the same known functions and one undetermined constant.

We now give a little more detail on the calculation of $\text{Im}M^{(1)}$. In the s channel ($\pi N \to \pi N$) the only intermediate state corresponding to one-loop Feynman diagrams is the πN state. Thus the s-channel discontinuity of $M^{(1)}$ is computed from the unitarity equation

$$\begin{aligned} \text{disc}_s M^{(1)}_{ab} &= i \sum (2\pi)^4 \delta^4(p+q-P_c) M^{(0)*}_{bc} M^{(0)}_{ac} \\ &= iG^4 \int (dp_1) \sum_{s_1} 2\pi\delta_+(p_1^2 - m^2) \int (dq_1) 2\pi\delta_+(q_1^2)(2\pi)^4\delta^4(p+q-P_c) \\ &\quad \times \bar{u}(p')\left[\tau_b\tau_c \frac{-\gamma q'}{s-m^2} + \tau_c\tau_b \frac{\gamma q_1}{(p'-q_1)^2 - m^2} + \delta_{bc}\frac{1}{m}\right] u(p_1) \\ &\quad \times \bar{u}(p_1)\left[\tau_c\tau_a \frac{-\gamma q}{s-m^2} + \tau_a\tau_c \frac{\gamma q_1}{(p-q_1)^2 - m^2} + \delta_{ac}\frac{1}{m}\right] u(p). \end{aligned} \tag{2.12}$$

The nine terms in (2.12) correspond to nine s-channel unitarity diagrams obtained by "squaring" the three NLσM tree diagrams of Fig. 2. They are shown in Fig. 3. The u-channel ($\pi N \to \pi N$) discontinuity follows from (2.12) by crossing

$$M_{ab}(p, q, p', q') = M_{ba}(p, -q', p', -q). \tag{2.13}$$

In the t channel ($\pi\pi \to N\bar{N}$) there are both $\pi\pi$ and $N\bar{N}$ states corresponding to one-loop Feynman diagrams. Thus we need the lowest-order chiral $\pi\pi$ scattering matrix element, and the lowest-order $N\bar{N}$ scattering matrix element, as well as the low-

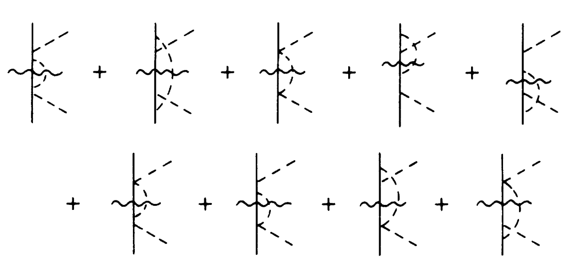

FIG. 3. s-channel unitarity diagrams.

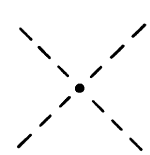

FIG. 4. NLσM tree diagrams for $\pi\pi \to \pi\pi$.

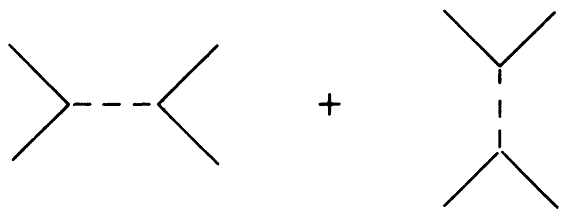

FIG. 5. NLσM tree diagrams for $N\bar{N}\to N\bar{N}$.

est-order $\pi\pi\to N\bar{N}$ matrix element, obtained from (2.6) by crossing. The lowest-order chiral $\pi\pi$ scattering matrix element is the Weinberg matrix element (Fig. 4),

$$M_{abcd}(q_1, q_2, q_3, q_4) = \frac{1}{f_\pi^2}[\delta_{ab}\delta_{cd}(q_1+q_2)^2 + \delta_{ac}\delta_{bd}(q_1-q_3)^2 + \delta_{ad}\delta_{bc}(q_1-q_4)^2]. \quad (2.14)$$

Note that the (conserved) axial-vector current derived from the Lagrangian (2.1′) is

$$A_{\mu,a} = -v_0\partial_\mu\phi_a - \hat{\chi}\partial_\mu\phi_a + \phi_a\partial_\mu\hat{\chi} + \bar{\psi}\gamma_\mu\gamma_5\tfrac{1}{2}\tau_a\psi. \quad (2.15)$$

So to lowest order $v_0 = f_\pi$ and (2.4) gives $f_\pi^{-2} = G^2/m^2$. In the limit $M\to\infty$, the $N\bar{N}\to N\bar{N}$ Born term computed from (2.1′) consists of just the usual pseudoscalar coupling one-pion exchange and annihilation diagrams (Fig. 5). Including both the $\pi\pi$ and $N\bar{N}$ intermediate states there are nine t-channel unitarity diagrams, three of which vanish by the G-parity extension of Furry's theorem (nucleon loops with an odd number of pions attached give zero). The six contributing diagrams are shown in Fig. 6.

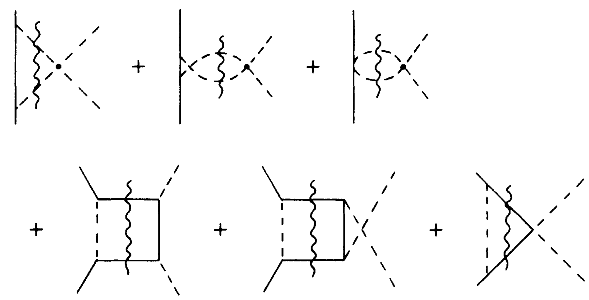

FIG. 6. t-channel unitarity diagrams.

The result of carrying out the integrations and spin and i-spin sums of (2.12), and the corresponding calculations for the u channel and t channel, is a set of functions which are recognized to be kinematical factors times the imaginary parts of one-loop Feynman integrals with two, three, or four scalar propagators. The box integrals have discontinuities in two channels, e.g., the second unitarity diagram in Fig. 3 and the fourth unitarity diagram in Fig. 6 correspond to the same (box) Feynman diagram.

As a check, the whole calculation can be done an alternate way which we briefly describe. Return to the original form of the chiral-invariant LσM Lagrangian (2.1). On the formal Lagrangian level the σ field is eliminated by imposing the chiral-invariant constraint

$$\vec{\phi}^2 + \chi^2 = v_0^2$$

or

$$\chi = (v_0^2 - \vec{\phi}^2)^{1/2}. \quad (2.16)$$

Then the chiral-invariant nonlinear σ-model Lagrangian is obtained:

$$\mathcal{L}_{\mathrm{NL}\sigma\mathrm{M}} = \tfrac{1}{2}(\partial\vec{\phi})^2 + \frac{1}{2(v_0^2-\vec{\phi}^2)}(\vec{\phi}\cdot\partial\vec{\phi})^2 + \bar{\psi}i\gamma\cdot\partial\psi - g_0(v_0^2-\vec{\phi}^2)^{1/2}\bar{\psi}\psi + ig_0\bar{\psi}\gamma_5\vec{\tau}\cdot\vec{\phi}\psi \quad (2.17a)$$

$$= \tfrac{1}{2}(\partial\vec{\phi})^2 + \bar{\psi}(i\gamma\cdot\partial - m_1)\psi + \frac{1}{2v_0^2}(\vec{\phi}\cdot\partial\vec{\phi})^2 + \frac{g_0^2}{2m_1}\bar{\psi}\psi\phi^2 + ig_0\bar{\psi}\gamma_5\vec{\tau}\cdot\vec{\phi}\psi + \cdots . \quad (2.17b)$$

The one-loop $\pi N\to\pi N$ Feynman diagrams generated by (2.17) are shown in Fig. 7. The corresponding Feynman integrals are, of course, divergent, but the imaginary parts can be computed unambiguously

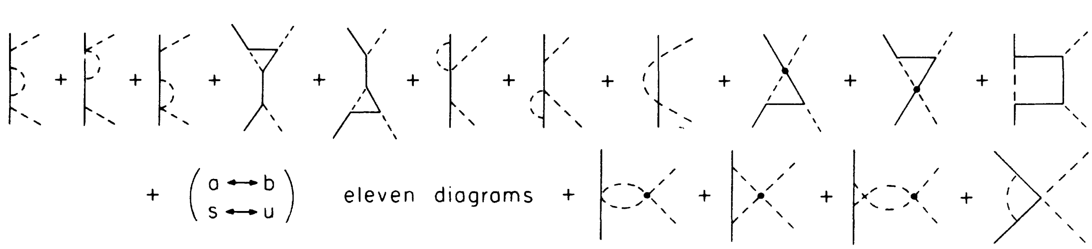

FIG. 7. One-loop NLσM Feynman diagrams (some diagrams which contribute only to renormalization constants are omitted).

by the Cutkosky rules and agree with the results of the iterated unitarity calculation. Then the one-loop Feynman integrals may be rendered convergent by subtracting polynomials in the external momenta. The Feynman integrals corresponding to the individual diagrams of Fig. 7 are given in Appendix A. Adding them all gives the one-loop amplitude:

$$\begin{aligned}
A^{(+)} &= \frac{G^2}{m} - G^4\left\{\frac{1}{16\pi^2 m}\left(a + b\frac{t}{m^2}\right) - \frac{3}{2m}\left[I^{(0)}_{\pi N}(s) + I^{(0)}_{\pi N}(u)\right] + m\left[\frac{1}{s}I^{(0)}_{\pi N}(s) + \frac{1}{u}I^{(0)}_{\pi N}(u)\right] + 2m\left[J_{N\pi N}(s) + J_{N\pi N}(u)\right]\right.\\
&\qquad \left. + 3\left[\mathcal{A}(s,t,u) + \mathcal{A}(u,t,s)\right] + \frac{t}{m^3}I^{(0)}_{\pi\pi}(t) + \frac{3}{m}I^{(0)}_{NN}(t) + \frac{2t}{m}K_{N\pi\pi}(t) + \frac{4t}{m}K_+(t)\right\},\\
A^{(-)} &= -G^4\left\{\frac{1}{m}\left[I^{(0)}_{\pi N}(s) - I^{(0)}_{\pi N}(u)\right] + \frac{m}{2}\left[\frac{1}{s}I^{(0)}_{\pi N}(s) - \frac{1}{u}I^{(0)}_{\pi N}(u)\right] + 2m\left[J_{N\pi N}(s) - J_{N\pi N}(u)\right]\right.\\
&\qquad \left. - \left[\mathcal{A}(s,t,u) - \mathcal{A}(u,t,s)\right] + \frac{s-u}{m}\left[4K_{++}(t) + 4K_+(t) + K_{N\pi\pi}(t)\right]\right\},\\
B^{(+)} &= G^2\left(\frac{1}{m^2-s} - \frac{1}{m^2-u}\right) - G^4\left\{\frac{b_P}{16\pi^2}\left(\frac{1}{m^2-s} - \frac{1}{m^2-u}\right) + \frac{1}{2m^2}\left[I^{(0)}_{\pi N}(s) - I^{(0)}_{\pi N}(u)\right] + \left[\frac{1}{s}I^{(0)}_{\pi N}(s) - \frac{1}{u}I^{(0)}_{\pi N}(u)\right]\right.\\
&\qquad - 2\left[\frac{1}{s-m^2}I^{(m^2)}_{\pi N}(s) - \frac{1}{u-m^2}I^{(m^2)}_{\pi N}(u)\right] + 2\left[J_{N\pi N}(s) - J_{N\pi N}(u)\right]\\
&\qquad \left. - 4m^2\left[\frac{1}{s-m^2}J^{(m^2)}_{N\pi N}(s) - \frac{1}{u-m^2}J^{(m^2)}_{N\pi N}(u)\right] + 3\left[\mathcal{B}(s,t,u) - \mathcal{B}(u,t,s)\right]\right\},\\
B^{(-)} &= G^2\left(\frac{1}{m^2-s} + \frac{1}{m^2-u}\right) - G^4\left\{\frac{b_P}{16\pi^2}\left(\frac{1}{m^2-s} + \frac{1}{m^2-u}\right) + \frac{c}{16\pi^2 m^2} + \frac{1}{2}\left[\frac{1}{s}I^{(0)}_{\pi N}(s) + \frac{1}{u}I^{(0)}_{\pi N}(u)\right]\right.\\
&\qquad - 2\left[\frac{1}{s-m^2}I^{(m^2)}_{\pi N}(s) + \frac{1}{u-m^2}I^{(m^2)}_{\pi N}(u)\right] - 2\left[J_{N\pi N}(s) + J_{N\pi N}(u)\right]\\
&\qquad - 4m^2\left[\frac{1}{s-m^2}J^{(m^2)}_{N\pi N}(s) + \frac{1}{u-m^2}J^{(m^2)}_{N\pi N}(u)\right]\\
&\qquad \left. - \left[\mathcal{B}(s,t,u) + \mathcal{B}(u,t,s)\right] + \frac{t}{m^2}K_+(t)\right\}.
\end{aligned} \tag{2.18}$$

The various functions in (2.18) are Feynman integrals and linear combinations of Feynman integrals defined in Appendix A. The "polynomial" terms a, bt/m^2, c, $b_P[(m^2-s)^{-1} \mp (m^2-u)^{-1}]$ are the terms required by the subtractions needed for the $I_{\pi\pi}$, $I_{\pi N}$, and I_{NN} integrals.

We now determine the conditions placed on these constants by the chiral LET (2.11) and coupling-constant renormalization. The chiral-limit threshold is: First $q^2 = \mu^2 = 0$ and $t = 0$; then $|\vec{q}|$, the center-of-mass three-momentum, approaches zero:

$$\text{(i) } q^2 = \mu^2 = 0, \quad t = 0, \qquad \text{(ii) } \rho \equiv |\vec{q}|/m \to 0. \tag{2.19}$$

[Note that this is different from the usual current-algebra and PCAC off-shell low-energy limit, which is: First $t = 0$ and $\vec{q} = 0$, then $q_0 \to 0$; and leads to a LET (Adler-Weisberger-Weinberg) giving the amplitude at the physical threshold up to correction terms of order μ^2/m^2.] For $q^2 = 0$ and $t = 0$, the pole terms of (2.18) are

$$\frac{1}{m^2-s} - \frac{1}{m^2-u} = -\frac{1}{m\omega}, \quad \frac{1}{m^2-s} + \frac{1}{m^2-u} = 0, \tag{2.20}$$

where $\omega = p \circ q/m$ and

$$\frac{\omega}{m} \underset{\mu=0}{=} \rho^2 + (\rho^4 + \rho^2)^{1/2} = \rho + \rho^2 + \tfrac{1}{2}\rho^3 + \cdots . \tag{2.21}$$

The chiral-limit threshold expansions of the integrals in (2.18) are given in Appendix B. Substitution of those results into (2.18) gives

$$\begin{aligned}
A^{(+)} &\to G^2\frac{1}{m} - \frac{G^4}{16\pi^2 m}\left[a - 1 + O(\rho^2)\right],\\
A^{(-)} &\to -\frac{G^4}{16\pi^2 m}\left[\rho + O(\rho^2)\right],\\
B^{(+)} &\to G^2\left(\frac{-1}{m\omega}\right) - \frac{G^4}{16\pi^2}\left[b_P\left(\frac{-1}{m\omega}\right) + O(\rho)\right],\\
B^{(-)} &\to -\frac{G^4}{16\pi^2 m^2}\left[c - 2 + O(\rho)\right].
\end{aligned} \tag{2.22}$$

Comparing with (2.11a) we find

$$a = 1 + b_p . \qquad (2.23)$$

Comparison with (2.11b) gives

$$(1 - g_A{}^2)\,\frac{\omega}{2f_\pi{}^2} = -\,\frac{G^4}{16\pi^2 m^2}\,[\,\omega(1 + c - 2) + O(\omega^2)] + O(G^6).$$

To proceed with this we write [see the expression for the axial-vector current, Eq. (2.15)]

$$g_A = 1 + G^2 g_A^{(2)} + \cdots . \qquad (2.24)$$

Substituting this, we find (2.11b) is satisfied if

$$\frac{G^2}{m^2} = \frac{1}{f_\pi{}^2} \qquad (2.25)$$

[the renormalized version of (2.4)] and

$$c - 1 = \frac{1}{G^2/16\pi^2}\,(G^2 g_A^{(2)}). \qquad (2.26)$$

Although $g_A^{(2)}$ must be computable in the LσM, we are avoiding detailed LσM calculations; so, since $g_A \simeq 1.25$ is not so different from 1, we will simply make the empirical estimate

$$G^2 g_A^{(2)} \simeq 0.25. \qquad (2.27)$$

We will shortly give a theoretical estimate, which gives a similar result. The condition that the residue of the nucleon pole terms is the renormalized πN coupling constant is written as

$$\left(G^2 - \frac{G^4}{16\pi^2}\,b_P + \cdots\right) = g^2. \qquad (2.28)$$

But $G^2 = m^2/f_\pi{}^2 = 97.5$ while $g^2 = 184$, so we do not want to make the approximation that the second-order term accounts for the entire difference. Instead we use the Goldberger-Treiman relation (which is exact in the chiral-invariant limit)

$$g_A m = g f_\pi \qquad (2.29)$$

or

$$g^2 = \frac{m^2}{f_\pi^2}\,g_A{}^2$$
$$= G^2(1 + 2G^2 g_A^{(2)} + \cdots). \qquad (2.29')$$

Comparison of (2.28) with (2.29′) gives

$$b_P = \frac{-2}{G^2/16\pi^2}\,(G^2 g_A^{(2)}). \qquad (2.30)$$

Comparing (2.30) with (2.26) gives the relation

$$c = 1 - \tfrac{1}{2} b_P, \qquad (2.31)$$

analogous to (2.23). The estimate (2.27) substituted into (2.30) gives the estimate

$$b_P \simeq -0.8. \qquad (2.32)$$

Next we provide a brief, but amusing, theoretical estimate of this constant [or, via Eq. (2.30), of $g_A^{(2)}$]. Since the results of this paper rest on the assumption that the [1, 1] Padé approximant provides a decent approximation to the sum of the chiral perturbation series, at least at moderately low energies, we apply the Padé algorithm to the summation of the series on the left-hand side of (2.28). Estimating this by the [1, 1] Padé approximant we have

$$g^2 \simeq \frac{G^4}{G^2 + (G^4/16\pi^2) b_P}. \qquad (2.33)$$

Substituting the experimental values of G^2 and g^2 quoted above, we solve for $b_P = -0.76$, in remarkably close agreement with the semiempirical estimate (2.32).

We comment on the relation[8] of the chiral low-energy limit considerations presented here to the better-known Adler-Weisberger-Weinberg low-energy theorem based on current algebra and PCAC. The difference comes from the nucleon-pole terms, which occur both in the derivation of the LET and in the tree diagrams calculated from the σ-model Lagrangians. The value of these nucleon-pole terms depends on whether one takes first $q^2 = \mu^2 \to 0$, then $|\vec{q}| \to 0$ (the chiral-invariant limit), or first $|\vec{q}| \to 0$, then $q_0 \to 0$ (the PCAC limit). In the PCAC limit, the nucleon-pole terms do not contribute to the LET and the $g_A{}^2$, present in (2.11b), does not appear, i.e.,

$$A^{(-)}(\omega, t=0) + \omega B^{(-)}(\omega, t=0) \underset{\text{PCAC lim}}{=} \frac{\omega}{2f_\pi^2} + O(\omega^2),$$

$$\omega = \frac{p \cdot q}{m} = \mu + O(\mu^2).$$

Correspondingly, the value of the tree-diagram pole terms in $B^{(-)}$ is different:

$$\frac{1}{m^2 - s} + \frac{1}{m^2 - u} \underset{\text{PCAC lim}}{=} \frac{2q^2}{4m^2\omega^2}$$
$$\underset{\text{PCAC lim}}{=} \frac{1}{2m^2}.$$

Making these modifications we find that the Born terms of (2.18) satisfy the LET, provided Eq. (2.25) is satisfied ($G = m/f_\pi$). The condition $a = 1 + b_P$ from the isotopic even amplitude is unmodified. But to determine the value of c from this limit one has to know the PCAC threshold-limit values of all the integrals of (2.18), which cannot be recovered from our calculations which have set $\mu = 0$ at an early stage. We conjecture that if one did compute all the integrals keeping $\mu \neq 0$, and set $|\vec{q}| = 0$ and then expanded in powers of μ/m, then the condition that the G^4 terms give no contribution of order μ/m would give directly the condition $c = 1 - \frac{1}{2} b_P$(2.31).

One point concerning the chiral limit $\mu=0$ needs mention. The pion loop integral, subtracted at zero four-momentum, has an "infrared" divergence when the pion mass goes to zero:

$$\lim_{\mu\to 0} I_{\pi\pi}^{(0)}(t)=\frac{1}{16\pi^2}\left[\ln\left(\frac{-t}{\mu^2}\right)-2\right].$$

This implies that the subtraction constant b in (2.18) must also have a logarithmic dependence on the pion mass.[2,6] We can remove this pion-mass dependence by choosing a different subtraction point for this integral, i.e., we subtract at $t=-m^2$,

$$I_{\pi\pi}^{(0)}(t)=I_{\pi\pi}^{(-m^2)}(t)+\frac{1}{16\pi^2}\left[\ln\left(\frac{m^2}{\mu^2}\right)-2\right],$$

$$I_{\pi\pi}^{(-m^2)}(t) \underset{\mu=0}{=} \frac{1}{16\pi^2}\ln\left(\frac{-t}{m^2}\right),$$

and we combine

$$b+\ln\left(\frac{m^2}{\mu^2}\right)-2=\bar{b} \text{ independent of } \mu. \tag{2.34}$$

Then the $A^{(+)}$ amplitude in (2.18) is rearranged as

$$A^{(+)}=\frac{G^2}{m}-G^4\left[\frac{1}{16\pi^2 m}\left(a+\bar{b}\,\frac{t}{m^2}\right)+\cdots +\frac{t}{m^3}I_{\pi\pi}^{(-m^2)}(t)+\cdots\right] \tag{2.18$'$}$$

and all terms in (2.18′) are well defined in the limit $\mu\to 0$. (See also the note at the end of Appendix A.)

III. THE ISOSPIN AND PARTIAL-WAVE AMPLITUDES AND PHASE SHIFTS

The isospin amplitudes are

$$M^{(1)}=M^{(+)}+2M^{(-)},\quad M^{(3)}=M^{(+)}-M^{(-)}. \tag{3.1}$$

Incorporating the rearrangement (2.18′), and making some rearrangements of the linear combinations of t-channel integrals in (2.18), we obtain from (2.18)

$$\begin{aligned}
A^{(1)}=\frac{G^2}{m}-G^4\Bigg\{&\frac{1}{16\pi^2 m}\left(a+\bar{b}\,\frac{t}{m^2}\right)+\left(\frac{1}{2m}+2\,\frac{m}{s}\right)I_{\pi N}^{(0)}(s)-\frac{7}{2m}I_{\pi N}^{(0)}(u)+2m\Big(3J_{N\pi N}(s)-J_{N\pi N}(u)\Big)\\
&+\mathfrak{A}(s,t,u)+5\mathfrak{A}(u,t,s)+\frac{3}{m}I_{NN}^{(0)}(t)+\frac{1}{m}\left[\frac{t}{m^2}+10+\frac{12s+52m^2}{t-4m^2}+\frac{48m^2(s+m^2)}{(t-4m^2)^2}\right]I_{\pi\pi}^{(-m^2)}(t)\\
&+\frac{1}{m}\left[4s+4t+24m^2+\frac{40m^2s+120m^4}{t-4m^2}+\frac{96m^4(s+m^2)}{(t-4m^2)^2}\right]K_{N\pi\pi}(t)\Bigg\},\\
A^{(3)}=\frac{G^2}{m}-G^4\Bigg\{&\frac{1}{16\pi^2 m}\left(a+\bar{b}\,\frac{t}{m^2}\right)+\left(-\frac{5}{2m}+\frac{m}{2s}\right)I_{\pi N}^{(0)}(s)+\left(-\frac{1}{2m}+\frac{3m}{2u}\right)I_{\pi N}^{(0)}(u)+4mJ_{N\pi N}(u)\\
&+4\mathfrak{A}(s,t,u)+2\mathfrak{A}(u,t,s)+\frac{3}{m}I_{NN}^{(0)}(t)+\frac{1}{m}\left[\frac{t}{m^2}+1-\frac{6s+2m^2}{t-4m^2}-\frac{24m^2(s+m^2)}{(t-4m^2)^2}\right]I_{\pi\pi}^{(-m^2)}(t)\\
&-\frac{1}{m}\left[2s-t+\frac{20m^2s+12m^4}{t-4m^2}+\frac{48m^4(s+m^2)}{(t-4m^2)^2}\right]K_{N\pi\pi}(t)\Bigg\},\\
B^{(1)}=G^2\left(\frac{3}{m^2-s}+\frac{1}{m^2-u}\right)&\\
-G^4\Bigg\{&\frac{b_P}{16\pi^2}\left(\frac{3}{m^2-s}+\frac{1}{m^2-u}\right)+\frac{2c}{16\pi^2m^2}+\left(\frac{1}{2m^2}+\frac{2}{s}\right)I_{\pi N}^{(0)}(s)-\frac{1}{2m^2}I_{\pi N}^{(0)}(u)\\
&-2J_{N\pi N}(s)-6J_{N\pi N}(u)-\frac{6}{s-m^2}\left[I_{\pi N}^{(m^2)}(s)+2m^2J_{N\pi N}^{(m^2)}(s)\right]-\frac{2}{u-m^2}\left[I_{\pi N}^{(m^2)}(u)+2m^2J_{N\pi N}^{(m^2)}(u)\right]\\
&+\mathfrak{B}(s,t,u)-5\mathfrak{B}(u,t,s)+\frac{2}{m^2}\left(1+\frac{4m^2}{t-4m^2}\right)\left[I_{\pi\pi}^{(-m^2)}(t)+2m^2K_{N\pi\pi}(t)\right]\Bigg\},\\
B^{(3)}=G^2\left(\frac{-2}{m^2-u}\right)-G^4\Bigg\{&\frac{b_P}{16\pi^2}\left(\frac{-2}{m^2-u}\right)-\frac{c}{16\pi^2m^2}+\left(\frac{1}{2m^2}+\frac{1}{2s}\right)I_{\pi N}^{(0)}(s)-\left(\frac{1}{2m^2}+\frac{3}{2u}\right)I_{\pi N}^{(0)}(u)+4J_{N\pi N}(s)\\
&+\frac{4}{u-m^2}\left[I_{\pi N}^{(m^2)}(u)+2m^2J_{N\pi N}^{(m^2)}(u)\right]+4\mathfrak{B}(s,t,u)-2\mathfrak{B}(u,t,s)\\
&-\frac{1}{m^2}\left(1+\frac{4m^2}{t-4m^2}\right)\left[I_{\pi\pi}^{(-m^2)}(t)+2m^2K_{N\pi\pi}(t)\right]\Bigg\}.
\end{aligned} \tag{3.2}$$

The kinematics involved in the πN partial-wave projections are well known. We have, for the s- and p-wave amplitudes,

$$f^{(2I)}_{0\,1/2}=\frac{E+m}{8\pi W}\left[A^{(2I)}_0+(W-m)B^{(2I)}_0\right]+\frac{E-m}{8\pi W}\left[-A^{(2I)}_1+(W+m)B^{(2I)}_1\right],$$

$$f^{(2I)}_{1\,3/2}=\frac{E+m}{8\pi W}\left[A^{(2I)}_1+(W-m)B^{(2I)}_1\right]+\frac{E-m}{8\pi W}\left[-A^{(2I)}_2+(W+m)B^{(2I)}_2\right],\qquad(3.3)$$

$$f^{(2I)}_{1\,1/2}=\frac{E+m}{8\pi W}\left[A^{(2I)}_1+(W-m)B^{(2I)}_1\right]+\frac{E-m}{8\pi W}\left[-A^{(2I)}_0+(W+m)B^{(2I)}_0\right],$$

where

$$A^{(2I)}_l(s)=\tfrac{1}{2}\int_{-1}^{1}dx\,P_l(x)A^{(2I)}(s,t,u),$$
$$B^{(2I)}_l(s)=\tfrac{1}{2}\int_{-1}^{1}dx\,P_l(x)B^{(2I)}(s,t,u).\qquad(3.4)$$

Each of the twelve functions $\mathrm{Re}A^{(2I)}_l(s)$, $\mathrm{Re}B^{(2I)}_l(s)$ are given as a sum of integrals, in addition to the polynomial and pole terms, obtained by substitution of (3.2) into (3.4). The integrals were evaluated numerically by computer. (Many can be done analytically, and were checked that way.) With the values of these integrals we have $\mathrm{Re}f^{(2I)}_{lJ}$ determined from (3.3). The imaginary parts of the $f^{(2I)}_{lJ}$ can be obtained from the previously computed imaginary parts of all of the integrals and (3.4) and (3.3). They satisfy perturbative unitarity

$$\mathrm{Im}f^{(2I)}_{lJ(\text{one-loop})}=q\,|f^{(2I)}_{lJ(\text{tree})}|^2.\qquad(3.5)$$

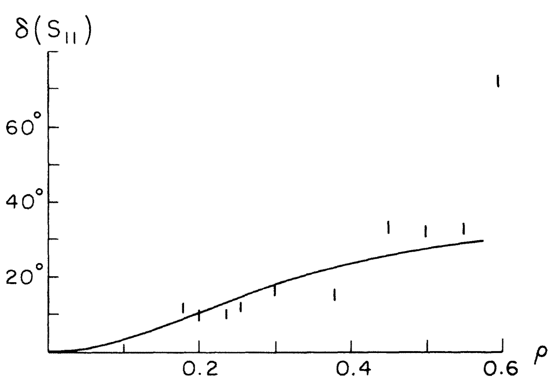

FIG. 8. Calculated S_{11} phase shift compared with values from the πN phase-shift analysis of Ref. 9. (The vertical dashes are not error bars.)

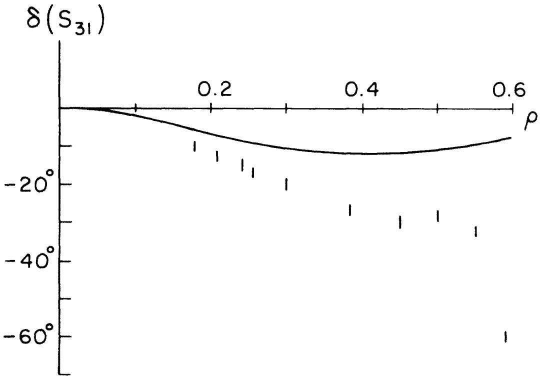

FIG. 9. S_{31} phase shift.

Our (standard) normalization is

$$f^{(2I)}_{lJ}=\frac{1}{q}\,e^{i\delta^{(2I)}_{lJ}}\sin\delta^{(2I)}_{lJ}.\qquad(3.6)$$

The [1, 1] Padé approximant is

$$f^{[1,1]}=\frac{f_{\text{tree}}^{\,2}}{f_{\text{tree}}-f_{\text{one-loop}}}\qquad(3.7)$$

for each I, l, J. It follows from (3.5) that $f^{[1,1]}$ satisfies elastic unitarity exactly:

$$\mathrm{Im}f^{[1,1]}=q\,|f^{[1,1]}|^2.\qquad(3.8)$$

Thus $f^{[1,1]}$ is of the form (3.6) and the phase shifts may be computed from

$$\tan\delta^{[1,1]}=\frac{\mathrm{Im}f^{[1,1]}}{\mathrm{Re}f^{[1,1]}}=q\,\frac{f_{\text{tree}}^{\,2}}{f_{\text{tree}}-\mathrm{Re}f_{\text{one-loop}}}.\qquad(3.9)$$

The results are plotted in Figs. 8–13.

IV. DISCUSSION

Comparing Figs. 8–13 with the six experimental[9] s- and p-wave, $I=\frac{1}{2}$ and $\frac{3}{2}$ phase shifts, one finds the following areas of qualitative agreement: The sign of each of the six phase shifts, just above threshold, is obtained correctly (including the

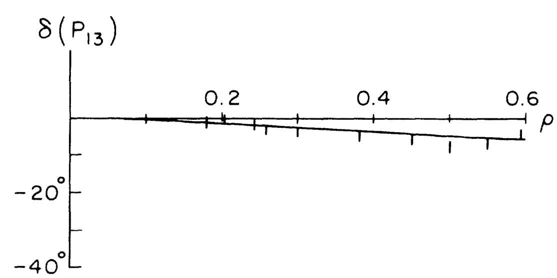

FIG. 10. P_{13} phase shift.

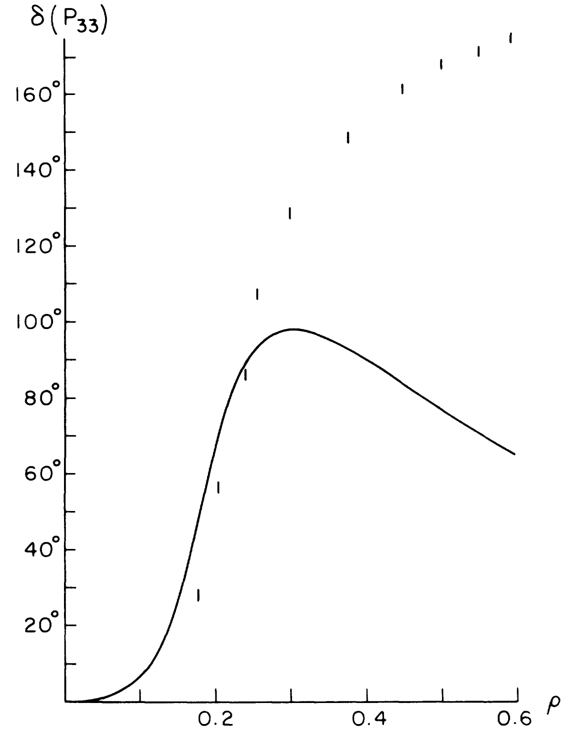

FIG. 11. P_{33} phase shift.

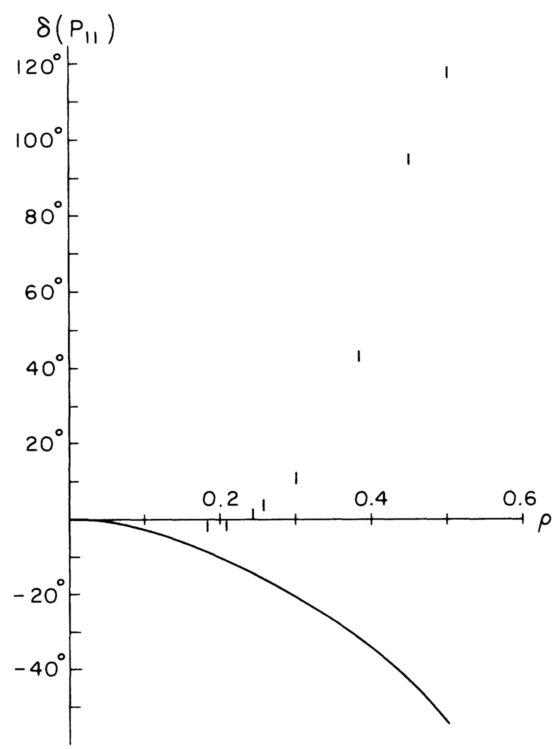

FIG. 12. P_{11} phase shift.

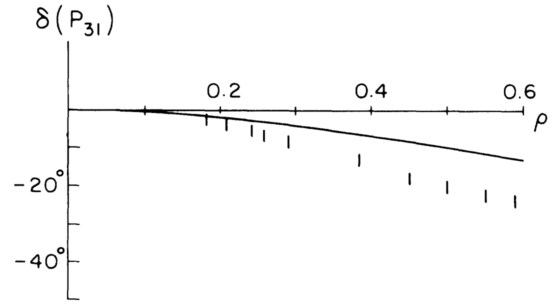

FIG. 13. P_{31} phase shift.

negative value for P_{11}). The P_{33} wave goes through 90° at the nominal mass of the $\Delta(1232)$; none of the other waves reach (positive or negative) 90° below $\rho=0.6$ ($W=1675$ MeV, for $\mu\neq 0$). Small experimental phase shifts are calculated to be small. The calculation has just one arbitrary parameter, the subtraction constant $\bar{b}$, which is adjusted ($\bar{b}=-1.4$) to make the P_{33} resonance occur at the correct value of $\rho=q/m$. That this can be done is not trivial; for example, we have observed that if one determines the constant b_P by a perturbative evaluation of Eq. (2.28) (replace the sum by the first two terms) one obtains $b_P=-1.43$ rather than -0.8, and for this value of b_P no value of the arbitrary constant $\bar{b}$ will produce a 33 resonance at $\rho=0.24$. On the other hand, it is clear that this calculation does not fit the πN phase shifts. The one spectacular disagreement is with P_{11} phase shift, for which the experimental phase shift changes sign less than 200 MeV above threshold and resonates at $\rho\sim 0.45$. Although the 33 resonance is obtained, the calculated phase shift turns over shortly above the resonance while the experimental phase shift continues to rise, and even for the other waves, for which there is rough agreement below $\rho=0.6$, it is clear that the experimental phase shifts have structure [e.g., the inelastic S_{11} (1535) resonance] not produced by the present approximate calculation. The P_{11} wave and the structure in the S_{11} wave are associated with strong inelasticity, and one cannot expect good results for them from the single-channel [1, 1] Padé approximant, which necessarily is purely elastic (satisfies elastic unitarity).

The difficulty with the P_{33} above the resonance can probably not be attributed to inelasticity since experimentally that wave remains elastic for 200 MeV above the resonance. To go beyond the present approximations in this case one has to think of higher-order Padé approximants, e.g. [2, 1] or [2, 2], and/or additional particles, e.g. $K\Lambda$ or $K\Sigma$ *virtual* intermediate states. However, our strategy now is to return to the $\pi\pi$ problem, where there is also a calculated phase shift (S_{00}) that turns over where the experimental phase shift continues to rise (above 700 MeV) and important inelastic effects associated with the $K\bar{K}$ threshold just below 1000 MeV. Here, the significant inelasticity is confined to a single two-body spin-zero channel, so that purely kinematic complications are much less than in the πN problem. If we succeed in significantly improving the $\pi\pi$ S_{00} wave calculation, e.g., produce the inelastic S^* resonance, we will return to the πN problem. In particular, the inelastic S_{11} (1535) resonance decays almost entirely into πN and ηN, a system which should be amenable to any two-body coupled-channel analysis developed for the $\pi\pi$, $K\bar{K}$ system.

We have made the following checks (in addition to those already mentioned) on our numerical results. A computer program to numerically evaluate all of the partial-wave integrals defined by (3.2) and (3.4), to combine them (3.4) and to compute the [1, 1] Padé phase shifts (3.9), was put together by one of us. That same calculation, for one value of ρ, was carried out without the computer, with an HP55, by the other of us. The results agreed, so we trust the results of the computer program for the other values of ρ. Another independent check was provided by the low-energy expansion. Each of the one-loop Feynman integrals, Eq. (2.18) and Appendix A, on the physical sheet may be expanded in powers of ρ and $\ln\rho$. This expansion, through order ρ, was necessary in order to determine the subtraction constants in (2.18) by comparison with the chiral LET. To get a better approximation, we have carried out this expansion through order ρ^2 (Appendix B). The resulting invariant functions $A^{(\pm)}$, $B^{(\pm)}$ are very much simpler than the "exact" expressions (2.18) and Appendix A, but should be a good approximation to them for $\rho \ll 1$. We have computed the [1, 1] phase shifts using the approximate low-energy expressions (a simple pocket-calculator numerical calculation—see Appendix B), and indeed the results of the full calculation do converge to the low-energy approximation results as ρ becomes small.

The existence of this low-energy expansion, and the exact low-energy theorem (in the chiral-invariant limit), leads to the following important observation. The signs of all six phase shifts just above the (chiral) threshold are determined by the Born term and the fixed constant b_P, i.e., they are independent of the arbitrary subtraction constant $\bar{b}$ [see (B7) and (B8)]. The leading terms for $\rho \to 0$ are independent of $\bar{b}$). The signs so determined are: S_{11} and P_{33} positive, S_{31}, P_{13}, P_{11}, and P_{31} negative— all, including P_{11}, in agreement with the experimentally determined phase shifts. Furthermore, because of the positivity of the square of the Born term, any phase shift computed from the [1, 1] Padé approximant (3.9) cannot change sign.[10] Thus one can state that no chiral-invariant, single-channel [1, 1] Padé calculation can produce the experimental sign change and 1470 resonance in the P_{11} wave.

Our results disagree with those of two previous Padé calculations alluded to in the Introduction. The first calculation, by Filkov and Palyushev[7] (FP), did not use a Lagrangian. Rather they started from a Born term consisting of the usual (pseudoscalar) nucleon-pole terms plus a constant in the $A^{(+)}$ amplitude. If that constant were taken to be m/f_π^2, then the Born term would be the same as the chiral-invariant NLσM Born term. FP did not choose the constant that way, rather they chose it to fit a combination of the experimental s- and p-wave scattering lengths. They then used iterated unitarity in the s- and u-channels and single-variable dispersion relations to compute the second-order amplitude. t-channel structure was parametrized by a pole with mass and residue treated as arbitrary parameters. They then constructed the [1, 1] Padé approximant for the partial-wave amplitudes and reported a rather impressive over-all fit to the s- and p-wave phase shifts, although they did not obtain the detailed inelastic structure in the S_{11} and S_{31} waves, and they had the wrong sign for the small P_{13} phase shift. The primary theoretical objection to their calculation is that since it does not incorporate the constraints of chiral invariance,[11] it cannot in principle be generalized to higher orders, because additional undetermined subtraction constants will be required. (The constant in $A^{(+)}$ corresponds to a nonrenormalizable interaction $\bar{\psi}\psi\phi^2$.) A more serious criticism was made by Bergere and Drouffe[7] (BD) who, in addition to a NLσM calculation (discussed below), also took the theoretical expressions of FP and tried to reproduce their numerical results. Instead, they found numerical results which bore no resemblence to those reported by FP.

BD did a NLσM calculation, using LσM regularization described in the introduction. We have argued that, at least in the chiral-invariant ($\mu = 0$) limit, our simpler one-loop calculation should be equivalent; and indeed, when we take the $\mu \to 0$ limit of their expressions they agree with the results given in our Appendix A, up to questions of subtractions—in particular, the imaginary parts are the same. However, we believe that the subtractions of BD do not satisfy the conditions imposed on our subtraction constants in Sec. II. The most easily seen disagreement is with respect to the subtraction constant b, defined in our Eq. (2.10) or (2.18). As pointed out at the end of Sec. II, this subtraction constant must have a particular dependence on $\ln(m^2/\mu^2)$ [Eq. (2.34)] such that the entire matrix element has a limit for $\mu \to 0$. In the calculations of BD this term arises from only one NLσM graph [their (B16′)], and their expression for this [their Eq. (28)] does not have a finite limit as $\mu \to 0$. Furthermore, BD report that by varying their subtraction constant (M for them), they can obtain a positive P_{11} phase shift and a negative P_{33} phase shift. As we have pointed out, this cannot happen in a single-channel chiral-invariant [1, 1] Padé calculation. The LσM regularization of the NLσM maintains the chiral invariance, so the BD calculation should depart from chiral invariance only through the nonzero pion mass. The presence of the pion mass does make a difference, e.g., in the chiral-invariant limit the πN s-wave scattering lengths are zero [the $f_{0\,1/2}$ of (B7) have no constant terms as $\rho \to 0$], while in the PCAC limit, one obtains scattering lengths $a^{(+)}$ proportional to μ and $a^{(-)}$ proportional to μ^2, i.e., small, and vanishing in the limit $\mu \to 0$. Similarly, the threshold behavior of the p-wave amplitudes is $\sim\rho$ in the chiral-invariant limit and $\sim\rho^2$ for $\mu \neq 0$.[12] Thus, we cannot make a conclusive statement about the $\mu \neq 0$ case, but we would be surprised if any of the phase shifts have a different sign when computed with $\mu \neq 0$ than they have when computed with $\mu = 0$.

APPENDIX A: ONE-LOOP FEYNMAN DIAGRAMS AND INTEGRALS

We label the Feynman diagrams of Fig. 7 as $1, 2, \ldots, 11$ (the s-channel diagrams), $1', 2', \ldots, 11'$ (the u-channel diagrams—not shown), and $12, \ldots, 15$ (the t-channel diagrams). The corresponding contributions to the invariant matrix element are:

$$M_{ab}(1) \doteq -G^4\bar{u}(p')(3\tau_b\tau_a)\left\{\frac{m}{2s}I^{(0)}_{\pi N}(s) + \gamma Q\left[\frac{1}{2s}I^{(0)}_{\pi N}(s)\right]\right\}u(p)\,, \tag{A1}$$

$$M_{ab}(2) = M_{ab}(3) \doteq -G^4\bar{u}(p')(\tau_b\tau_a)\left\{-\frac{s+m^2}{2ms}I^{(0)}_{\pi N}(s) + \gamma Q\left[-\frac{1}{2s}I^{(0)}_{\pi N}(s)\right]\right\}u(p)\,, \tag{A2}$$

$$M_{ab}(4) = M_{ab}(5) \doteq -G^4\bar{u}(p')(\tau_b\tau_a)\left\{mJ_{N\pi N}(s) + \gamma Q\left[-\frac{1}{s-m^2}I^{(m^2)}_{\pi N}(s) - \frac{2m^2}{s-m^2}J^{(m^2)}_{N\pi N}(s)\right]\right\}u(p)\,, \tag{A3}$$

$$M_{ab}(6) = M_{ab}(7) \doteq 0\,, \tag{A4}$$

$$M_{ab}(8) \doteq -G^4\bar{u}(p')(\delta_{ab})\left\{\frac{3s+m^2}{2ms}I^{(0)}_{\pi N}(s) + \gamma Q\left[\frac{s+m^2}{2m^2 s}I^{(0)}_{\pi N}(s)\right]\right\}u(p)\,, \tag{A5}$$

$$M_{ab}(9) = M_{ab}(10) \doteq -G^4\bar{u}(p')(\tau_a\tau_b)\left\{-\frac{1}{m}I^{(0)}_{\pi N}(s) + \gamma Q\left[J_{N\pi N}(s)\right]\right\}u(p)\,, \tag{A6}$$

$$M_{ab}(11) \doteq -G^4\bar{u}(p')(\tau_b\tau_a + 2\tau_a\tau_b)\left\{\mathcal{K}_A(s,t) + \frac{2m}{t-4m^2}I^{(4m^2)}_{NN}(t) + \gamma Q\left[\mathcal{K}_B(s,t) - K_{NNN}(t)\right]\right\}u(p)\,, \tag{A7}$$

$$Q = \tfrac{1}{2}(q+q')\,.$$

The matrix elements $M_{ab}(1'), \ldots, M_{ab}(11')$ are obtained from the above by the substitution

$$a \leftrightarrow b\,, \quad s \leftrightarrow u\,, \quad Q \leftrightarrow -Q\,; \tag{A8}$$

$$M_{ab}(12) \doteq -G^4\bar{u}(p')(\delta_{ab})\left[\frac{t}{m^3}I^{(0)}_{\pi\pi}(t)\right]u(p)\,, \tag{A9}$$

$$M_{ab}(13) = M_{ab}(14) \doteq -\frac{1}{2}G^4\bar{u}(p')\Bigg((\delta_{ab})\left[2\frac{t}{m}K_{N\pi\pi}(t) + 4\frac{t}{m}K_+(t)\right] + \frac{1}{2}[\tau_b,\tau_a]\left\{\frac{s-u}{m}[K_{N\pi\pi}(t) + 4K_+(t) + 4K_{++}(t)] + \gamma Q\left[\frac{t}{m^2}K_+(t)\right]\right\}\Bigg)u(p)\,, \tag{A10}$$

$$M_{ab}(15) \doteq -G^4\bar{u}(p')(\delta_{ab})\left[\frac{3}{m}I^{(0)}_{NN}(t)\right]u(p)\,. \tag{A11}$$

We have used $m^2/f_\pi^2 = G^2$. The equal sign with a dot means equal up to the arbitrary polynomial [and constant times $1/(s-m^2)$] implied by the substractions made to obtain convergent integrals. The arbitrary constants and polynomial are restored in the combined invariant matrix element (2.18). The imaginary parts of (A1) through (A11) agree with the imaginary parts obtained from the iterated unitarity Eq. (2.12), etc.

The Feynman integrals appearing in these formulas are as follows:

$$I_{ab}(s) = i\int (dk)\frac{1}{[(k+p)^2-m_a^2][(k-q)^2-m_b^2]},$$

$$(dk) = \frac{d^4k}{(2\pi)^4}, \quad (p+q)^2 = s \tag{A12}$$

$$I_{ab}^{(c)}(s) = I_{ab}(s) - I_{ab}(c), \tag{A13}$$

$$K_{abc}(s) = i\int (dk)\frac{1}{(k^2-m_a^2)[(k+p)^2-m_a^2][(k-q)^2-m_c^2]}. \tag{A14}$$

The combination

$$J_{N\pi N}(s) = K_{N\pi N}(s) + \frac{1}{s-m^2}I_{\pi N}^{(m^2)}(s) \tag{A15}$$

has a finite first derivative at the branch point $s=m^2$ ($\mu=0$). The combinations

$$K_+(t) = \frac{1}{t-4m^2}\left[I_{\pi\pi}^{(-m^2)}(t) + 2m^2K_{N\pi\pi}(t)\right], \tag{A16}$$

$$K_{++}(t) = \frac{1}{2(4m^2-t)}\left[\tfrac{1}{2}I_{\pi\pi}^{(-m^2)}(t) - m^2K_{N\pi\pi}(t) - 6m^2K_+(t)\right] \tag{A17}$$

occur in the scalar decompositions of the vector and tensor integrals

$$i\int (dl)\frac{l^\mu}{(l^2-m^2)(l+p)^2(l+p')^2} = (p+p')^\mu K_+(t), \tag{A18}$$

$$i\int (dl)\frac{l^\mu l^\nu}{(l^2-m^2)(l+p)^2(l+p')^2} = g^{\mu\nu}m^2K_0(t) + p_+^\mu p_+^\nu K_{++}(t) + p_-^\mu p_-^\nu K_{--}(t), \tag{A19}$$

$$p_\pm = p \pm p'$$

$$m^2K_0(t) = \frac{t}{4}K_+(t). \tag{A20}$$

The integral $K_{--}(t)$ does not contribute to the matrix element.

$$\mathcal{K}_A(s,t) + \gamma Q\mathcal{K}_B(s,t) = (s-m^2)i\int (dk)\frac{-\gamma k}{k^2[(k+p)^2-m^2][(k+p')^2-m^2][(k+p+q)^2-m^2]}\bigg|_{\gamma p=\gamma p'=m}, \tag{A21}$$

$$\mathcal{K}_A(s,t) = \frac{m}{2[(s-m^2)^2+st]}\left\{(s-m^2)[2(s-m^2)+t]L(s,t) + 2(s-m^2)^2K_{N\pi N}(s) + t(s-m^2)K_{NNN}(t)\right\}, \tag{A22a}$$

$$\mathcal{K}_B(s,t) = \frac{1}{2[(s-m^2)^2+st]}\left\{(s-m^2)(t-4m^2)L(s,t) - 2(s^2-m^4)K_{N\pi N}(s) + (s-m^2)[2(s-m^2)+t]K_{NNN}(t)\right\}, \tag{A22b}$$

where

$$L(s,t) = (s-m^2)H(s,t) - K_{\pi NN}(t), \tag{A23}$$

$$H(s,t) = i\int (dk)\frac{1}{(k^2)[(k+p)^2-m^2][(k+p')^2-m^2][(k+p+q)^2-m^2]}. \tag{A24}$$

Note that $H(s,t)$ and $K_{\pi NN}(t)$ are separately infrared divergent in the limit $m_\pi \to 0$, but the combination $L(s,t)$ exists for $m_\pi = 0$. In (2.18) we have used the combinations

$$\mathcal{A}(s,t,u) = \mathcal{K}_A(s,t) + \frac{2m}{t-4m^2}I_{NN}^{(4m^2)}(t), \tag{A25a}$$

$$\mathcal{B}(s,t,u) = \mathcal{K}_B(s,t) - K_{NNN}(t). \tag{A25b}$$

Our formulas (A21)–(A25) agree with the $m_\pi \to 0$ limits of the expressions for the πN "box" integrals given by Brunet.[13]

APPENDIX B: CHIRAL THRESHOLD EXPANSION

The chiral ($\mu=0$) threshold expansions of the Feynman integrals are done from the Feynman-parameter form of the integrals. As in high-energy asymptotic expansions, one has to perform $n-1$ of the n Feynman-parameter integrations before one can expand the integrand in powers of $\rho = |\vec{q}|/m$. For several of the integrals it was most convenient to expand first in the variable

$$\xi = \frac{s-m^2}{m^2}, \tag{B1}$$

and then reexpand using

$$\xi = 2\rho + 2\rho^2 + \rho^3 + \cdots \, . \tag{B2}$$

The momentum-transfer variable is of order ρ^2,

$$\mathrm{t} = \frac{t}{m^2} = -2\rho^2(1-\cos\theta)\,; \tag{B3}$$

$$16\pi^2 \,\mathrm{Re}\, I_{\pi N}^{(0)}(s) = -1 + (\xi - \xi^2)\ln\xi + \cdots \, ,$$

$$16\pi^2 \frac{m^2}{s-m^2}\,\mathrm{Re}\, I_{\pi N}^{(m^2)}(s) = (1 - \xi + \xi^2)\ln\xi + \cdots \, ,$$

$$16\pi^2 m^2\,\mathrm{Re}\, J_{N\pi N}(s) = 1 - \tfrac{1}{4}\xi + \tfrac{1}{9}\xi^2 + (-\tfrac{1}{2}\xi + \tfrac{2}{3}\xi^2)\ln\xi + \cdots \, ,$$

$$16\pi^2 \frac{m^4}{s-m^2}\,\mathrm{Re}\, J_{N\pi N}^{(m^2)}(s) = -\tfrac{1}{4} + \tfrac{1}{9}\xi + (-\tfrac{1}{2} + \tfrac{2}{3}\xi)\ln\xi + \cdots \, ,$$

$$16\pi^2 I_{\pi N}^{(0)}(u) = -1 - \mathrm{t} + (-\xi - \mathrm{t} - \xi^2)\ln\xi + \cdots \, ,$$

$$16\pi^2 \frac{m^2}{u-m^2} I_{\pi N}^{(m^2)}(u) = \frac{\mathrm{t}}{\xi} - \mathrm{t} + (1 + \xi + \xi^2 + \mathrm{t})\ln\xi + \cdots \, ,$$

$$16\pi^2 m^2 J_{N\pi N}(u) = 1 + \tfrac{1}{4}\xi + \tfrac{1}{9}\xi^2 + \tfrac{3}{4}\mathrm{t} + (\tfrac{1}{2}\xi + \tfrac{2}{3}\xi^2 + \tfrac{1}{2}\mathrm{t})\ln\xi + \cdots \, ,$$

$$16\pi^2 \frac{m^4}{u-m^2} J_{N\pi N}^{(m^2)}(u) = -\tfrac{1}{4} - \tfrac{1}{9}\xi - \tfrac{1}{2}\frac{\mathrm{t}}{\xi} + (-\tfrac{1}{2} - \tfrac{2}{3}\xi)\ln\xi + \cdots \, ,$$

$$16\pi^2 I_{NN}^{(0)}(t) = -\tfrac{1}{6}\mathrm{t} + \cdots \, , \tag{B4}$$

$$16\pi^2 I_{\pi\pi}^{(-m^2)}(t) = \ln(-\mathrm{t})\, ,$$

$$16\pi^2 m^2 K_{N\pi\pi}(t) = \frac{\pi^2}{2\sqrt{-\mathrm{t}}} + \tfrac{1}{2}\ln(-\mathrm{t}) + \cdots \, ,$$

$$16\pi^2 m\,\mathrm{Re}\,\mathcal{K}_A(s,t) = -\tfrac{1}{4}\xi + \tfrac{1}{9}\xi^2 + (\tfrac{1}{2}\xi - \tfrac{1}{3}\xi^2)\ln\xi + \cdots \, ,$$

$$16\pi^2 m\,\mathcal{K}_A(u,t) = \tfrac{1}{4}\xi + \tfrac{1}{9}\xi^2 - \tfrac{1}{4}\mathrm{t} + (-\tfrac{1}{2}\xi - \tfrac{1}{3}\xi^2 - \tfrac{1}{2}\mathrm{t})\ln\xi + \cdots \, ,$$

$$16\pi^2 \frac{2m^2}{t-4m^2} I_{NN}^{(4m^2)}(t) = -1 - \tfrac{1}{6}\mathrm{t} + \cdots \, ,$$

$$16\pi^2 m^2\,\mathrm{Re}\,\mathcal{K}_B(s,t) = \tfrac{1}{36}\xi + \tfrac{1}{6}\xi\ln\xi + \cdots \, ,$$

$$16\pi^2 m^2\,\mathcal{K}_B(u,t) = -\tfrac{1}{36}\xi - \tfrac{1}{6}\xi\ln\xi + \cdots \, ,$$

$$16\pi^2 m^2 K_{NNN}(t) = \tfrac{1}{2} + \tfrac{1}{24}\mathrm{t} + \cdots \, .$$

In (B4) we have not expanded all integrals through order ρ^2, because not all are needed that far. All integrals which appear only in the invariant functions $B^{(\pm)}$ are multiplied by at least one power of ρ in the partial-wave amplitudes (3.3). And for $\rho \to 0$, the coefficients of the singular integral $K_{N\pi\pi}(t)$ in $A^{(1)}, A^{(3)}, B^{(1)}, B^{(3)}$ of (3.2) are

$$A^{(1)}:\ \tfrac{1}{2}\xi\mathrm{t}\, ,$$

$$A^{(3)}:\ -\tfrac{1}{4}\xi\mathrm{t}\, ,$$

$$B^{(1)}:\ -\mathrm{t}\, ,$$

$$B^{(3)}:\ \tfrac{1}{2}\mathrm{t}\, .$$

Substituting these results into (3.2), and using (B2), we have

$$\mathrm{Re}\,A^{(1)} = \frac{G^2}{m} - \frac{G^4}{16\pi^2 m}\left[a - 1 + 2\rho - \frac{14}{9}\rho^2 + \left(\bar{b} - \frac{3}{4}\right)\mathrm{t} - \frac{\pi^2}{2}\rho\sqrt{-\mathrm{t}} - \frac{4}{3}\rho^2\ln 2\rho + \cdots\right],$$

$$\mathrm{Re}\,A^{(3)} = \frac{G^2}{m} - \frac{G^4}{16\pi^2 m}\left[a - 1 - \rho - \frac{41}{9}\rho^2 + \left(\bar{b} - \frac{3}{2}\right)\mathrm{t} + \frac{\pi^2}{4}\rho\sqrt{-\mathrm{t}} - \frac{4}{3}\rho^2\ln 2\rho + \cdots\right],$$

$$\mathrm{Re}\,B^{(1)} = \frac{G^2}{m^2}P^{(1)} - \frac{G^4}{16\pi^2 m^2}\left(b_P P^{(1)} + 2c - 4 + \frac{5}{9}\rho + \frac{\pi^2}{2}\sqrt{-\mathrm{t}} + \frac{4}{3}\rho\ln 2\rho + \cdots\right),$$

$$\mathrm{Re}\,B^{(3)} = \frac{G^2}{m^2}P^{(3)} - \frac{G^4}{16\pi^2 m^2}\left(b_P P^{(3)} - c + 2 + \frac{5}{9}\rho - \frac{\pi^2}{4}\sqrt{-\mathrm{t}} + \frac{4}{3}\rho\ln 2\rho + \cdots\right), \tag{B5}$$

where

$$\frac{1}{m^2}P^{(1)} = \frac{3}{m^2 - s} + \frac{1}{m^2 - u} = \frac{1}{m^2}\left(-\frac{1}{\rho} + 1 - \frac{1}{2}\rho - \frac{\mathrm{t}}{4\rho^2} + \frac{\mathrm{t}}{2\rho} + \frac{\mathrm{t}^2}{8\rho^3} + \cdots\right),$$

$$\frac{1}{m^2}P^{(3)} = -\frac{2}{m^2 - u} = \frac{1}{m^2}\left(-\frac{1}{\rho} + 1 - \frac{1}{2}\rho + \frac{\mathrm{t}}{2\rho^2} - \frac{\mathrm{t}}{\rho} - \frac{\mathrm{t}^2}{4\rho^3} + \cdots\right). \tag{B6}$$

Putting in the values $a = 1 + b_P$ and $c = 1 - \frac{1}{2}b_P$, and substituting (B5) and (B6) into (3.3) and (3.4), we have the chiral threshold expansions of the partial-wave amplitudes:

$$\begin{aligned}
8\pi f^{(1)}_{0\,1/2} &= \frac{G^2}{m}\left(2\rho - \tfrac{7}{3}\rho^2 + \cdots\right) - \frac{G^4}{16\pi^2 m}\left[\left(-1 - \tfrac{4}{3}b_P - 4\bar{b}\right)\rho^2 + \cdots\right],\\
8\pi f^{(3)}_{0\,1/2} &= \frac{G^2}{m}\left(-\rho + \tfrac{5}{3}\rho^2 + \cdots\right) - \frac{G^4}{16\pi^2 m}\left[\left(-1 + \tfrac{7}{6}b_P - 4\bar{b}\right)\rho^2 + \cdots\right],\\
8\pi f^{(1)}_{1\,3/2} &= \frac{G^2}{m}\left(-\tfrac{1}{3}\rho + \tfrac{1}{6}\rho^2 + \cdots\right) - \frac{G^4}{16\pi^2 m}\left[-\tfrac{1}{3}b_P\rho + \left(-1 + \tfrac{1}{6}b_P + \tfrac{4}{3}\bar{b}\right)\rho^2 + \cdots\right],\\
8\pi f^{(3)}_{1\,3/2} &= -\frac{G^2}{m}\left(\tfrac{2}{3}\rho - \tfrac{1}{3}\rho^2 + \cdots\right) - \frac{G^4}{16\pi^2 m}\left[\tfrac{2}{3}b_P\rho + \left(-2 - \tfrac{1}{3}b_P + \tfrac{4}{3}\bar{b}\right)\rho^2 + \cdots\right],\\
8\pi f^{(1)}_{1\,1/2} &= \frac{G^2}{m}\left(-\tfrac{4}{3}\rho + \tfrac{5}{3}\rho^2 + \cdots\right) - \frac{G^4}{16\pi^2 m}\left[-\tfrac{4}{3}b_P\rho + \left(-3 + \tfrac{2}{3}b_P + \tfrac{4}{3}\bar{b}\right)\rho^2 + \cdots\right],\\
8\pi f^{(3)}_{1\,1/2} &= \frac{G^2}{m}\left(-\tfrac{1}{3}\rho - \tfrac{1}{3}\rho^2 + \cdots\right) - \frac{G^4}{16\pi^2 m}\left[-\tfrac{1}{3}b_P\rho + \left(-1 + \tfrac{1}{6}b_P + \tfrac{4}{3}\bar{b}\right)\rho^2 + \cdots\right],
\end{aligned} \tag{B7}$$

where $b_P \simeq -0.8$ and $\bar{b}$ is the one arbitrary parameter. Finally, substitute (B7) into (3.9) to compute the phase shifts in this approximation:

$$\begin{aligned}
\tan S_{11} &= \frac{(G^2/4\pi)\rho^2(1 - \frac{7}{6}\rho)^2}{1 - \frac{7}{6}\rho + (G^2/16\pi^2)[(-\frac{1}{2} - \frac{2}{3}b_P - 2\bar{b})\rho]}, \quad \tan S_{31} = \frac{-(G^2/4\pi)\frac{1}{2}\rho^2(1 - \frac{5}{3}\rho)^2}{1 - \frac{5}{3}\rho + (G^2/16\pi^2)[(1 - \frac{7}{6}b_P + 4\bar{b})\rho]},\\
\tan P_{13} &= \frac{-(G^2/4\pi)\frac{1}{6}\rho^2(1 - \frac{1}{2}\rho)^2}{1 - \frac{1}{2}\rho + (G^2/16\pi^2)[b_P + (3 - \frac{1}{2}b_P - 4\bar{b})\rho]}, \quad \tan P_{33} = \frac{(G^2/4\pi)\frac{1}{3}\rho^2(1 - \frac{1}{2}\rho)^2}{1 - \frac{1}{2}\rho + (G^2/16\pi^2)[b_P + (-3 - \frac{1}{2}b_P + 2\bar{b})\rho]},\\
\tan P_{11} &= \frac{-(G^2/4\pi)\frac{2}{3}\rho^2(1 - \frac{5}{4}\rho)^2}{1 - \frac{5}{4}\rho + (G^2/16\pi^2)[b_P + (\frac{9}{4} - \frac{1}{2}b_P - \bar{b})\rho]}, \quad \tan P_{31} = \frac{-(G^2/4\pi)\frac{1}{6}\rho^2(1 + \rho)^2}{1 + \rho + (G^2/16\pi^2)[b_P + (3 - \frac{1}{2}b_P - 4\bar{b})\rho]}.
\end{aligned} \tag{B8}$$

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[10]An exception could occur if the Born term had a zero at exactly the energy for which the [1, 1] Padé demoninator goes through zero. However, the zeros of the Born term are not adjustable, so there is no way to make this happen at any desired energy in any particular partial wave.

[11]In addition to the Born term not being chiral invariant, FP used g as the perturbation-expansion parameter. Chiral invariance requires the perturbation-expansion parameter to be $G = m/f_\pi$.

[12]The usual $\delta_l \sim \rho^{2l+1}$, and its breakdown for $\mu \to 0$, can be seen, e.g., from the Q_l functions which arise from the partial-wave integrations. For the u-channel pole one encounters $Q_l[-1 + (2m\omega - \mu^2)/2p^2]$, which is $\sim(\rho^2)^l$ for $\mu \neq 0$ and ρ^l for $\mu = 0$.

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