Augmented quantum field theory: A proposal to extend conventional formulations

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An alternative approach to scalar field quantization is proposed and analyzed, particularly for $(\varphi^4)_n$ models, $n \geq 2$. Without altering the classical equation of motion at all, the action is "augmented" by an additional term that in effect induces a new measure in a functional integration approach to quantization. Guided by specialized soluble models, a lattice-space formulation is proposed for covariant theories for which in the continuum limit the truncated four-point correlation function is non-negative in contrast to the conventional formulation. Besides suggesting nontrivial behavior for $n \geq 4$, the augmented models lead to new noncanonical solutions for $n = 2, 3$. All solutions of the augmented models are disconnected from those of the conventional approach in the sense that the augmented models pass to a pseudofree model differing from the free model as the nonlinear coupling constant vanishes.

I. INTRODUCTION

The quantum field theory of self-interacting scalar fields has long served as a theoretical laboratory for developing and testing new methods of formulation and solution. In recent years the superrenormalizable $P(\varphi)_2$ and $(\varphi^4)_3$ models have been rigorously established outside of perturbation-theoretic limitations.[1] The construction of these models has proceeded along lines suggested by conventional renormalized perturbation theory, and thus it is perhaps no great surprise that the results largely conform with the heuristic expectations of perturbation theory. On the other hand, predictions for theories such as $(\varphi^4)_n$, $n \geq 4$, based on perturbation theory, seem to be at variance with presently proposed nonperturbative arguments based on renormalization-group calculations.[2] It is the conclusion of these latter arguments that the ultimate theory is *free* whether one deals with renormalizable ($n = 4$) or so-called nonrenormalizable models ($n \geq 5$). Even the presently constructed superrenormalizable models $(\varphi^4)_n$, $n = 2, 3$, seem to possess an inherent limitation in that there is, for these models, a maximum (finite) value allowed for the renormalized coupling constant λ_{ren}.[3]

The present situation clearly poses a dilemma. Should one really believe that models such as $(\varphi^4)_2$ and $(\varphi^4)_3$ are constrained by upper limits on λ_{ren}, and in addition that models such as $(\varphi^4)_n$, $n \geq 4$, are nothing more than free theories? Or are these results an inevitable consequence of certain limitations inherent in the conventional approach to these problems?

In this paper we outline an *alternative approach* to scalar field quantization that leads to different solutions for superrenormalizable and renormalizable models and potentially provides a nontrivial solution for nonrenormalizable models. All indications point to the overall consistency of our approach, but existence questions have not yet been completely settled.

When one breaks with tradition, one needs both alternative ideas and plausible motivation. We find it expedient to present the alternative ideas along with heuristic motivation initially (Sec. II), and to argue for their most basic potential validity and to give a concrete (lattice-space) formulation subsequently (Sec. III).

II. HEURISTIC FORMULATION AND DISCUSSION OF THE AUGMENTED MODELS

Classical preliminaries

The classical theory under discussion, restricted for convenience to a quartic interaction, is described by the action

$$I = \int [\tfrac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \tfrac{1}{2}m^2\varphi^2 - \lambda\varphi^4]dx \tag{1}$$

($dx \equiv d^n x$), and the classical equation of motion

$$(\Box + m^2)\varphi = -4\lambda\varphi^3 \tag{2}$$

is derived from stationarity of the action.

Recall that in theories with contraints (which these models are not) auxiliary variables in the form of Lagrange multipliers can be introduced into the action in a linear fashion, and are used to impose the constraints by insisting that the action be stationary with respect to their variation. In the functional integral approach to quantum field theory such Lagrange variables linear in the action lead to δ functionals that impose the constraints on the remaining variables of integration. Our models do not have constraints in the usual sense, but we wish to modify the action in a not unrelated fashion.

Rather than Eq. (1) we propose to adopt the *augmented* classical action

$$I' = \int [\tfrac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \tfrac{1}{2}m^2\varphi^2 - \lambda\varphi^4 - \tfrac{1}{2}\chi^2\varphi^2]dx\,, \tag{3}$$

where $\chi(x)$ is an auxiliary field variable. Stationarity of the action with respect to arbitrary variations of φ and χ leads to the two equations of motion

$$(\Box + m^2)\varphi = -4\lambda\varphi^3 - \chi^2\varphi\,, \tag{4}$$

$$\chi\varphi^2 = 0. \tag{5}$$

The field χ enters the action in a nonlinear fashion and that is reflected by the appearance of χ in both equations of motion. Evidently, $\chi\varphi^2=0$ implies that $\chi^2\varphi=0$, so that *the field φ still satisfies Eq.* (2). Wherever $\varphi(x)\neq 0$ it follows that $\chi(x)\equiv 0$; but if $\varphi(x)\equiv 0$ on some open set, then we may take $\chi(x)\neq 0$ on that set. Of course, one completely consistent solution is to take $\chi(x)\equiv 0$ for all space-time. While the change we have introduced has no effect on the classical theory, *this change has a profound influence on the quantum theory.*

Quantum formulation

We formulate the quantum theory of these models in the language of functional integrals.[4] For convenience, we analyze the Euclidean-space formulation for the generating functional of time-ordered Green's functions. (The Minkowski-space form could be used equally well.) For the conventional formulation of the model we consider the expression

$$S(h) = \mathfrak{N}\int \exp\left(i\int h\Phi\,dx - \int\{\tfrac{1}{2}[(\nabla\Phi)^2 + m^2\Phi^2] + \lambda\Phi^4\}dx\right)\mathfrak{D}\Phi\,, \tag{6}$$

while for the augmented theory we consider

$$S'(h) = \mathfrak{N}\int \exp\left(i\int h\Phi\,dx - \int\{\tfrac{1}{2}[(\nabla\Phi)^2 + (m^2+\mathrm{X}^2)\Phi^2] + \lambda\Phi^4\}dx\right)\mathfrak{D}\Phi\,\mathfrak{D}\mathrm{X}\,. \tag{7}$$

In these expressions $h=h(x)$ denotes the "external field" test function, $\Phi=\Phi(x)$, $\mathrm{X}=\mathrm{X}(x)$ are Euclidean-space field variables, and $(\nabla\Phi)^2 = \sum(\partial\Phi/\partial x_k)^2$. Normalization is adjusted so that $S(0)\equiv 1\equiv S'(0)$. Equations (6) and (7) are heuristic and formal, as are essentially all such functional integrals; nevertheless, the difference therein contains the essence of our proposed modification of conventional scalar field theory.

First a few general remarks. Formally, (7) is relativistically covariant in Euclidean space just as is its conventional counterpart (6). The field X may be taken to have mass dimension 1, and one may describe the proposed modification as a superposition of the conventional theory for all statistically independent, local mass possibilities equal to or greater than m^2. In the conventional theory the parameter m^2 represents the bare mass, but it would be premature to judge its significance in the augmented theory. From a formal point of view it is certainly possible that $m^2=0$ as far as the integration in either (6) or (7) goes.

Another crucial point to note about (7) is the fact that as $\lambda\to 0^+$ the generating functional does *not* reduce to that of the free theory but rather to a "pseudofree" theory characterized by

$$S_0'(h) \equiv \mathfrak{N}\int \exp\left\{i\int h\Phi\,dx - \tfrac{1}{2}\int[(\nabla\Phi)^2 + (m^2+\mathrm{X}^2)\Phi^2]dx\right\} \times \mathfrak{D}\Phi\,\mathfrak{D}\mathrm{X}\,, \tag{8}$$

which differs *fundamentally* from the free field generating functional

$$S_0(h) \equiv \mathfrak{N}\int \exp\left\{i\int h\Phi\,dx - \tfrac{1}{2}\int[(\nabla\Phi)^2 + m^2\Phi^2]dx\right\}\mathfrak{D}\Phi = \exp\left[-\tfrac{1}{2}\int (p^2+m^2)^{-1}|\tilde{h}(p)|^2dp\right]. \tag{9}$$

There is every reason to expect, at least on formal grounds, that (7) admits an asymptotic expansion in λ which is evidently an expansion about the pseudofree theory and not about the free one. Elsewhere we have argued that nonrenormalizable interactions may well be continuous perturbations and possibly have asymptotic perturbation series about suitable *pseudofree* theories.[5] Here in Eqs. (7) and (8) we give specific (albeit formal) proposals for what those interacting and pseudofree theories are for scalar fields.

Generality of the proposal

Although we have confined attention for simplicity only to $(\varphi^4)_n$ models, the form of the pseudofree theory can be argued to be the same for other models, e.g., $(\varphi^6)_n$, $(\varphi^8)_n$, or even $P(\varphi)_n$ (see below). Moreover, we explicitly propose Eq. (7) as a basis for formally defining a theory for models that are normally regarded as (super) renormalizable, such as $(\varphi^4)_n$, $n=2,3,4$, etc. Conceivably such a prescription, when carefully formulated, may overcome some of the limitations of conventional treatments of these models.

Eliminating the auxiliary field

Equations (7) and (8) are formulated with the X field integrations still to be performed. Alternative expressions arise if we carry out the X integrations. So far we have deliberately left unspecified the formal meaning of $\mathfrak{D}\Phi$ and $\mathfrak{D}\mathrm{X}$ [save for an implicit prescription used in evaluating

[(9)]. For the free theory one traditionally adopts the formal prescription

$$\mathfrak{D}\Phi \equiv \prod_x d\Phi(x)\,, \tag{10}$$

or alternatively that $\mathfrak{D}\Phi$ possesses formal *local translational* invariance,

$$\mathfrak{D}\Phi \equiv \mathfrak{D}(\Phi + \Lambda)\,, \tag{11}$$

for arbitrary $\Lambda(x)$, which then rather directly leads to (9). Likewise, we shall assume that

$$\mathfrak{D}X \equiv \prod_x dX(x)\,. \tag{12}$$

With this understanding we can formally carry out the X-field integration in (7) and (8) to yield

$$S'(h) \equiv \mathfrak{N}\int \exp\left(i\int h\Phi dx - \int \{\tfrac{1}{2}[(\nabla\Phi)^2 + m^2\Phi^2] + \lambda\Phi^4\}dx\right)\mathfrak{D}'\Phi\,, \tag{13}$$

and for the pseudofree model

$$S_0'(h) \equiv \mathfrak{N}\int \exp\left\{i\int h\Phi\, dx - \tfrac{1}{2}\int [(\nabla\Phi)^2 + m^2\Phi^2]dx\right\} \times \mathfrak{D}'\Phi\,. \tag{14}$$

In these expressions

$$\begin{aligned}\mathfrak{D}'\Phi &\equiv \mathfrak{N}_0\int \exp\left(-\tfrac{1}{2}\int X^2\Phi^2 dx\right)\mathfrak{D}\Phi\,\mathfrak{D}X \\ &= \left(\prod_x |\Phi(x)|^{-1}\right)\mathfrak{D}\Phi \\ &= \prod_x |\Phi(x)|^{-1} d\Phi(x)\,.\end{aligned} \tag{15}$$

Clearly $\mathfrak{D}'\Phi$ is no longer translation invariant in the sense of (11), and so (14) does not lead to the free theory result. Instead $\mathfrak{D}'\Phi$ is invariant under *local multiplication*, namely under the transformation $\Phi(x) \to S(x)\Phi(x)$, $S(x) > 0$.

Remark. It should be observed that the formal definition of $\mathfrak{D}'\Phi$ at each x, namely $|\Phi(x)|^{-1}d\Phi(x)$, is not locally integrable at $\Phi(x) = 0$. How this problem is dealt with will be discussed in the next section.

Heuristic motivation

Of course, a certain reluctance to accept (13) may be expected. One is familiar with (6), which is an expression of the form of (13) but with a different significance for the elemental field differential. The traditional, translation-invariant form $\mathfrak{D}\Phi$ is required for models having canonical commutation relations. However, for nonrenormalizable theories, or more specifically for theories with infinite field strength renormalization, the existence of canonical commutation relations may be highly suspect. As previously argued, whether or not one includes the X field the same *classical* theory arises. The identity of the classical theory can also be seen in the traditional quantum-to-classical limit where $\hbar$ is treated as small by appealing to "stationary-phase"-type arguments in (7). In an equation such as (13) the classical theory emerges because the "effective action" agrees with the correct form apart from a term $O(\hbar)$. In the case of the nonrenormalizable interactions $(\varphi^p)_n$, $p > 2n/(n-2)$, the complete failure of conventional renormalized perturbation theory suggests that the translation-invariant form (11) may be inappropriate. Outside of perturbation theory, it can be argued that a nonrenormalizable interaction generally acts as a partial "hard core" in relation to histories otherwise allowed by the free action (see below).[5] Characterization of that hard-core behavior is *essential* to formulating the theory even for arbitrarily small coupling constant. The proposed formal change or modification offered in this paper takes the form of replacing $\mathfrak{D}\Phi$ by $\mathfrak{D}'\Phi$, the multiplication-invariant form. (This formally simple change leads to related changes in the definition of the integrand, but that aspect is spelled out in the next section.) There is a certain elegant simplicity in noting that the form of the requisite change follows only from inserting a formal masslike interaction with the auxiliary field $X(x)$.

Form invariance

It is noteworthy that the strength of the interaction with X is immaterial. In particular, let the term $\int X^2\Phi^2 dx$ in the action be taken as $g\int X^2\Phi^2 dx$, $g > 0$. Then if follows that the functional $S'(h)$ is *independent* of g and *identical* to that proposed for $g = 1$. This result is found by a simple scale change of X with an overall numerical factor canceling in the normalization factor $\mathfrak{N}$. And interestingly the resultant theory is even unchanged if $\int X^2\Phi^2 dx$ is replaced by $\int X^4\Phi^4 dx$, $\int X^6\Phi^6 dx$, etc., since such interactions lead to the same expression for $\mathfrak{D}'\Phi$.

Why the augmented theory might work

We have noted that the expression (8) may well serve as the pseudofree expression for a large class of interactions. In retrospect, the formal reason for this is not too difficult to see. According to our viewpoint, a nonrenormalizable interaction acts as it does because of the partial hard-core effect of the interaction relative to the free term.[5] Consider the covariant cases where $(dx \equiv d^n x)$

$$W_0 = \tfrac{1}{2}\int [(\nabla\Phi)^2 + m^2\Phi^2]dx\,, \tag{16}$$

$$W_1 = \int |\Phi|^p dx \tag{17}$$

denote the free and interacting Euclidean actions, respectively. Sobolev-type arguments assert that $W_1 \leq K W_0^{p/2}$ for some finite K provided $p \leq 2n/(n-2)$; but this type of relation fails to hold whenever $p > 2n/(n-2)$.[6] As an example, if $n=4$ and $p>4$, then for the particular field

$$\Phi(x) = |x|^{-\gamma}\exp(-x^2), \tag{18}$$

one finds that $W_0 < \infty$ and $W_1 = \infty$ whenever $1 > \gamma \geq 4/p$.

How the proposed modification deals with such fields will become more apparent in the next section, but a certain heuristic argument can be presented here. Namely, with the augmented form of the pseudofree action

$$W_0' = \tfrac{1}{2}\int [(\nabla\Phi)^2 + m^2\Phi^2 + X^2\Phi^2]dx, \tag{19}$$

or with the equally effective augmented pseudofree action

$$W_0' = \tfrac{1}{2}\int [(\nabla\Phi)^2 + m^2\Phi^2 + |X|^p|\Phi|^p]dx, \tag{20}$$

a field such as (18) that is "allowed" by W_0 and "forbidden" by W_1 will also be "forbidden" by W_0' since X^2 can stand for any other substitute including $|\Phi|^{p-2}$ in (19) or simply 1 in (20).

Of course, the foregoing is a very picturesque characterization, and what *really* happens is described in a very different fashion. Inclusion of the field X and its subsequent integration to generate $\mathfrak{D}'\Phi$ (or starting initially with $\mathfrak{D}'\Phi$) leads to a random field of fundamentally different, *non*-Gaussian (and therefore noncanonical) character that has the property that (renormalized) *local products can be properly defined.* On the other hand, *no* such local products can be defined from a basically Gaussian random field in the parametric range $p > 2n/(n-2)$ which characterizes nonrenormalizable interactions.

III. FURTHER DEVELOPMENT OF AUGMENTED MODELS

It is one thing to propose a formal functional integral such as (6), (7), or (13), and it is quite another thing to develop a useful computational technique let alone provide a rigorous mathematical meaning. For the conventional formalism, the free theory is Gaussian, local field products must as a consequence be defined as Wick products, and an expansion of the interaction in a power series of the coupling constant leads naturally to a series of terms composed of integrals over products of two-point functions (since all Gaussian correlation functions are expressible in terms of the two-point function). As a result a graphical interpretation (in the manner of Feynman) proves to be a convenient aid. Divergences that may arise are avoided by one or more cutoffs, and the parameters of the model are related to the cutoffs in a way (hopefully) that meaningful results survive as the cutoffs are removed.

For the augmented theory the pseudofree models are *non*-Gaussian, and so all the conventional rules are suspect. Local products need not be given as Wick products, expansion of the interaction in powers of λ does not lead to terms composed simply of products of two-point functions, a Feynman graphical treatment is irrelevant, pseudofree propagators are not free propagators, conventional renormalization prescriptions may not be applicable, etc. Certainly, new rules need to be developed if one is to proceed via a perturbation development of (13) in a power series in λ. It should be emphasized that in spite of all the rigorous developments in the Bogoliubov-Parasiuk-Hepp-Zimmermann (BPHZ) framework,[7] this particular formalism is tied to Gaussian free fields and has nothing to say for non-Gaussian pseudofree fields.

In lieu of an alternative set of rules to generate a perturbation theory we proceed in a more constructive fashion.

Lattice-space formulation: conventional theory

Cutoffs in the form of lattices have long been used[8] and have recently experienced a revival. Space-time lattices for theories formulated in Euclidean space-time are particularly convenient.[9] In many ways such models resemble problems in classical statistical mechanics, and certain correlation inequalities first established in statistical mechanics have recently been used with great success in field theory.[10]

Let us first make a few remarks regarding lattice-space treatment of conventional theories. In such an approach, Eq. (6) is approximated as

$$\begin{aligned} S(\{h_k\},\epsilon) &\equiv \langle e^{i\sum h_k\Phi_k\Delta}\rangle \\ &\equiv N\int \exp[i\textstyle\sum h_k\Phi_k\Delta - \tfrac{1}{2}\sum \epsilon^{-2}(\Phi_{k*}-\Phi_k)^2\Delta \\ &\quad -\tfrac{1}{2}m_0^2\textstyle\sum \Phi_k^2\Delta - \lambda_0\sum\Phi_k^4\Delta]\prod d\Phi_k, \end{aligned} \tag{21}$$

where ϵ denotes a lattice dimension, $\Delta \equiv \epsilon^n$ is the volume of a single cell, and h_k and Φ_k are the average fields at the kth lattice site (k denotes an n-fold index). The sums extend over all lattice sites on a (hyper) cubic lattice, and k^* is cryptic for the n nearest "larger" sites to the site k. The quadratic term (m_0^2) also includes any Wick-ordered contribution from the quartic term (λ_0), and N denotes a normalization such that $S(\{0\}, \epsilon) \equiv 1$. Clearly, the individual terms in this expression are Riemann sum approximations to the integrals in question, with Δ representing the volume of integration. Convergence of such an expression as $\epsilon \to 0^+$, or $\Delta \to 0^+$, has been established for $n=2, 3$, where in the latter case the parameter m_0^2 contains a contribution that is $O(\ln\Delta)$ representing the mass renormalization.[11] For $n \geq 4$ this same formula may be used as a starting point allowing for arbitrary renormalizations of m_0^2, λ_0, and also of the coefficient of the "gradient" terms. It is for such an approach that one finds either free behavior[2] by renormalization-group techniques or an Ising-type behavior.[12]

If the terms representing the derivatives are expanded out within the exponent, one observes that the only connection between lattice sites is through a (ferromagnetic) term of the form $\sum \epsilon^{-2} \Phi_{k^*}\Phi_k \Delta$. The quartic form of the remainder of the exponent leads—independently of the sign of m_0^2—to various correlation inequalities among which one has the basic (first Griffiths-Kelly-Sherman and Lebowitz) inequality[13]

$$\begin{aligned} 0 &\leq \langle \Phi_{k_1}\Phi_{k_2}\Phi_{k_3}\Phi_{k_4}\rangle \\ &\leq \langle \Phi_{k_1}\Phi_{k_2}\rangle\langle \Phi_{k_3}\Phi_{k_4}\rangle + \langle \Phi_{k_1}\Phi_{k_3}\rangle\langle \Phi_{k_2}\Phi_{k_4}\rangle \\ &\quad + \langle \Phi_{k_1}\Phi_{k_4}\rangle\langle \Phi_{k_2}\Phi_{k_3}\rangle , \end{aligned} \tag{22}$$

where $\langle\ \rangle$ is as defined by (21). This particular relation can be put to use in two ways. Assuming convergence of the two-point function as $\Delta \to 0^+$, the uniform bound on the four-point function guarantees (by compactness) a convergent subsequence for the four-point function.[14] Of course, there is no assertion that such convergence leads to anything but a free theory. But in any event, including the nontrivial examples $(\varphi^4)_2$ and $(\varphi^4)_3$, the renormalized coupling constant λ_{ren} (given by the four-point vertex at zero momentum) is linearly related to the four-point function evaluated at zero momentum and thus inevitable constrained by the left and right sides of (22).[3]

Gradient-free model

Before leaving the conventional approach, we turn our attention briefly to certain highly artificial models defined as in (6) or (21) but *without* the gradient terms. In the lattice-space form one considers the expression

$$\begin{aligned} \bar{S}(\{h_k\}, \epsilon) &\equiv \langle e^{i\sum h_k \Phi_k \Delta}\rangle \\ &\equiv \bar{N} \int \exp[i\sum h_k\Phi_k\Delta - \tfrac{1}{2}m_0^2 \sum \Phi_k^2\Delta \\ &\qquad - \lambda_0 \sum \Phi_k^4 \Delta] \prod d\Phi_k . \end{aligned} \tag{23}$$

Here m_0^2 and $\lambda_0 \geq 0$ denote arbitrary functions of Δ chosen for consistency, and $\bar{N}$ is adjusted so that $\bar{S}(\{0\}, \epsilon) \equiv 1$. The simplicity of this expression permits a rather general study. Note first that the Φ_k are independent, identically distributed variables and that only the real part contributes. By suitable scaling ($\Phi = u/\Delta$) it follows that

$$\bar{S}(\{h_k\}, \epsilon) \equiv \prod_k \bar{S}(h_k, \Delta), \tag{24}$$

where

$$\bar{S}(h, \Delta) \equiv \int \cos(hu) e^{-F(u,\Delta)} du , \tag{25}$$

$$F(u, \Delta) \equiv \tfrac{1}{2}m_0^2 \Delta^{-1} u^2 + \lambda_0 \Delta^{-3} u^4 + C_0 , \tag{26}$$

and C_0 is chosen so that

$$\int e^{-F(u,\Delta)} du \equiv 1 . \tag{27}$$

If, as $\Delta \to 0^+$, we assume that

$$\Delta^{-1} \int u^2 e^{-F(u,\Delta)} du \to A, \quad 0 < A < \infty \tag{28}$$

then with the presumed form for F it is inevitable for $p > 2$ that

$$\Delta^{-1} \int |u|^p e^{-F(u,\Delta)} du \to 0 . \tag{29}$$

Consequently, in the limit $\Delta \to 0^+$,

$$\bar{S}(\{h_k\}, \epsilon) \to \bar{S}_0(h) \equiv \exp\left[-\tfrac{1}{2}A \int h^2(x) dx\right] . \tag{30}$$

This result is of course just a consequence of a central-limit-type theorem. It indicates that the continuum limit of the lattice form (23) for the interacting gradient-free artificial model is invariably the *free* gradient-free model for some mass parameter.

We note in addition that the lattice-space form of this artificial model satisfies the inequality (22), regardless of the fact that the off-diagonal two-point function vanishes. This artificial model forms an important link in developing the proposed form for the augmented model.

Lattice-space formulation: augmented theory

For the lattice-space form of the augmented theory formally given in (13), we provisionally adopt the expression

$$\begin{aligned} S'(\{h_k\}, \epsilon) &\equiv \langle e^{i\sum h_k \Phi_k \Delta}\rangle' \\ &\equiv N' \int \exp[i\sum h_k\Phi_k\Delta - \tfrac{1}{2}\sum \epsilon^{-2}(\Phi_{k^*} - \Phi_k)^2\Delta - \tfrac{1}{2}m_0^2\sum \Phi_k^2\Delta - \lambda_0 \sum \Phi_k^4\Delta] \prod d\Phi_k / |\Phi_k|^{1-2\Delta} . \end{aligned} \tag{31}$$

Note that the formal measure [cf. (15)] $|\Phi(x)|^{-1}d\Phi(x)$ has been taken as $|\Phi_k|^{-(1-2\Delta)}d\Phi_k$. More properly the exponent here should read $-(1-2b\Delta)$, where b is an arbitrary positive constant with dimensions $(\text{length})^{-n}$; *here and elsewhere we choose units such that* $b\equiv 1$. The proposed choice of exponent ensures the integrability of Φ_k near zero. Specifically, for any B, $0<B<\infty$, we have

$$\int_{-B}^{B}\frac{d\Phi}{|\Phi|^{1-2\Delta}}=\Delta^{-1}B^{2\Delta}\simeq\Delta^{-1}, \tag{32}$$

where the last relation holds for Δ sufficiently small.

The parameters m_0^2, λ_0, and N' are to be determined by consistency and normalization arguments, and k^* has the same meaning as in (21).

Gradient-free model

As the next step in our development, we treat the artificial gradient-free model as an augmented theory. Thus let us consider the expression

$$\begin{aligned}\overline{S}'(\{h_k\},\epsilon)&\equiv\langle e^{i\sum h_k\Phi_k\Delta}\rangle'\\&\equiv\overline{N}'\int\exp(i\textstyle\sum h_k\Phi_k\Delta-\frac{1}{2}m_0^2\sum\Phi_k^2\Delta-\lambda_0\sum\Phi_k^4\Delta)\prod d\Phi_k/|\Phi_k|^{1-2\Delta}.\end{aligned}\tag{33}$$

Independence of each variable, reality, and suitable scaling ($\Phi=u/\Delta$) lead to the relations

$$\overline{S}'(\{h_k\},\epsilon)\equiv\prod_k\overline{S}'(h_k,\Delta), \tag{34}$$

where

$$\overline{S}'(h,\Delta)\equiv\int\cos(hu)e^{-F'(u,\Delta)}\Delta du/|u|^{1-2\Delta}, \tag{35}$$

$$F'(u,\Delta)\equiv\tfrac{1}{2}m_0^2\Delta^{-1}u^2+\lambda_0\Delta^{-3}u^4+C_0'. \tag{36}$$

Note that we have made explicit a factor Δ which for small Δ approximately normalizes [see (32)] the measure $du/|u|^{1-2\Delta}$ in any interval including $u=0$. Indeed the interval itself may even increase or decrease as a power of Δ, and the approximate normalization for small Δ still holds. The parameter C_0' is chosen so that $\overline{S}'(0,\Delta)\equiv 1$, and given the preceding remarks it follows that

$$C_0'\to 0 \text{ as } \Delta\to 0^+. \tag{37}$$

Next introduce new (renormalized) parameters according to

$$m^2\equiv m_0^2\Delta^{-1},\quad \lambda\equiv\lambda_0\Delta^{-3} \tag{38}$$

with m^2 and λ independent of Δ (recall that we have chosen $b\equiv 1$). Then it follows that

$$\begin{aligned}\overline{S}'(h,\Delta)&=1-\int[1-\cos(hu)]e^{-F'(u,\Delta)}\Delta du/|u|^{1-2\Delta}\\&\simeq 1-\Delta\int[1-\cos(hu)]e^{-(1/2)m^2u^2-\lambda u^4}du/|u|,\end{aligned}\tag{39}$$

where the last expression is correct to leading order in Δ. Consequently, in the limit that $\Delta\to 0^+$,

$$\overline{S}'(\{h_k\},\epsilon)\to\overline{S}'(h)\equiv\exp\left(-\int dx\int\{1-\cos[h(x)u]\}e^{-(1/2)m^2u^2-\lambda u^4}du/|u|\right). \tag{40}$$

Note that with this answer we have bypassed the central-limit theorem and have avoided the free-model limit. The reason for this behavior stems from the fact that, unlike the previous case in (28) and (29), it follows for all $p\geq 2$ (indeed all $p>0$) that

$$\Delta^{-1}\int|u|^pe^{-F'(u,\Delta)}\Delta du/|u|^{1-2\Delta}\to A_p,\qquad 0<A_p<\infty. \tag{41}$$

In turn this property comes about since, roughly speaking,

$$e^{-F'(u,\Delta)}\frac{\Delta}{|u|^{1-2\Delta}}\simeq\delta(u)+\Delta G(u), \tag{42}$$

a structure that cannot be obtained within the confines of the conventional lattice formulation.

In summary, the gradient-free model has satisfactory limiting behavior in the augmented lattice-space form, while it possesses *no* nontrivial limit when taken in the conventional lattice-space form. In the augmented formulation suitable *multiplicative* renormalizations of the parameters m_0^2 and λ_0 exist, and there is *no* Wick ordering. This situation arises because of the very different nature of the field variables in the augmented approach as compared to those of the conventional approach. In this regard *critical use* is made of the Δ factor that enters into $d\Phi/|\Phi|^{1-2\Delta}$ and so into the normalization in (35); if instead one had chosen $d\Phi/|\Phi|^{0.99}$, say, then a central-limit-type theorem would have been unavoidable even here as well.

In other words, the lattice volume Δ enters and is used in very different ways in the conventional and augmented formulations, and that very difference is what leads to the different renormalization of the model parameters. To see this clearly one has only to note that in the conventional lattice-space approach the Δ appearing in each term of the action eventually becomes the volume element of the integration as evidently is the case in (30). On the other hand, it is the Δ in the overall normalization, required by the Δ in the exponent in the augmented measure, that eventually becomes the volume element of the integration in (40).

The previous discussion has implicitly assumed that $\lambda > 0$ but it applies equally well for $\lambda = 0$. Such a discussion pertains to the gradient-free form of the pseudofree model (14) analyzed as an augmented model. The result of such an analysis is given as $\Delta \to 0^+$ by

$$\overline{S}_0'(\{h_k\}, \epsilon) \to \overline{S}_0'(h) \equiv \exp\left(-\int dx \int \{1-\cos[h(x)u]\} e^{-(1/2)m^2u^2} du/|u|\right), \tag{43}$$

and evidently (43) also follows from (40) as $\lambda \to 0^+$. This expression for the pseudofree gradient-free model is very different from that for the free gradient-free model [cf. (30)].

No bound on λ_{ren}

It is especially noteworthy that for the augmented models *the basic inequality* (22) *does not apply*. In fact for even infinitely divisible characteristic functionals [which (40) and (43) are] *all truncated correlation functions are non-negative*.[15] This means that for any pregiven value of λ and m^2, there is a $\Delta(\lambda, m^2) > 0$ such that for all $\Delta < \Delta(\lambda, m^2)$,

$$\begin{aligned} 0 \le &\langle \Phi_{k_1}\Phi_{k_2}\rangle'\langle \Phi_{k_3}\Phi_{k_4}\rangle' \\ &+ \langle \Phi_{k_1}\Phi_{k_3}\rangle'\langle \Phi_{k_2}\Phi_{k_4}\rangle' + \langle \Phi_{k_1}\Phi_{k_4}\rangle'\langle \Phi_{k_2}\Phi_{k_3}\rangle' \\ \le &\langle \Phi_{k_1}\Phi_{k_2}\Phi_{k_3}\Phi_{k_4}\rangle', \end{aligned} \tag{44}$$

where $\langle\ \rangle'$ is defined by (33). Recent results of Ellis and Newman[16] in fact show that (44) holds for all $\Delta < \frac{1}{2}$ (i.e., $\Delta < \frac{1}{2}b^{-1}$) for the pseudofree model ($\lambda = 0$). These results hold only for the gradient-free model, but in the lattice-space formulation introduction of the gradients increases (from zero) the *ferromagnetic* coupling among cell fields. Given that the truncated four-point correlation function is non-negative [true for $\Delta < \Delta(\lambda, m^2)$] the increase of any ferromagnetic coupling will preserve that inequality.[17] Consequently, (44) applies not only to the lattice models without gradients in (33) but also to the lattice models with gradients in (31), at least for sufficiently small Δ. Therefore, the limitation on λ_{ren} previously encountered no longer applies in the augmented approach.

Remark. On the assumption that the limit $\Delta \to 0^+$ exists for the augmented formulation [say as in (31) or in (51) below], then (44) implies that the solutions of the augmented $(\varphi^4)_2$ and $(\varphi^4)_3$ models will *not* lead to the conventional solutions of constructive quantum field theory for these models. Instead the augmented approach will provide alternative, noncanonical solutions for these models with λ_{ren} unconstrained by (22).

One is reminded here of the situation in Schrödinger wave mechanics for the ordinary differential equation

$$H_\lambda = -\frac{d^2}{dx^2} + x^2 + \frac{\lambda}{x^2}, \quad \lambda \ge 0.$$

Insisting on the existence of even-parity eigenfunctions that do not vanish at $x = 0$ is one way to generate a set of solutions so long as $\lambda < \frac{3}{4}$, but for $\lambda \ge \frac{3}{4}$ all eigenfunctions (both odd and even parity) must vanish at $x = 0$. One could arrange to extend the solutions valid above $\lambda \ge \frac{3}{4}$ to smaller λ values, which would then give a second set of solutions for $\lambda < \frac{3}{4}$. These second solutions would not reduce to the free (harmonic-oscillator) solutions as $\lambda \to 0^+$, but to solutions of the oscillator formulated with Dirichlet boundary conditions at $x = 0$. This behavior reflects the fact that at $x = 0$ the differential operator H_λ is limit circle for $\lambda < \frac{3}{4}$ and limit point for $\lambda \ge \frac{3}{4}$.[18]

Background remarks

The model here called gradient-free has been discussed previously in the literature. It has been analyzed as a conventional theory—indeed effectively as a lattice-cutoff theory—and it was recognized that no nontrivial result emerges as $\Delta \to 0^+$.[19] Alternatively, the same formal model has been analyzed in a completely cutoff-free fashion with results obtained on the grounds of symmetry and general principles alone.[20] The results of this analysis agree with the limiting relations presented in (40) and (43). There is no ambiguity in that derivation save for one arbitrary parameter b chosen as unity that enters (40) and (43) by the substitutions $b\,dx$, bm^2, and $b^3\lambda$ for the indicated variables, the need for which can be seen on dimensional grounds. The arbitrary parameter b can be regarded simply as a reflection of the arbitrariness in the choice of the unit of length. This is, of course, the same parameter b discussed below Eq. (31).

It is no accident that the augmented model results agree with those of the no-cutoff approach. In fact, recognizing that the conventional lattice-space approach failed to give acceptable behavior, the no-cutoff solution [e.g., (40) or (43)] was evaluated for functions $h(x)$ that were constant within each cell of a (hyper) cubic lattice of cell volume Δ. Subsequently, a Fourier transform over each cell test field h_k was made to discover the appropriate form of the implied cutoff formulation for the formal path integral. In this way the result $d\Phi_k/|\Phi_k|^{1-2\Delta}$ (rather than $d\Phi_k$) for the elementary cell field measure was initially ascertained. Clearly one way to realize the new measure is to observe that for some numerical factor of proportionality M,

$$\frac{d\Phi_k}{|\Phi_k|^{1-2\Delta}}=M\int\exp[-\tfrac{1}{2}X_k^{2/(1-2\Delta)}\Phi_k^2\Delta]dX_k d\Phi_k. \qquad (45)$$

It is this kind of relation that ultimately leads to the formal expressions that appear as (7) or (8) (modulo the gradient terms).

The foregoing argument constitutes the first step in choosing the form of the augmented model, namely, it is the form which when cut off on a lattice leads in the limit $\Delta\to 0^+$ to the correct, no cutoff result for the gradient-free model. The next step is to argue why the same augmented model may work for models when gradient terms are present.

Role of the gradient terms: preliminaries

It is useful to regard the lattice-space formulation in (21) and (31) from the point of view of classical statistical mechanics for continuous "spin" variables. Gradient-free models then represent completely independent or uncorrelated spins, each of which is characterized by a certain distribution. Gradient terms introduce coupling or correlation between spins, and the coupling term enters in such a way (ferromagnetic) as to promote "alignment," e.g., the product $\Phi_1\Phi_2$ becomes preferentially positive. At lower "temperatures" (represented by smaller lattice spacings) the tendency for alignment becomes even stronger, and in an expression such as (21) or (31) it means that $\Phi_1-\Phi_2\approx 0$ with greater probability.

A simpler and more familiar example can be provided by the following *one*-dimensional example. Consider the probability measure induced on a sequency of real random variables by the expression

$$N\exp[-\tfrac{1}{2}\sum\epsilon^{-1}(x_{k+1}-x_k)^2-\tfrac{1}{2}\sum\epsilon x_k^2]\prod dx_k. \qquad (46)$$

We identify $x(t_k)\equiv x_k$, $t_k\equiv k\epsilon$ and regard t as time. With the gradient *absent* the (weak) limit $\epsilon\to 0^+$ describes white noise, a generalized stochastic process $x(t)$ not pointwise defined. With the gradient *present* the (weak) limit $\epsilon\to 0^+$ describes an Ornstein-Uhlenbeck process (locally rather like Brownian motion), a process which is concentrated on *continuous* sample paths. Thus in the limit the gradient terms have introduced a certain degree of *smoothness* (relative to white noise) through the tendency of the gradient terms to make neighboring variables comparable with a high degree of probability.

Heuristically, the form of the exponent in (46) is suggested by a uniform time lattice Riemann sum approximation to the expression $\frac{1}{2}\int[\dot{x}^2(t)+x^2(t)]dt$. But apart from that there is no *a priori* reason that the coefficient of the gradient term should be ϵ^{-1}. Any other negative power such as $\epsilon^{-\beta}$, $\beta>0$, forces neighboring x values together. If so, then what really distinguishes $\beta=1$?

To show that $\beta=1$ let us examine the expectation value of $(x_K-x_1)^2$ in the distribution (46). In particular, let us evaluate the simpler but equally useful expression (appropriate to Brownian motion) given by

$$\frac{\int(x_K-x_1)^2\exp\left[-\frac{1}{2}\sum_1^{K-1}\epsilon^{-\beta}(x_{k+1}-x_k)^2\right]\delta(x_1)\prod dx_k}{\int\exp\left[-\frac{1}{2}\sum_1^{K-1}\epsilon^{-\beta}(x_{k+1}-x_k)^2\right]\delta(x_1)\prod dx_k}=\epsilon^{\beta}\frac{\int(y_K+y_{K-1}+\cdots+y_3+y_2)^2e^{-\Sigma y_k^2/2}\prod dy_k}{\int e^{-\Sigma y_k^2/2}\prod dy_k}$$

$$=\epsilon^{\beta}(K-1)\frac{\int y^2e^{-y^2/2}dy}{\int e^{-y^2/2}dy}. \qquad (47)$$

In the limit $\epsilon\to 0$, $K\to\infty$, such that $0<t\equiv K\epsilon<\infty$, it follows that only $\beta=1$ provides acceptable behavior for this quantity. This result for β was ultimately determined by the particular K dependence of the expectation of $(x_K-x_1)^2$. A different K dependence would have required a value for β different from that suggested by the Riemann sum approximation; in another language that difference in β values would reflect the need for a nontrivial field strength renormalization.

When considering higher dimensional ($n\geqslant 2$) lattices, some of the foregoing remarks should be kept in mind. Generally, there will be analogous smoothing introduced by the presence of gradient terms, but it may not be sufficient to result in random fields that are concentrated on continuous functions. Equally important is the proper $\Delta(\equiv\epsilon^n)$ dependence of the coefficient of the gradient term which along with $m_0{}^2$ and λ_0 represent adjustable renormalization parameters chosen for overall consistency.

Relevance of gradient-free results for models with gradients

As we have seen the gradients tend to introduce correlation between neighboring field values. For Gaussian variables such smoothing tends to favor the construction of local (Wick) products, but the smoothing may not be adequate if the field power or space-time dimension is too great. For the gradient-free models the non-Gaussian augmented formulation is *fully* consistent with the construction of local products.[20] Now we discuss the relation of augmented models with gradients to those without gradients.

It is first instructive to compare in a formal sense the free and interaction actions W_0 and W_1 in (16) and (17) that typically make up the exponent in a functional integral. In the nonrenormalizable regime $p>2n/(n-2)$ the interaction is *not* controlled by the free term. Clearly the converse is also true, for one could imagine a field with a high degree of modulation, e.g., a smooth field multiplied by $\cos(\exp|x|^{-2})$, for which the gradient term diverges. Such modulation is in principle suppressed by the correlation of neighboring field values just as it is in the conventional treatment with Gaussian variables (even for free fields). On the other hand, a field of "bounded variation" seems such that the gradient term is locally dominated by the gradient-free term. Consider the field

$$\Phi(x)=|x|^{-\gamma}\exp(-x^2) \tag{48}$$

previously considered in (18). Near $x=0$ and up to a factor,

$$(\nabla\Phi)^2(x)\simeq|x|^{-2\gamma-2}, \tag{49}$$

$$\Phi^4(x)\simeq|x|^{-4\gamma}. \tag{50}$$

For $n=4$, fields with $\gamma\geq 1$ have divergent W_0, but they also have divergent W_1. For $n=5$, fields with $\gamma\geq\frac{3}{2}$ have divergent W_0, but W_1 diverges already whenever $\gamma\geq\frac{5}{4}$, etc. Thus it seems heuristically reasonable to assert that for $(\varphi^4)_n$, $n\geq 4$, fields of bounded variation are controlled by the gradient-free terms, while fields of unbounded variation are controlled by the gradient terms as they already are in the conventional treatment. The domination by the gradient-free terms for fields of bounded variation is what suggests that the form of the basic measure of the augmented model which is suitable for gradient-free models should also be suitable for models with gradients. For $(\varphi^4)_n$, $n=2,3$, such domination of the gradient term by the gradient-free terms does not hold, but perhaps that does not preclude the compatibility of the gradients with the augmented formulation. (See the earlier remark in reference to H_λ.) For the reasons sketched above we have chosen the same basic cell measure in (31) when gradients are present that proved successful for (33) when gradients were absent.

Proposal for the lattice formulation of the augmented model

With the foregoing discussion as motivation our proposal for the lattice-space form of the generating functional for the augmented $(\varphi^4)_n$ model is given by

$$S'(\{h_k\},\epsilon)\equiv\langle e^{i\sum h_k\Phi_k\Delta}\rangle'$$
$$\equiv N'\int\exp[i\sum h_k\Phi_k\Delta-\tfrac{1}{2}Z\sum\epsilon^{-2}(\Phi_{k*}-\Phi_k)^2\Delta-\tfrac{1}{2}m_0{}^2\sum\Phi_k{}^2\Delta-\lambda_0\sum\Phi_k{}^4\Delta]\prod d\Phi_k/|\Phi_k|^{1-2\Delta}. \tag{51}$$

This formula is essentially equivalent to that proposed in (31) apart from the additional renormalization parameter $Z=Z(\Delta)$ for the gradient term. It is not easy to predict just what behavior to expect for the free parameters in (51). One possibility is that the parametrization of the gradient-free models is still applicable, namely that $m_0{}^2=m^2\Delta$ $(=m^2b\Delta)$, $\lambda_0=\lambda\Delta^3$ $(=\lambda b^3\Delta^3)$, along with the trial proposal that $Z=\Delta$ $(=b\Delta)$ as suggested by the quadratic nature of the gradient term. At the very least, such a parametrization and a rescaling $(\Phi=u/\Delta)$, which worked for the gradient-free theories, provides a possible starting point. In such a reparametrization (51) becomes

$$S'(\{h_k\},\epsilon)\equiv\langle e^{i\sum h_k\Phi_k\Delta}\rangle'$$
$$\equiv N_0'\int\exp[i\sum h_ku_k-\tfrac{1}{2}z\sum\epsilon^{-2}(u_{k*}-u_k)^2-\tfrac{1}{2}m^2\sum u_k{}^2-\lambda\sum u_k{}^4]\prod du_k/|u_k|^{1-2\Delta}. \tag{52}$$

Here the parameters m^2, λ, and z may also retain residual dependence on Δ, but if the analogy with the gradient-free analysis is valid these parameters may be treated as constants (with $z=1$). Clearly in the latter case the only Δ that remains to become the volume element in the limit $\Delta\to 0^+$ is the Δ that appears in the elementary measure and the normalization constant N_0'. On the other hand, the presence of the gradients may change substantially the renormalization behavior so that m^2, λ, and z would be different than assumed. Of course, that is not serious since we still are proposing solutions for nonrenormalizable models ($n\geq 5$) that depend on only a finite number of parameters. Even for the renormalizable cases ($n\leq 4$) we are proposing noncanonical models. Clearly, the demonstration of consistency as $\Delta\to 0^+$ of expressions such as (51) or (52) would open new roads of study for quantum field theory.

Positivity and domination of correlation functions

As one argument to analyze (51) or (52) further we appeal to a simple but convergent "high-temperature" expansion and deduce several consequences. For this purpose, let $\langle\;\rangle'$ denote the average with respect to (51), and let $\langle\;\rangle'_{(0)}$ denote an average similar in structure to (51) except that the ferromagnetic coupling term

$$J\equiv Z\sum \epsilon^{-2}\Phi_{k^*}\Phi_k\Delta \tag{53}$$

is omitted from the exponent. A superscript T denotes truncated correlation function as defined for example by expansion of the identity

$$\langle e^{\Sigma h_k\Phi_k\Delta}-1\rangle'^T\equiv \ln\langle e^{\Sigma h_k\Phi_k\Delta}\rangle'. \tag{54}$$

Then it follows that

$$\langle\Phi_{k_1}\Phi_{k_2}\cdots\Phi_{k_p}\rangle'^T=\sum_r (r!)^{-1}\langle\Phi_{k_1}\Phi_{k_2}\cdots\Phi_{k_p}J^r\rangle'^T_{(0)}. \tag{55}$$

Note that $\langle\;\rangle'_{(0)}$ is an average of the kind that is defined in (33) with an alternative meaning for the quadratic term in the exponent. For such averages we pointed out that all truncated correlation functions of the form $\langle\;\rangle'^T_{(0)}$ are non-negative for sufficiently small Δ. Consequently, it follows for sufficiently small Δ that

$$\langle\Phi_{k_1}\Phi_{k_2}\cdots\Phi_{k_p}\rangle'^T\geq 0 \tag{56}$$

for the average $\langle\;\rangle'$ defined in (51). For $p=4$ this result provides an alternative proof of (44) suitable for the augmented models with the gradients included.

Inspection of (40) shows that the expressions for the truncated correlation functions valid for sufficiently small Δ are majorized by their pseudofree values ($\lambda=0$). Consequently, it follows for $\langle\;\rangle'_\lambda\equiv\langle\;\rangle'$ defined in (51) and for sufficiently small Δ that

$$\langle\Phi_{k_1}\Phi_{k_2}\cdots\Phi_{k_p}\rangle'^T_{\lambda=0}\geq\langle\Phi_{k_1}\Phi_{k_2}\cdots\Phi_{k_p}\rangle'^T_{\lambda>0}\geq 0. \tag{57}$$

One might gain the impression that such a relation would lead to a universal constraint on the renormalized coupling constant for $p=4$, but that is incorrect for we have not yet exhibited the dependence of these expressions on the arbitrary parameter b that remains at our disposal. If we now make b explicit along the lines suggested, it follows for sufficiently small Δ that

$$\langle\Phi_{k_1}\Phi_{k_2}\cdots\Phi_{k_p}\rangle'^T_{\lambda=0}=b^{1-p/2}\langle\Phi_{k_1}\Phi_{k_2}\cdots\Phi_{k_p}\rangle'^T_{\lambda=0(b=1)}. \tag{58}$$

For $p=4$ the left side of (57) varies as b^{-1}, which can be made as large as desired (or as small). (The limit $b\to\infty$ leads to a Gaussian field; but this limit, again an inevitable consequence of central-limit-type theorems, in no way corresponds to the limit $\lambda\to 0^+$.)

Finally, we note the domination as forms of the two-point correlation with gradients by the corresponding one without gradients. This relation is familiar for covariant free theories [see (9)] for which

$$\begin{aligned}\left\langle\left(\int h\Phi\,dx\right)^2\right\rangle_{\text{grad}}&\equiv\int (p^2+m^2)^{-1}|\tilde h(p)|^2dp\\ &\leq\int (m^2)^{-1}|\tilde h(p)|^2dp\\ &\equiv\left\langle\left(\int h\Phi\,dx\right)^2\right\rangle_{\text{no grad}}.\end{aligned} \tag{59}$$

The general validity of this relation for even theories is easily demonstrated.

For the interacting augmented models, with Δ sufficiently small but nonzero, we assert on the basis of $\langle\;\rangle'$ defined in (51) and $\langle\;\rangle'_{\text{no}}\equiv\langle\;\rangle'$ defined in (33) that

$$\langle(\Sigma h_k\Phi_k\Delta)^2\rangle'_{\text{no}}\geq\langle(\Sigma h_k\Phi_k\Delta)^2\rangle'. \tag{60}$$

In this expression the parameters $m_0{}^2$ and λ_0 are the same on each side. Given (38) for the choice of parameter renormalization, we have seen that the left side of (60) converges as $\Delta\to 0^+$, and according to (40) yields

$$\int h^2(x)dx\int u^2e^{-(1/2)m^2u^2-\lambda u^4}du/|u|. \tag{61}$$

For this special choice of renormalization in (38) the result in (61) provides an upper bound for (60). Consequently, the convergence as $\Delta\to 0^+$ (at least for a subsequence) of the two-point function for

$\langle\,\rangle'$ defined in (51) is ensured for the proposed parameter renormalization in (38). Interestingly, even this bound is enough to guarantee (weak) convergence (for a subsequence) of the probability measure in (51).[21] But the resultant measure may be trivial, i.e., either free or just concentrated at $\Phi_k \equiv 0$. Nontrivial results may result if another choice for Z is adopted. But if that proves insufficient it will be necessary in addition to try parametrizations other than that in (38), perhaps even renormalization of b itself. Clearly these questions merit further study.

IV. CONCLUSION

In this paper we have discussed an alternative formulation for the quantum field theory of covariant self-interacting scalar fields and compared it with the conventional approach. Motivation for our proposals stems from basic limitations that appear inherent in the conventional approach. As we have stressed both the augmented and conventional formulations can be viewed as quantizations of one and the same classical theory. On the surface, the formal structure of the augmented formulation compared to the conventional formulation, as represented by Eqs. (13) and (6), respectively, is relatively simple, i.e., merely replacing one formal measure for another. Anyone familiar with functional-integration techniques is aware that in certain cases (e.g., gravitation) just which measure is correct has long been a subject of debate. Admittedly, there has been little debate on the choice of a formal measure for "simple" self-interacting scalar theories, but perhaps conventional presumptions are too restrictive. A change of measure as we propose does not correct for improper mathematical handling of the conventional formalism; in truth, the augmented formulation represents new physics.

Even if one is willing to consider alternative measures for self-interacting scalar fields, what then is the appropriate choice to make? In this paper we have strongly argued for the replacement of the translation-invariant form $\mathfrak{D}\Phi$ by the multiplication- or scale-invariant form $\mathfrak{D}'\Phi$, expressions which are heuristically defined in (10) and (15), respectively. The successes of constructive quantum field theory have confirmed that the translation-invariant form is appropriate for "small" n, i.e., $n=2,3$ (not to mention $n=1$ which characterizes anharmonic quantum-mechanical oscillators). On the other hand, the scale-invariant form is exactly what is needed for the gradient-free models. Elsewhere we have argued that in a certain sense gradient-free models are related to covariant models for which $n=\infty$.[22] This remark is at least plausible when one notes on the basis of standard perturbation theory that for any n a gradient-free model exhibits even stronger divergences than the same model with the gradients present. Consequently, it is suggestive that the scale-invariant measure suitable for the gradient-free formulation holds for "large" n. Moreover, as discussed in the text, the augmented approach may also provide viable alternatives in ranges where the conventional formulation applies.

Even beyond the technical validity of the scale-invariant measure for gradient-free models there is another point of a conceptual nature in its favor: The modification of the conventional treatment introduced by adopting $\mathfrak{D}'\Phi$ for $\mathfrak{D}\Phi$ is qualitative and profound not just quantitative and mild. This point has been stressed already in our discussion of the lattice-space formulation with regard to the ultimate disposition of the parameter Δ which represents the cell volume in the limit $\Delta \to 0^+$. The fundamental difference in this regard between the augmented and conventional approaches leads to alternative prescriptions to define local field products. No new prescriptions for local field products would follow from mild or conservative modifications of the measure.

A further important consequence of the augmented formulation is the validity for sufficiently small Δ of "reverse" correlation inequalities such as given in (44). Directly applicable to the gradient-free models, this inequality also remains valid in the presence of the gradients owing to their ferromagnetic coupling nature. The inequality (44) valid for the augmented models if Δ is small enough is to be contrasted with the more common inequality in (22) that applies for the conventional formulation. Although (22) implies a convergent subsequence for the four-point function, given two-point convergence, the renormalized coupling constant λ_{ren} is constrained to a finite interval. On the other hand, for the augmented formulation (44) cannot be used to guarantee a convergent four-point function, but if it exists the renormalized coupling constant λ_{ren} is unconstrained. This result makes the augmented formulation potentially interesting not only for $(\varphi^4)_n$ models, $n \geq 4$, but also for new formulations of $(\varphi^4)_2$ and $(\varphi^4)_3$ models.

The formal analysis of Sec. II is useful for heuristic purposes, but it is not suited for computation or to establish existence. A perturbation analysis constructed along conventional lines seems out of the question, even though we expect (13) to have an asymptotic expansion in λ. Instead we have proposed a lattice-space formulation with the relevant generating functional given in (51). Convergence as $\Delta \to 0^+$ (with suitable boundary conditions and as the overall lattice volume goes to infinity) of this expression for a suitable choice of parameters should provide a Euclidean-space-covariant model. Passage to real time (Minkowski space), determination of (free) asymptotic fields, characterization of the scattering matrix, etc., would proceed along lines well established for conventional, constructive, or axiomatic formulations.

Generalization to other models

It is tempting to conclude with a few remarks suggesting alternative approaches to models other than $(\varphi^4)_n$ which have been our main concern. We expect that a parallel analysis applies for $(\varphi^p)_n$ for any even p, or suitable $P(\varphi)_n$ based on the augmented formulation and the same pseudofree models [e.g., (14)]. For $n=2$ these models should also differ from the conventional models of constructive quantum field theory. The relevant lattice-space formulations are evident generalizations of those presented in (51) or (52).

For the two remaining examples we content ourselves with only indicating possible first steps in an analysis analogous to Eq. (7).

If one deals with a self-interacting $O(N)$-invariant, N-component scalar field $\vec{\Phi}(x)$, then one should consider the conventional action *augmented* by the term

$$\tfrac{1}{2}\int \vec{X}^2(x)\vec{\Phi}^2(x)dx\,, \qquad (62)$$

which involves an N-component auxiliary field $\vec{X}(x)$. Such a proposal generates a measure $\mathfrak{D}'\vec{\Phi}$ which is invariant under arbitrary local $O(N)$ rotations and dilatations.[23]

If one deals with a charged spinor field $\psi(x)$ having $\frac{1}{2}N$ components, then a local four-fermion coupling (Fermi coupling), which for $n>2$ is nonrenormalizable by conventional standards, may be approached in an alternative fashion if one takes the conventional action *augmented* by the term

$$\int \vec{\chi}^2(x)\overline{\psi}(x)\psi(x)dx\,, \qquad (63)$$

where $\vec{\chi}(x)$ is an N-component auxiliary field. Euclidean-space fermion fields are sufficiently troublesome as to suggest that this example be formulated directly as a Minkowski-space functional integral.

Just these two additional examples should suggest the richness in new formulations of old models that becomes possible when augmented action functionals are considered. It would be striking indeed if the weak interactions could be formulated in a consistent way along the lines suggested here "just" by a change of measure in function space and without the need to postulate additional particles.

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