Varieties of instability of a boson field in an external potential

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A relativistic scalar field is quantized in a one-dimensional "box" comprising two broad electrostatic potential wells. As the potential difference increases, the phenomena found long ago by Schiff, Snyder, and Weinberg in such a model occur: merging of mode frequencies and disappearance of the vacuum as a discrete state, followed by appearance of complex frequencies and unboundedness below of the total energy. However, a new effect appears for some values of the potential: The discrete vacuum (with the associated particle interpretation) reappears, but the energy remains unbounded below because some negative-norm modes have greater frequencies than some positive-norm modes. That is, a particle-antiparticle pair can have energy less than that of empty space. As the outer walls of the box approach infinity, this situation goes over into the boson Klein "paradox," marked by nonuniqueness of the vacuum (spontaneous breaking of time-reversal symmetry) and coexistence of positive- and negative-norm continuum modes at the same frequency. These phenomena are of interest in connection with current work on pair creation in external gravitational fields, especially black holes.

This paper has three purposes: (1) to report a curious and unexpected property of the eigensolutions of the Klein-Gordon equation in a deep potential well, (2) to clarify the relationship between the Klein paradox[1] and the effect discovered by Schiff, Snyder, and Weinberg[2] (SSW), and (3) to draw to the attention of both general relativists and field theorists the close connection between the present intense activity concerning "particle creation by black holes" and the existing lore concerning strong electrostatic potentials in quantum field theory.

The Klein-Gordon equation with a time-independent external electrostatic potential reduces, after separation of the time variable, to the equation

$$[\omega_j - eA_0(\vec{x})]^2\phi_j(\vec{x}) = (-\nabla^2 + m^2)\phi_j(\vec{x}) \tag{1}$$

for normal modes ϕ_j with time dependence $e^{-i\omega_j t}$. (We set $\hbar = c = 1$.) Ordinarily in the corresponding quantum field theory the sign of the "norm"

$$\epsilon_j = i\int dx\{\phi_j^*(\partial_t + ieA_0)\phi_j - [(\partial_t + ieA_0)\phi_j]^*\phi_j\} \tag{2}$$

determines whether a normalizable solution ϕ_j represents a particle (+) or an antiparticle (−). If the potential differences of the problem exceed the threshold for pair creation, $|e\Delta A_0| \gtrsim 2m$, then the behavior of the solutions generally makes ordinary second quantization inapplicable. The simplest example is a step-function potential in one spatial dimension,

$$eA_0 = -V \text{ for } -L_- < x < 0, \qquad eA_0 = 0 \text{ for } 0 < x < L_+, \tag{3}$$

where V is constant. This model already exhibits a variety of phenomena.

If $L_+ = L_- = \infty$, the boson version of the Klein "paradox" arises. An unambiguous separation of the ϕ_j into particle modes and antiparticle modes becomes impossible, because solutions of the two types are degenerate at the same frequency ω and hence admit Bogolubov transformations mixing annihilation operators with creation operators. Among the infinitude of rival "vacuum" states therefore associated with inequivalent choices of basis modes are some states for which one calculates fluxes of energy and charge outward from the potential jump. Although this accords entirely with the usual interpretation that pairs are created and ejected by the potential step, there is nothing in the field theory preventing one from considering a state with purely inward fluxes or with no net flux. (The justification for choosing purely outward fluxes is evidently a statistical one, having to do with the plausibility of various initial conditions.) The situation may be described as spontaneous breakdown of time-reversal symmetry. The Klein effect has excited renewed interest since generalizations of it were discovered in quantum field theories set in the intense gravitational potentials near black holes.[3]

A formally different, but presumably physically related, effect was discovered by SSW (Ref. 2) in the case that either or both of $L_\pm$ are finite and the field is required to vanish at that boundary. They found that as the strength of the potential V increases, bound particle and antiparticle modes "annihilate" each other in pairs and pairs of normalizable solutions with complex frequencies appear. Let us review how this situation comes about.

The real frequencies in the SSW model as func-

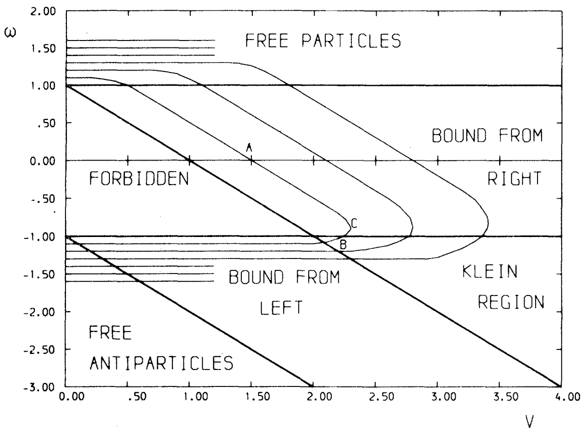

FIG. 1. Real frequency levels computed by SSW for $m=1$ and $L_+ \gg L_-$ (schematic).

tions of V are graphed schematically in Fig. 1. The qualitative behavior of an eigenfunction in the spatial regions $x<0$ and $x>0$ is determined by the relation of the corresponding frequency to the values $\pm m$ and $\pm m - V$, respectively. Thus, solutions corresponding to points in the part of Fig. 1 labeled "bound from right" are oscillatory in the left region ($x<0$) but exponentially decaying on the right. The reverse is true of solutions "bound from the left." "Free" and "Klein" solutions are oscillatory everywhere.

At $V=0$ the spectrum of ω is, of course, that of a free boson field quantized in a finite box. As L_+ increases, the spacing between the frequency levels decreases, resulting in a continuous spectrum when $L_+ = \infty$. The particle frequencies remain separated from the antiparticle frequencies, however, by a mass gap of width $2m$.

As the potential is turned on, the mode frequencies change. In particular, "bound states" grow out of the particle continuum into the mass gap. At the point labeled A the frequency ω of the lowest-lying particle mode becomes negative. In second quantization this means that a particle can exist with negative energy (relative to the vacuum state); in fact, since arbitrarily many particles can be present in that mode, the total energy operator of the second-quantized theory is not bounded below. No catastrophic instability of the system thereby results, however, since the absolute law of charge conservation prevents the vacuum from decaying into such a state, even under the influence of any physically permissible external perturbation. In fact, the negativity of the energy can be removed simply by a gauge transformation which adds a constant to all the frequencies in the theory. To put it another way, charge conservation allows the energy observable to be redefined by the addition of a term proportional to the total charge, because the energy difference between two states of different charge of a closed system cannot be measured. (The redefinition must be applied consistently to *all* charged fields, including any external perturbing field whose quanta might carry charge away from the subsystem of original interest.)

At the point B one of the antiparticle modes moves into the region of Fig. 1 containing wave functions bound into the left potential well. This phenomenon may appear paradoxical, since one would expect a particle of charge $-e$ to be repelled, not attracted, by the negative potential. Klein and Rafelski[4] have pointed out the physical explanation of this effect. For such a mode the oscillatory part of the wave function inside the well makes a positive contribution ϵ_- to the norm (2), but the exponential part on the right-hand side contributes a negative term, ϵ_+, with $|\epsilon_+| > \epsilon_-$, so that the total norm, ϵ, is negative—corresponding to an antiparticle mode. However, the energy of a state contains a term proportional to $\int dx\, e\rho A_0$, where the charge density $\rho(x)$, for a one-particle state, is just the integrand of Eq. (2). In the present case, with the chosen gauge, only the part of the function inside the potential well contributes to this integral; the result is $-\epsilon_- V$, which is negative. This term makes the total energy of the antiparticle less than m (i.e., $\omega > -m$), so that such an antiparticle bound state really can exist. In other words, the charge density of the state is *polarized* by the potential, so that the total charge is negative, but the charge density near the potential is positive and hence contributes a negative (expectation value of) binding energy. The modes in the Klein region, with free-particle-like wave functions of opposite charge inside and outside the well, may be regarded as extreme instances of this polarization. (It should be understood that an extended charge structure is being attributed here not to the particle itself, but only to its wave function.)

The really significant effects begin at the value V_c of V corresponding to the point labeled C in Fig. 1. There the bound-antiparticle mode appears to coalesce with the lowest bound-particle mode. The graphs of the two frequencies as functions of V form a single smooth curve with a vertical tangent at C, corresponding to a frequency ω_c. At that point there is only one solution of Eq. (1) satisfying the boundary conditions, and its norm is $\epsilon = 0$. [The missing eigenfunction survives as a solution of an inhomogeneous form of Eq. (1)—see Refs. 5 and 6.] Let us call this situation a *singular mode*. As V increases beyond V_c there are eigenfunctions corresponding to a conjugate pair of complex ω's, which grow out from ω_c into the

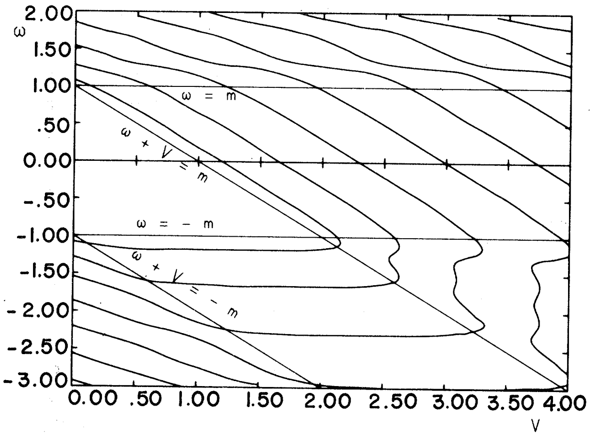

FIG. 2. Frequency levels for $m=1$ and $L_+=L_-$.

complex plane. Qualitatively similar behavior is observed at larger V for successive higher-lying pairs of particle and antiparticle modes.

Schroer and Swieca[5] have shown that singular and complex-frequency modes can be quantized, although not in the usual way. The Hilbert space of the quantized field theory does not contain a discrete vacuum (ground state). In addition, when complex frequencies are present, the total energy of the field is not bounded below; the states of such a theory would be unstable against perturbations (e.g., coupling to other fields which exist in nature). Despite the formal completeness of this quantization procedure, it has been objected that the resulting theory is physically irrelevant because its predictions would be profoundly changed if the dynamics of the electromagnetic field itself were considered or if other interactions[7,4] were included. The same opinion exists about the Klein paradox.[8] Nevertheless, the study of unstable linear systems seems worth pursuing, both because of the light that may be cast on related nonlinear problems and because of their intrinsic intellectual interest. [Making the quantum theory of such systems rigorous is an unsolved mathematical problem, because of the lack of a spectral theorem for Eq. (1) except under conditions which prevent the effects in question from occurring.]

This paper reports a reinvestigation of the SSW model (3) with a modern computer. SSW interpreted x as the radial coordinate of a spherically symmetric problem, with $-L_-$ marking the center; they therefore studied only $L_+=\infty$ or $L_+ \gg L_-$ and reported results of the type shown in Fig. 1. In the present work the case $L_+=L_-$ has been treated, in the hope of elucidating the connection between the SSW effect and the Klein paradox or its black-hole analogs. (Because of the central singularity, the radial coordinate of a black hole is best regarded as extending to $-\infty$ as the hole is approached.)

The calculation of the normal modes and the determination of the signs of their norms [Eq. (2)] were done essentially as described in Ref. 2; information was also obtained from a useful equation given by Klein and Rafelski [Ref. 4, Eq. (2.17)]. Details are given in Ref. 6. The results are presented in Figs. 2 and 3. In contrast with the case studied by SSW (Fig. 1), the turnaround of the frequency curves occurs not in the region of "bound states," but rather in the "Klein region" ($m-V<\omega<-m$), where the $\phi_j(x)$ act like particle modes at $x<0$ and like antiparticle modes at $x>0$. This is no surprise, since symmetry implies that the quantities ω (related to the wavelength of the oscillations of ϕ_j at $x>0$) and $\omega+V$ (related to the wavelength at $x<0$) must play parallel roles in this case.

What is surprising is that each of the curves (except the first) has more than one point of vertical tangency. Whenever one of these "wiggles" is convex toward the left, it must mark the *disappearance* of a pair of complex frequencies. This behavior is sketched in Fig. 3, an enlarged plot of the Klein region of Fig. 2, with eA_0 redefined from Eq. (3) by the addition of a constant, $\frac{1}{2}V$. This trivial gauge transformation increases every frequency ω by $\frac{1}{2}V$, but leaves the physics unchanged; its purpose is to show the symmetry of this system under charge conjugation.

It turns out (from the Klein-Rafelski equation or by direct numerical calculation) that near a point of vertical tangency the slope of the curve of frequencies $\omega_j(V)$ has the opposite sign from the norm ϵ_j of the corresponding solutions. Therefore, immediately beyond the point V at which a pair of complex frequencies disappears, there exists a

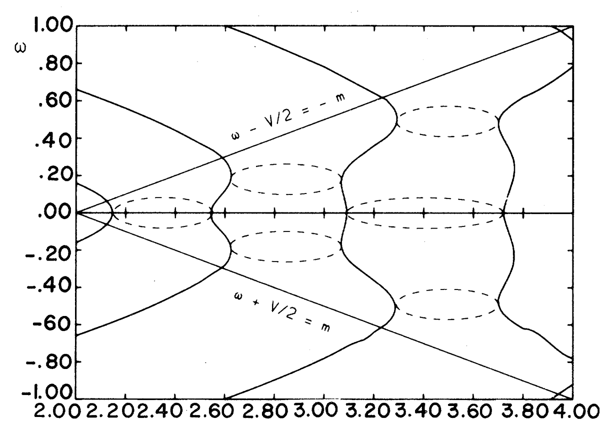

FIG. 3. Enlargement of the Klein region of Fig. 2 in a charge-symmetric gauge. The dashed lines qualitatively represent extensions of the curves into the complex ω plane.

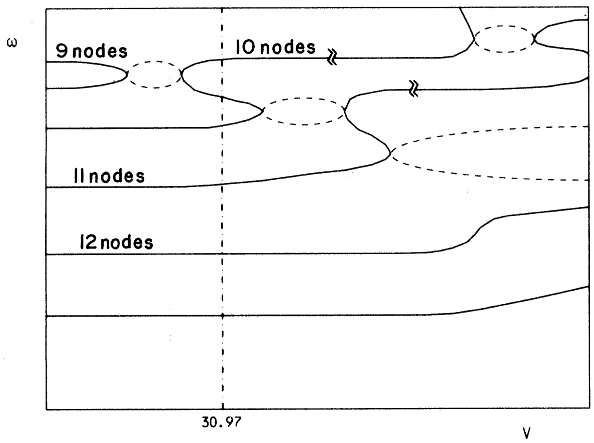

FIG. 4. Frequency levels for $L_+ = 10\,L_-$, $\omega \approx -1.1$, $V \approx 30.97$ (schematic).

negative-norm solution with a frequency higher than that of some positive-norm solution. [For example, in Fig. 3 one can see at $V = 2.6$, starting at the top, two positive solutions, then a negative one ($\omega \approx 0.1$), a positive one ($\omega \approx -0.1$), and two negative ones.] Since the frequencies are real, these modes can be quantized in the usual way. (Indeed, if there are no other complex frequencies or singular modes at that value of V, then the field theory has a discrete vacuum state, even though the contrary was true for smaller V.) On the other hand, in a state with a definite number of particles N_j in a mode ϕ_j, the contribution of that mode to the total energy is $\operatorname{sign}(\epsilon_j)\,\omega_j N_j$. It follows that when a positive-norm and a negative-norm mode appear in the abnormal frequency order, the energy spectrum of the quantized field theory is not bounded below, since each pair of such quanta makes a (gauge-independent) contribution equal to the difference of the frequencies.

This phenomenon of negative-energy particle-antiparticle pairs must be added to the catalog of "catastrophes" which may befall a boson field in an external potential; the others are the Klein paradox, complex frequencies, and the singular behavior at a point of vertical tangency in an $\omega_j(V)$ graph.

A model with $L_+ = 10\,L_-$ has also been computed. Here the first few frequencies behave as in Fig. 1, but at higher frequencies, belonging to functions with about 10 nodes, behavior like that in Fig. 3 is detected (see Fig. 4, where positive and negative solutions appear in the abnormal order near $V = 30.97$, for instance). The conclusion is that the disappearance of complex frequencies and the existence of negative-energy pairs would be found also in the original SSW model with $L_- \ll L_+$ (but $L_+ \neq \infty$), if one looked at large enough quantum numbers.

In retrospect, the nature of Fig. 3 might have been predicted from a knowledge of the Klein paradox. In that case there are no bounded solutions of complex frequency, but for every ω in the Klein interval $(1 - V < \omega < -1)$ there are both a positive-norm solution and a negative-norm solution (not normalizable, but bounded). With effort one can envision this as a limiting case of Fig. 3 as $L_- = L_+ \to \infty$, with the curves becoming closer together and acquiring more wiggles until they fill the entire diagram. It clearly is not a limit of the original SSW picture (Fig. 1), where the particle and antiparticle modes are separated by a horizontal curve. (In fact, resolving that discrepancy was a major motivation of this research.)

A detailed account of this work, including an exposition of the quantization of abnormal modes and a review of the relevance of boson instabilities to black-hole physics, is available from the author.[6]

Note added in proof. The relationship between the SSW and Klein effects has been clarified further by R. M. Wald (unpublished). He attributes the exponential growth of the complex-frequency modes of the field in the SSW situation to stimulated emission by each side of the potential well under the influence of the Klein emission from the other side. As D. W. Sciama (unpublished) has remarked, this means that a deep potential well acts as a laser.

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[1]O. Klein, Z. Phys. 53, 157 (1929); J. Schwinger, Phys. Rev. 82, 664 (1951). Further references are given by A. I. Nikishov, Nucl. Phys. B21, 346 (1970).

[2]L. I. Schiff, H. Snyder, and J. Weinberg, Phys. Rev. 57, 315 (1940). See also Ref. 4, where the effect is demonstrated for more general potentials.

[3]Pair creation by a charged black hole [G. W. Gibbons, Commun. Math. Phys. 44, 245 (1975)] is an electromagnetic Klein paradox; pair creation by a rotating black hole [W. G. Unruh, Phys. Rev. D 10, 3194 (1974)]

is a gravitational analog of the Klein paradox. The gravitational analog of the SSW effect would be expected to arise in a space with an ergosphere but no horizon. There have been attempts to describe the Hawking effect [S. W. Hawking, Commun. Math. Phys. 43, 199 (1975)] in language reminiscent of the Klein or SSW effects, but that situation is rather different from anything encountered in flat-space field theories, so that analogies may be as misleading as helpful. The stabilization of the electromagnetic systems when (physically necessary) nonlinear interactions are included (e.g., Refs. 7 and 4) suggests that similar qualitative changes will occur in the gravitational models when the dynamics of the gravitational field is considered.

[4]A. Klein and J. Rafelski, Phys. Rev. D 11, 300 (1975); 12, 1194 (1975).
[5]B. Schroer and J. A. Swieca, Phys. Rev. D 2, 2938 (1970).
[6]S. A. Fulling, report (unpublished).
[7]A. B. Migdal, Zh. Eksp. Teor. Fiz. 61, 2209 (1971) [Sov. Phys.—JETP 34, 1184 (1972)].
[8]J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley, Cambridge, Mass., 1955), pp. 311, 312.