Causality and asymptotic behavior in electroproduction

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We consider Regge and scaling behavior in electroproduction employing the Deser-Gilbert-Sudarshan representation. Under a certain analyticity assumption we connect the limits $q^2 \to -\infty$ and $q^2 \to 0$ of the integral of the proton structure function $W_2(\nu, q^2)$ over positive ν and, as a consequence, obtain a sum rule on the scaling function $F_2(\omega)$. This sum rule appears to be in reasonable agreement with experiment.

I. INTRODUCTION

The use of causality to derive certain sum rules on the invariant functions arising in the decomposition of matrix elements of current commutators was introduced and discussed by Schroer and Stichel,[1] Meyer and Suura,[2] and Gervais.[3]

Causality sum rules involving Bjorken scaling functions in electroproduction were derived by Leutwyler and Stern.[4] Under the assumption of scaling, these are equivalent[5] to those of Meyer and Suura.[2] The extension of the work of Meyer and Suura to the nonforward case and a thorough investigation of the role of causality in sum rules originally attributed to current algebra or light-cone commutators of the quark model were recently undertaken.[5,6]

The convergence of the fixed-mass limit of the causality sum rules was discussed by Leutwyler and Otterson.[7] Their idea of causal Regge subtractions could also be used to derive convergent fixed-mass sum rules from current algebra.[8] For this purpose it is possible to use causal Regge forms introduced by Brandt[9] in a causal Regge expansion of a structure function in electroproduction.

The causal Regge expansion was used by Brandt and Ng[10] in an attempt to derive a sum rule on the electroproduction scaling function $F_2(\omega)$. This derivation was subsequently criticized by several authors[11-13]; the point was that a certain term in the causal Regge expansion of the structure function $W_2(\nu, q^2)$ was missing in the form assumed by Brandt and Ng. However, there is a definite restriction on the causal spectral function $\sigma(a, b)$ in the DGS[-14] representation of $W_2(\nu, q^2)$ which arises on imposing scaling, namely[9]

$$\int_0^\infty \sigma(a, b)da = 0 \tag{1.1}$$

and which may be further specialized on using the Regge expansion of $\sigma(a, b)$. It thus appears plausible that under certain conditions a corresponding restriction on $W_2(\nu, q^2)$ may exist. It is our purpose in this paper to investigate this possibility.

We find that use of the causal Regge form together with the scaling condition (1.1) and a certain analyticity assumption give the equation

$$2\int_0^\infty \overline{W}_2(\nu, q^2)d\nu = \alpha - q^2\int_0^\infty \frac{\eta(x)}{x-q^2}\,dx\ , \tag{1.2}$$

where $\overline{W}_2(\nu, q^2)$ is the proton structure function $W_2(\nu, q^2)$ minus its causal Pomeron contribution, α is an arbitrary constant, and $\eta(x)$ is a function such that $\eta(x)=0$ for $x \leqslant 0$, $\lim_{x\to\infty} x\eta(x)=0$, and

$$\int_0^\infty \eta(x)dx = 0. \tag{1.3}$$

These equations then imply our basic result:

$$\lim_{q^2\to-\infty}\int_0^\infty W_2(\nu, q^2)d\nu = \lim_{q^2\to 0}\int_0^\infty \overline{W}_2(\nu, q^2)d\nu. \tag{1.4}$$

From this result follows the sum rule

$$\int_0^1 [F_2(\omega) - F_2(0)]\frac{d\omega}{\omega} = 1 \tag{1.5}$$

for the proton electroproduction scaling function $F_2(\omega)$. To specialize (1.4) to (1.5) we have used a definite form for the causal Pomeron contribution.[9] However, the sum rule (1.5) completely exhausts the content of the result (1.4).

It will be evident from our proof of the result (1.4) that its validity is not dependent on the specific Regge expansion introduced in Sec. II [Eq. (2.2)], but in this respect essentially only requires that $\overline{\sigma}(a, 0)$ is finite. This is a mild requirement for Regge behavior. Much more stringent conditions on $\sigma(a, b)$, near $b=0$, must in fact be placed to ensure Regge asymptotics.[15]

The specific analyticity assumption we make is that the contribution of $\overline{\sigma}(a, 0)$ to $W_2(\nu, q^2)$ has at most a δ singularity at $\nu = -\frac{1}{2}q^2$, since this is the singularity of $W_2(\nu, q^2)$ at this point. A list of all the conditions we have used in obtaining the sum rule (1.5) is given at the beginning of the discussion in Sec. V. In this section we also give remarks on the neutron case and compare the sum rule (1.5) to similar results from the quark model.

The main work of the paper is done in Secs. II and III. Section II introduces the conditions of Regge behavior and scaling on the causal representation and leads to the singularity in $\bar{\sigma}(a, 0)$ at $a=0$. Further considerations on this singularity in Sec. III yield the sum rule (1.5).

An experimental estimate of the left-hand side of (1.5) is attempted in Sec. IV. It is found to be in reasonable agreement with the data, preferring a value of $F_2(0)$ around 0.06.

II. CASUALITY, SCALING, AND REGGE BEHAVIOR

Causality implies that the structure function $W_2(\nu, q^2)$ in electroproduction satisfies the DGS[14] representation

$$W_2(\nu, q^2) = -q^2 \int_0^\infty da \int_{-\infty}^\infty db\, \sigma(a,b)\theta(1-|b|) \times \delta(q^2+2b\nu-a)\epsilon(\nu+b). \tag{2.1}$$

It was noted by Brandt[9] that the behavior of the spectral function $\sigma(a,b)$ near $b=0$ determines the Regge behavior of the structure function $W_2(\nu, q^2)$. An expansion of $\sigma(a,b)$, near $b=0$, consistent with Regge behavior for $W_2(\nu, q^2)$ is

$$\sigma(a,b) = \sigma_P(a)\ln|b| + \sigma_0(a) + \sigma_1(a)|b|^{1/2} + \cdots. \tag{2.2}$$

It is the term $\sigma_0(a)$ which is missing from the analysis of Brandt and Ng.[10]

For $\nu>0$ and $q^2<0$, the representation (2.1) may be written in the form

$$W_2(\nu, q^2) = \frac{-q^2}{2\nu}\int_0^\infty da\, \sigma\left(a, \frac{a-q^2}{2\nu}\right) \times \theta\left(1-\frac{a-q^2}{2\nu}\right). \tag{2.3}$$

It follows that scaling of νW_2 implies that

$$\int_0^\infty da\, \sigma(a,b) = 0 \tag{2.4}$$

i.e.,

$$\int_0^\infty da\, \sigma_i(a) = 0, \quad i = P, 0, 1, \ldots. \tag{2.5}$$

From (2.3) and (2.4) one finds that the scaling function $F_2(\omega)$ is given by[13]

$$F_2(\omega) = \tfrac{1}{2}\omega\theta(1-\omega)\int_0^\infty da\, a\sigma'(a,\omega) - \tfrac{1}{2}\delta(1-\omega)\int_0^\infty da\, a\sigma(a,1), \quad \omega>0 \tag{2.6}$$

where $\sigma'(a,b) = (\partial\sigma/\partial b)(a,b)$.

On closer inspection of the representation (2.3) we now note that the necessary condition for the scaling of νW_2 is in fact more stringent than (2.4). It is given by

$$\lim_{\nu\to\infty} \nu \int_0^\nu \sigma(a,b)da = 0. \tag{2.7}$$

Condition (2.4) is then a consequence of (2.7). For the terms $\sigma_i(a)$ in the Regge expansion, condition (2.7) gives

$$\lim_{\nu\to\infty} \nu \int_0^\nu da\, \sigma_i(a) = 0, \quad i = P, 0, 1, \ldots. \tag{2.8}$$

Defining $\overline{W}_2(\nu, q^2)$ to be $W_2(\nu, q^2)$ minus its causal Pomeron part, in (2.2) given by the first term, one obtains from (2.1), (2.2), and (2.4) the equation[16]

$$\int_0^\infty \overline{W}_2(\nu, q^2)d\nu = \tfrac{1}{2}q^2\int_0^\infty da\, \sigma_0(a)\ln(a-q^2). \tag{2.9}$$

It is this equation which we shall use as the basis for the derivation of our main result. We show in the Appendix that, with the Pomeron given by (2.2),

$$\lim_{q^2\to 0}\int_0^\infty \overline{W}_2(\nu, q^2)d\nu = G_E{}^2(0), \tag{2.10}$$

so that it vanishes for the neutron and equals unity for the proton case which is under consideration. From this, one concludes that, for the proton, the term $\sigma_0(a)$ cannot vanish identically, since in this case

$$\tfrac{1}{2}\int_0^\infty \sigma_0(a)\ln(a-q^2)da \sim \frac{1}{q^2}, \quad \text{as } q^2\to -0. \tag{2.11}$$

This equation shows, in fact, that *the function* $\sigma_0(a)$ *must be singular at* $a=0$.

Before we leave this section we remark that from the representation (2.1) and the scaling condition (2.4) one also obtains the equation

$$\int_0^\infty \frac{W_2(\nu, -2\xi\nu-\eta)}{2\xi\nu+\eta}d\nu = -\tfrac{1}{2}\int_0^\infty da\, \sigma(a,\xi)\ln(a+\eta), \quad \xi>0, \quad \eta\geq\xi^2 \tag{2.12}$$

a relation that fails to hold for $\xi=0$, giving way to the Pomeron-subtracted relation (2.9) at fixed mass. We also note that from (2.12) follows the causality sum rule

$$\int_{-\infty}^\infty \frac{W_2(\nu, -2\xi\nu-\eta)}{2\xi\nu+\eta}d\nu = 0 \tag{2.13}$$

of Leutwyler and Stern.[4]

In the next section we use scaling and a plausible assumption on the zero-mass singularity in $\sigma_0(a)$ to obtain the result (1.4).

III. SUM RULE

When the scaling conditions (2.8) are satisfied, the expansion (2.2) generates from the representation (2.1) a series for $W_2(\nu, q^2)$ such that each term is causal and scales. In particular the contribution of $\sigma_0(a)$ to νW_2 scales. If we define a function $\psi(x)$ by

$$\psi(x) = \int_0^x \sigma_0(a)da, \qquad (3.1)$$

and $\psi(x) = 0$ for $x < 0$, we find that the contribution of $\sigma_0(a)$ to νW_2 is $-\frac{1}{2}q^2\psi(2\nu + q^2)$, when $q^2 < 0$ and $\nu > 0$. From Eq. (2.6) this scales to $\frac{1}{2}\alpha\delta(1-\omega)$ where

$$\alpha = -\int_0^\infty a\sigma_0(a)da. \qquad (3.2)$$

We now define a function $\eta(x)$ by

$$\psi(x) = \alpha\delta(x) + \eta(x) . \qquad (3.3)$$

Then $\eta(x) = 0$ for $x < 0$ and the scaling of ψ, $x\psi(\lambda x) \sim \alpha\delta(\lambda)$ gives

$$\lim_{x\to\infty} x\eta(\lambda x) = 0 , \qquad (3.4)$$

where λ is arbitrary (including $\lambda = 0$).

Now the structure function $W_2(\nu, q^2)$ has only a $\delta(q^2 + 2\nu)$ singularity at $\nu = -\frac{1}{2}q^2$. We shall assume that the contribution of $\sigma_0(a)$ to $W_2(\nu, q^2)$, $-q^2/2\nu\psi(2\nu + q^2)$, does not possess a worse singularity at $\nu = -\frac{1}{2}q^2$. Thus $\eta(x)$ can have at most a $\delta(x)$ singularity at $x = 0$. The scaling condition (3.4) then implies that, under such conditions, $\eta(x)$ is in fact finite at $x = 0$ and that

$$\eta(0) = 0 \text{ and } \lim_{x\to\infty} x\eta(x) = 0. \qquad (3.5)$$

Equation (3.3) then gives

$$\sigma_0(x) = \alpha\delta'(x) + \eta'(x). \qquad (3.6)$$

Further, from (3.2), (3.5), and (3.6) one deduces that

$$\int_0^\infty \eta(x)dx = 0 , \qquad (3.7)$$

for integration by parts gives

$$\int_0^\infty \eta(x)dx = -\int_0^\infty x\eta'(x)dx$$
$$= -\int_0^\infty x\sigma_0(x)dx + \alpha\int_0^\infty x\delta'(x)dx$$
$$= 0.$$

Substituting for $\sigma_0(a)$ from (3.6) into (2.9) we obtain, after an integration by parts and use of (3.5),

$$\int_0^\infty \overline{W}_2(\nu, q^2)d\nu = \tfrac{1}{2}\alpha - \tfrac{1}{2}q^2\int_0^\infty \frac{\eta(x)}{x-q^2}dx. \qquad (3.8)$$

We now observe that the limit of the right-hand side of this equation as $q^2 \to 0$ exists and is given by $\frac{1}{2}\alpha$ since $\eta(0) = 0$. The limit as $q^2 \to -\infty$ also exists and is again given by $\frac{1}{2}\alpha$ since (3.7) holds. We are thus led to the general sum rule

$$\lim_{q^2\to-\infty}\int_0^\infty \overline{W}_2(\nu, q^2)d\nu = \lim_{q^2\to 0}\int_0^\infty \overline{W}_2(\nu, q^2)d\nu. \qquad (3.9)$$

An *alternative proof* of this result follows. By direct integration of Eq. (2.6) one has

$$\int_0^\infty \overline{F}_2(\omega)\frac{d\omega}{\omega} = \tfrac{1}{2}\alpha, \qquad (3.10)$$

where α is given by (3.2), since $\sigma_0(a) = \overline{\sigma}(a, 0)$. Define $\eta'(x)$ by Eq. (3.6). Then

$$\int_0^x \sigma_0(a)da = \alpha\delta(x) + \int_0^x \eta'(a)da,$$

and the scaling property gives

$$\lim_{x\to\infty} xf(\alpha x) = 0, \ \forall\alpha,$$

$$\text{where } f(x) = \int_0^x \eta'(a)da.$$

Assuming that the contribution of σ_0 to W_2 has at most a δ singularity at $\nu = -\frac{1}{2}q^2$, $f(x)$ must be finite at $x = 0$ and $f(0) = 0$. We may therefore take $\eta(0) = 0$ to complete the definition of $\eta(x)$. Equations (3.5) and (3.6) then hold and (3.8) follows. From (3.8) one has

$$\tfrac{1}{2}\alpha = \lim_{q^2\to 0}\int_0^\infty \overline{W}_2(\nu, q^2)d\nu ,$$

which, with (3.10), completes the proof.

Before we proceed to discuss the consequences of the sum rule (3.9), it is important to point out here that its validity is not dependent on the special form of the causal Regge expansion (2.2). In this respect we only require that $\overline{\sigma}(a, 0)$ be finite. It is then possible to write W_2 in the form

$$\overline{W}_2 = \overline{W}_2^{(0)} + \overline{W}_2^{(1)} , \qquad (3.11)$$

where $\overline{W}_2^{(0)}$ is the contribution of $\overline{\sigma}(a, 0)$ such that both $\overline{W}_2^{(0)}$ and $W_2^{(1)}$ separately scale. Equation (3.11) then completely replaces the expansion (2.2) for the purpose of the derivation of the sum rule (3.9). It is of course also implied that the subtracted causal Pomeron contribution, W_2^P, already scales since W_2 does. However, the Pomeron term need not have the specific form given in (2.2), and the result (3.9) holds with any scaling causal Pomeron function subtracted from W_2, i.e., any causal function $W_2^P(\nu, q^2)$ such that νW_2^P scales and

$$\lim_{\nu\to\infty} \nu(W_2 - W_2^P) = 0. \qquad (3.12)$$

We now specialize the sum rule (3.9) by taking for the Pomeron the particular form given in (2.2)

$$\sigma_P(a,b)=\sigma_P(a)\ln|b|. \qquad (3.13)$$

It is shown in the Appendix that, in this case, Eq. (2.10) holds. The sum rule (3.9) then reads

$$\int_0^\infty F_2(\omega)\frac{d\omega}{\omega}=G_E{}^2(0)=1, \qquad (3.14)$$

where $\overline{F}_2(\omega)=F_2(\omega)-F_2^P(\omega)$. But $F_2^P(\omega)$ may be explicitly obtained, for this case, from (2.6) and (3.13). One finds

$$F_2^P(\omega)=\tfrac{1}{2}\theta(1-\omega)\int_0^\infty da\, a\sigma_P(a),$$

i.e.,

$$F_2^P(\omega)=F_2(0)\theta(1-\omega), \quad \omega>0 \qquad (3.15)$$

since $\overline{F}(\omega)\to 0$ as $\omega\to 0$. We thus finally obtain from (3.9) the sum rule

$$\int_0^1[F_2(\omega)-F_2(0)]\frac{d\omega}{\omega}=1, \qquad (3.16)$$

for the proton electroproduction structure function.

In the next section we give an experimental estimate of the left-hand side in (3.16). A complete set of the conditions required for its validity and some remarks on the neutron case are given in Sec. V.

IV. EXPERIMENTAL ESTIMATE

Experimental estimates of integrals of the type appearing in the sum rule (1.5), from the currently available data, have proved to be rather difficult, due to uncertainties in the small ω behavior. In particular the estimate of the integral in the present sum rule is very much dependent on the value of $F_2(0)$. The data for νW_2 show some apparent apprximate constancy for $0<\omega<0.1$ around the value 0.3. This is taken by some authors as the experimental value of $F_2(0)$. However, one notes that the asymptotic expression

$$F_2(\omega)\sim\alpha_1+\alpha_2\omega^{1/2}+\cdots, \text{ for small } \omega \qquad (4.1)$$

implies that

$$F_2'(\omega)\sim\tfrac{1}{2}\alpha_2\omega^{-1/2}+\cdots, \qquad (4.2)$$

so that $F_2'(\omega)\to\infty$ as $\omega\to 0$. Thus unless $\alpha_2=0$, the apparent constancy of $F_2(\omega)$ for $0<\omega<0.1$ is misleading and the approximate constant value in this range should not be taken as $F_2(0)$.

The approach of Close and Gunion[17] accords to this point of view. We shall therefore use their fits to the data in the small-ω region, particularly since their parameters appear to be in agreement with more recent estimates.[18] Following these authors we write

$$F_2(\omega)=\alpha_1+\alpha_2\omega^{1/2}+\alpha_3\omega^{3/2}, \quad 0\le\omega\le\omega_0 \qquad (4.3)$$

where $\omega_0<0.1$, and obtain

$$\begin{aligned}\int_0^1\overline{F}_2(\omega)\frac{d\omega}{\omega}&=\int_0^{\omega_0}\overline{F}_2(\omega)\frac{d\omega}{\omega}\\&\quad+\int_{\omega_0}^1F_2(\omega)\frac{d\omega}{\omega}-F_2(0)\int_{\omega_0}^1\frac{d\omega}{\omega}\\&=\int_{\omega_0}^1F_2(\omega)\frac{d\omega}{\omega}+2\omega_0{}^{1/2}(\alpha_2+\tfrac{1}{3}\omega_0\alpha_3)\\&\quad-\alpha_1\ln\omega_0{}^{-1}. \end{aligned}\qquad (4.4)$$

The authors of Ref. 17 estimate the parameters α_i in four different fits to the data giving

Fit	α_1	α_2	α_3
I	0.12	0.462	4.02
II	0.06	0.618	4.64
III	0.05	0.645	4.75
IV	0.07	0.663	3.67.

(4.5)

A recent experimental determination[19] gives

$$\int_{0.04}^{0.8}F_2(\omega)\frac{d\omega}{\omega}=0.81, \qquad (4.6)$$

with an estimated systematic error of 5%. Since $F_2(\omega)\simeq 0$ for $0.8<\omega<1$, we shall take this to estimate the first term on the right-hand side of Eq. (4.4) for $\omega_0=0.04$. The other two terms may then be calculated for the four fits of Ref. 17 from (4.5), yielding

$$\begin{aligned}\int_0^1\overline{F}_2(\omega)\frac{d\omega}{\omega}&\simeq 0.81+\tfrac{2}{5}(\alpha_2+\tfrac{1}{75}\alpha_3)-2\alpha_1\ln 5\\&\simeq 0.63, 0.89, 0.93, 0.87,\end{aligned} \qquad (4.7)$$

for the fits I–IV, respectively. The sum rule (1.5) thus appears to be in reasonable agreement with experiment, preferring a value of $F_2(0)$ around 0.06. This is in line with the value $F_2(0)\simeq 0.07$ obtained from the generalized vector-dominance model.[18] Kim and Rodenberg[18] also obtain $\alpha_2=0.55$.

Brandt and Ng[10] estimate the same integral using $F_2(0)=0.285$, and obtain 0.02 for its value. But with $F_2(0)=0.285$, $\overline{F}_2(\omega)\simeq 0$ for $0<\omega<0.04$ so that their estimate may be read in the form

$$\int_{0.04}^1[F_2(\omega)-0.285]\frac{d\omega}{\omega}\simeq 0.02, \qquad (4.8)$$

giving

$$\int_{0.04}^1F_2(\omega)\frac{d\omega}{\omega}\simeq 0.94. \qquad (4.9)$$

If we use this expression in (4.4), in place of (4.6), we obtain

$$\int_0^1\overline{F}_2(\omega)\frac{d\omega}{\omega}\simeq 0.76, 1.02, 1.06, 1.00, \qquad (4.10)$$

from the four fits of Ref. 17, improving the agreement with experiment.

V. DISCUSSION

1. We start by reviewing the conditions under which we have obtained the sum rule (1.5). These are the following:

(a) causality, which gives the representation (2.1);

(b) regge behavior, which requires that

$$\sigma_0(a) = \lim_{b\to 0} \bar{\sigma}(a, \boldsymbol{b})$$

exists;

(c) scaling, which yields the condition (2.7) and also requires that the integrals on the right-hand side of (2.6) exist;

(d) analyticity, which shows that $\bar{\sigma}(x, 0)$ and $\psi(x)$ are singular at $x=0$;

(e) an assumption on the singularity of the term $-(q^2/2\nu)\psi(2\nu+q^2)$ in $W_2(\nu, q^2)$ at $\nu=-\frac{1}{2}q^2$, namely it is a δ-singularity, since this is the singularity of $W_2(\nu, q^2)$ at this point.

2. Our consideration of the singularity structure of $\psi(x)$ at $x=0$ was important for obtaining the sum rule (1.5). In the neutron case this singularity is nonexistent and one cannot therefore link $\psi(x)$ at $x=0$ to $W_2(\nu, q^2)$ at $\nu=-\frac{1}{2}q^2$. To see that $\psi(x)$ is nonsingular at $x=0$ in the neutron case consider Eq. (2.9), as $q^2\to 0$. Then

$$\lim_{q^2\to 0} \frac{2}{q^2}\int_0^\infty \bar{W}_2(\nu, q^2)d\nu \equiv \lim_{q^2\to 0}\int_0^\infty \sigma_0(a)\ln(a-q^2)da \tag{5.1}$$

exists, so that $x\sigma_0(x)\ln x\to 0$ as $x\to 0$ and $\psi(x)$, defined by (3.1), cannot be singular at $x=0$. In fact $\psi(0)=0$ and $\psi(x)$ does not contribute to the elastic term in $W_2(\nu, q^2)$.

One may artificially add to $W_2(\nu, q^2)$ in this case a well-defined term putting in such a singularity by hand, make the assumption (d) about it, and subtract the new term at the end, obtaining a sum rule similar to (1.5), for $F_2^{en}(\omega)$, with the right-hand side equal to zero. However, it is clear that this sum rule is much less compelling than (1.5) and the basis for its validity is rather weak. Such a sum rule would imply $F_2(0)\neq 0$.[10]

3. Sum rules similar to (1.5) have been obtained in various forms from some versions of the quark model.

(i) The Gottfried sum rule,[20] obtained from the naive nonrelativistic quark model, reads

$$\int_0^\infty W_2(\nu, q^2)d\nu = 1. \tag{5.2}$$

This is obviously stronger than our basic result (1.4). Further, on the basis of Regge behavior, the left-hand side is divergent.

(ii) The sum rule (5.2) also follows from the naive relativistic quark-parton model,[21] together with

$$\int_0^\infty W_2^{en}(\nu, q^2)d\nu = \tfrac{2}{3}, \tag{5.3}$$

for the neutron. The left-hand side of (5.3) is again divergent, and the sum rule is in fact not satisfied at $q^2=0$. However, one should note that the quark parton model is generally restricted to the deep-inelastic region, so that (5.2) and (5.3) should really be read for scaling functions. As such the quark parton model is not suitable for checking the result (1.4), of which the sum rule (1.5) is a direct consequence.

(iii) The general quark-parton model[21] yields neither of the sum rules (5.2), (5.3). However, the "valence + sea"[22] specialization of the general model gives in place of (5.2) and (5.3), respectively,

$$\int_0^1 [F_2(\omega)-F_{sea}(\omega)]\frac{d\omega}{\omega} = 1 \tag{5.4}$$

and

$$\int_0^1 [F_2^{en}(\omega)-F_{sea}(\omega)]\frac{d\omega}{\omega} = \tfrac{2}{3}, \tag{5.5}$$

where the sea contribution describes the diffraction or Pomeron part, so that

$$F_{sea}(\omega)\to F_2(0) \text{ as } \omega\to 0. \tag{5.6}$$

Equation (5.4), by itself, does not restrict $F_2(\omega)$ since $F_{sea}(\omega)$ is unspecified. However, it is convergent and when combined with the sum rule (1.5) gives

$$\int_0^1 [F_{sea}(\omega)-F_2(0)]\frac{d\omega}{\omega} = 0 \tag{5.7}$$

a restriction on $F_{sea}(\omega)$ indicating that, in a sense, $F_2(0)$ averages $F_{sea}(\omega)$ over the whole range.

4. Separating the elastic contribution in (3.8), with $\alpha=2$, and taking the limit $q^2\to 0$, after division by q^2, we obtain

$$G'(0) - \frac{1}{4\pi^2\alpha}\int_{m_\pi+(1/2)m_\pi^2}^\infty \bar{\sigma}_T^\gamma(\nu)\frac{d\nu}{\nu} + \frac{1}{4\pi^2\alpha}\int_0^{m_\pi+(1/2)m_\pi^2}\sigma_P^\gamma(\nu)\frac{d\nu}{\nu} = -\tfrac{1}{2}\int_0^\infty \eta(x)\frac{dx}{x}, \tag{5.8}$$

where $G'(0)$ is dG/dq^2 at $q^2=0$ (see Appendix).

With $m_\pi = 0$ this gives the sum rule

$$\frac{1}{4\pi^2\alpha}\int_0^\infty \sigma_T^\gamma(\nu)\frac{d\nu}{\nu} = G'(0) + \tfrac{1}{2}\int_0^\infty \eta(x)\frac{dx}{x}\,. \qquad (5.9)$$

Now

$$G'(0) = 2G_E'(0) + \tfrac{1}{4}(1-\mu_p{}^2), \qquad (5.10)$$

so that $G'(0) < 0$ since $\mu_p > 1$ and the data[23] show that $G_E'(0) < 0$. An estimate of the left-hand side indicates that it is positive. It thus appears that

$$\int_0^\infty \eta(x)\frac{dx}{x} > 0, \qquad (5.11)$$

implying that $\eta(x) \neq 0$.

5. Equation (3.8) is of the same form as sum rules on structure functions that hold in the presence of fixed absorptive poles, such as causality sum rules[7] or current-algebra sum rules when modified to include such poles.[8] It therefore appears to be interesting, and may prove instructive, to pursue this analogy further and explore the connection of $\bar\sigma(a, 0)$ to the existence of fixed poles in electroproduction.

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APPENDIX

Separating the elastic contribution we write

$$\int_0^\infty \overline{W}_2(\nu,q^2)d\nu = G(q^2) + \int_{\nu_0}^\infty \overline{W}_2(\nu,q^2)d\nu - \int_0^{\nu_0} W_2^P(\nu,q^2)d\nu, \qquad (A1)$$

where

$$G(q^2) = (1-\tfrac{1}{4}q^2)^{-1}[G_E{}^2(q^2) - \tfrac{1}{4}q^2G_M{}^2(q^2)]\,, \qquad (A2)$$

$\nu_0 = \frac{1}{2}(2m_\pi + m_\pi{}^2 - q^2)$, and W_2^P is the causal Pomeron contribution to W_2. In the limit $q^2 \to 0$,

$$\int_{\nu_0}^\infty \overline{W}_2(\nu,q^2)d\nu \sim \frac{-q^2}{4\pi^2\alpha}\int_{m_\pi+(1/2)m_\pi{}^2}^\infty \bar\sigma_T^\gamma(\nu)\frac{d\nu}{\nu}\,, \qquad (A3)$$

where $\bar\sigma_T^\gamma(\nu)$ is the total photoabsorption cross section $\sigma_T^\gamma(\nu)$ minus its causal Pomeron part $\sigma_P^\gamma(\nu)$. The integral on the right-hand side of (A3) is convergent and the contribution of this term in (A1) vanishes at $q^2 = 0$. For the third term

$$\int_0^{\nu_0} W_2^P(\nu,q^2)d\nu \sim \frac{-q^2}{4\pi^2\alpha}\int_0^{m_\pi+(1/2)m_\pi{}^2} \sigma_P^\gamma(\nu)\frac{d\nu}{\nu}\,, \qquad (A4)$$

where

$$\sigma_P^\gamma(\nu) = 2\pi^2\alpha\int_0^{2\nu}\sigma_P(a)\ln\frac{a}{2\nu}\,da. \qquad (A5)$$

Using the scaling condition (2.5) for $\sigma_P(a)$, one can rewrite $\sigma_P^\gamma(\nu)$ in the form

$$\sigma_P^\gamma(\nu) = \sigma_T^\gamma(\infty)[1-\phi(2\nu)] + \phi_2(2\nu)\ln 2\nu, \qquad (A6)$$

where

$$\phi_1(\nu) = \frac{2\pi^2\alpha}{\sigma_T^\gamma(\infty)}\int_\nu^\infty \sigma_P(a)\ln a\,da\,, \qquad (A7)$$

$$\phi_2(\nu) = 2\pi^2\alpha\int_\nu^\infty\sigma_P(a)da, \qquad (A8)$$

and

$$\phi_1(0) = 1, \quad \phi_2(0) = 0. \qquad (A9)$$

Note that $\phi_1(0) = 1$ defines $\sigma_T^\gamma(\infty)$ from (A7) and that $\sigma_P^\gamma(\infty) = \sigma_T^\gamma(\infty)$ is ensured by condition (2.7).

If we now assume that $\phi_2(\nu) \sim \nu^\epsilon$, $\epsilon > 0$, as $\nu \to 0$, we see that $\sigma_P^\gamma(\nu) \to 0$ as $\nu \to 0$ and the integral on the right-hand side of (A4) is a finite constant. The third term in (A1) then vanishes at $q^2 = 0$ and we obtain Eq. (2.10) from (A1) and (A2).

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