Multiplicity dependence of transverse momentum in hadronic collisions*

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By using a simple eikonal model of hadron structure, along with the uncertainty relation, we obtain a successful empirical parametrization of the dependence of the transverse momentum on multiplicity for charged secondaries in high-energy proton-proton collisions.

I. INTRODUCTION

In two recent papers[1,2] the present authors have successfully parametrized several quantities relating to emitted particles in hadron-hadron collisions, using an extremely simple model. Indeed, the point of view has been not to make the model as realistic as possible, but to make it as simple as possible, employing essentially only one parameter to see how much data can be parametrized in this way and to compare this to what can be achieved with more elaborate theories and models.

Although, fundamentally, we view the hadron as composed of constituents of a certain "size," for the present discussion we confine our attention to energies where the hadron constituents participate collectively, so that a hadron-hadron collision is characterized only by the overall "size" of the hadron. The hadron is then seen as a continuous distribution of matter, for which distribution we assume the simplest form—namely, a step function, of radius R.

When two hadrons approach one another, no interaction takes place if the impact parameter, b, exceeds $2R$. If $b<2R$ an interaction occurs and particles are emitted. The hadronic spheres are imagined to move through one another and secondaries are produced in the volume overlap. We use a version of the eikonal approximation to give the relative probability for interaction and emission of secondaries.

Indeed, as the impact parameter decreases, the volume overlap increases and, according to the model, so does the probability for emission of increasing numbers of secondaries. In Ref. 2 we related the multiplicity of emitted charged secondaries to the impact parameter in this way and, in particular, calculated the mean multiplicity as well at higher moments of the multiplicity distribution.

For a given impact parameter, and hence a given overlap, there can be defined a characteristic length, $\Delta x_\perp$, that is a measure of the overlap. In Ref. 1, using a preliminary version of the definition of $\Delta x_\perp$, we applied the uncertainty relation in its simplest form to find the momentum conjugate to this $\Delta x_\perp$, and associated that with the rms value of the transverse momentum of emitted particles.

In this paper, using our model of the hadron, the eikonal approximation, and the uncertainty relation, we display the dependence of the transverse momentum of charged secondaries on their multiplicity. We calculate $\Delta p_\perp(n)$ at three values of the beam momentum, $|\vec{p}_L| = 28$, 102, and 405 GeV/c. Comparison with experiment at 28 GeV/c (the only value in this range at which data of sufficient detail exist) yields agreement to within about 3%. Intuitively satisfying qualitative agreement is obtained between theory and experiment in the following features: At a fixed beam momentum, the transverse momentum decreases with increasing multiplicity. For a given multiplicity, the transverse momentum exhibits a slow increase with beam momentum.

In Sec. II we outline in greater detail the assumptions of the model and in Sec. III, we define and calculate the quantity, $\Delta x_\perp$, conjugate to the transverse momentum. Comparison with experiment is presented in Sec. IV, and Sec. V contains a discussion of our model and its results, along with a comparison with other eikonal models.

II. ASSUMPTIONS OF THE MODEL

Consider two hadrons—protons, to be specific—each a homogeneous sphere of hadronic matter, parametrized by radius R. In a given collision at a fixed beam momentum a particular charged multiplicity n is observed when the impact parameter ranges from a maximum value for that multiplicity, $b^M(n)$, to a minimum value for that multiplicity, $b_m(n)$, namely

$$b^M(n) > b \geq b_m(n). \tag{2.1}$$

The range of n extends from 2 to the maximum value, n_M, of the charged secondary multiplicity observed at this given beam momentum.[3] Of course $b_m(n_M)=0$, $b^M(2)=2R$, and $b_m(n-2)=b^M(n)$.

The values of $b^M(n)$ and $b_m(n)$ are specified by the requirement that equal increments of n correspond to equal increments of the density-overlap function $W(b)$, defined as

$$\begin{aligned} &W(b_m(0))=0,\\ &\cdots\\ &W(b_m(n))=n\epsilon, \qquad (2.2)\\ &\cdots\\ &W(b_m(n_M))=n_M\epsilon, \end{aligned}$$

where ϵ is a constant.

Thus, for two multiplicities, n' and n'',

$$\frac{n'}{n''}=\frac{W(b_m(n'))}{W(b_m(n''))}. \qquad (2.3)$$

The density-overlap function, $W(b)$, represents the relative probability that an interaction takes place at impact parameter b, and is given according to the eikonal approximation[4] by

$$W(b)=\int z_t z_p dA, \qquad (2.4)$$

where z_t is the "thickness" of the target through which the projectile moves at a given point in the plane normal to the beam direction; z_p is the corresponding "thickness" of the projectile. The integral extends over the area that is the projection of the maximum volume of overlap in this plane at this impact parameter.[5] A straightforward calculation yields[6]

$$W(b)=(4R^2-b^2)[\pi R^2-\tfrac{1}{2}b(4R^2-b^2)^{1/2} - 2R^2\sin^{-1}(b/2R)]. \qquad (2.5)$$

We assume that to a given impact parameter there corresponds a well-defined rms value of the transverse momentum of emitted secondaries determined by the amount of overlap at that impact parameter, as specified below. Calling this value of the transverse momentum $\Delta p_\perp(b)$ for the present, since the multiplicity n has been associated with a range of impact parameters from $b_m(n)$ to $b^M(n)$, the expression for the transverse momentum in a reaction with a particular multiplicity can be formally written as a weighted average over this range of impact parameters,

$$\Delta p_\perp(n)=\frac{\int_{b_m(n)}^{b^M(n)}\Delta p_\perp(b)W(b)b\,db}{\int_{b_m(n)}^{b^M(n)}W(b)b\,db}. \qquad (2.6)$$

The final assumption is that the transverse momentum associated with a fixed impact parameter, $\Delta p_\perp(b)$, introduced above, is the variable conjugate to the linear measure of overlap, $\Delta x_\perp(b)$, at that impact parameter, and is given by the optimal form of the uncertainty relation,

$$\Delta p_\perp(b)\Delta x_\perp(b)=\tfrac{1}{2}\hbar. \qquad (2.7)$$

In the next section we calculate $\Delta x_\perp(b)$ geometrically. When this is done, Eq. (2.6) will give the desired result for $\Delta p_\perp(n)$ explicitly since, for a given beam momentum and a given multiplicity, Eq. (2.2) fixes the limits of integration and Eq. (2.7) yields $\Delta p_\perp(b)$.

III. CALCULATION OF $\Delta x_\perp(b)$

The physical meaning of $\Delta x_\perp(b)$ is that it is the coordinate that is conjugate to $\Delta p_\perp(b)$, the rms value of the transverse momentum or emitted particles. In a qualitative sense $\Delta x_\perp(b)$ is a measure of the characteristic linear dimension of the production region for a given impact parameter. While there can be some discretion in the precise method of calculating this quantity, we make the following specification.

As the two spheres move through one another at a fixed impact parameter, the volume of overlap varies over time; it is within this volume that particle production takes place. $\Delta x_\perp$ is the mean value of the radial coordinate of an area which is the projection of the time-varying volume of overlap onto a plane normal to the projectile's motion. The calculation of $\Delta x_\perp(b)$, the full details of which are presented elsewhere,[7] is a satisfying geometrical exercise, only the results of which are summarized here.

Referring to Fig. 1, we note that as the projectile moves from left to right, two well-defined regions determine the projections of the overlap volume onto the XY plane.

(a) Region I [$\sin^{-1}(b/2R)\le\theta\le\frac{1}{2}\pi$]. At the lower limit of θ [$\theta_m=\sin^{-1}(b/2R)$], the spheres just begin to make contact. In this region the projection of the overlap volume onto the plane of symmetry AA' is a circle of radius r, and the projection of the overlapping volume onto the XY plane is an

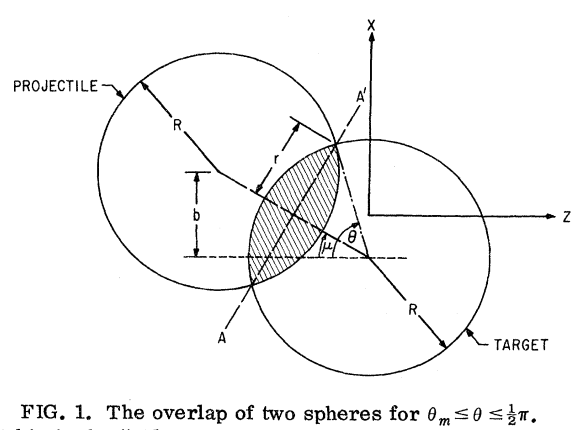

FIG. 1. The overlap of two spheres for $\theta_m\le\theta\le\frac{1}{2}\pi$. This is the "side view"; the projectile is traveling in the $+Z$ direction.

ellipse,

$$\frac{X^2}{r^2\cos^2\mu}+\frac{Y^2}{r^2}=1. \tag{3.1}$$

The area of the ellipse is

$$A(b,\theta)=\pi r^2(b,\theta)\cos\mu(b,\theta). \tag{3.2}$$

(b) Region II ($\frac{1}{2}\pi<\theta\leq\pi-\theta_m$). In this region the projection of the overlap volume in the XY plane is no longer an ellipse, but a figure that varies in area from

$$A(\tfrac{1}{2}\pi)=\pi R^{1/2}(R-\tfrac{1}{2}b)^{3/2} \tag{3.3}$$

at $\theta=\frac{1}{2}\pi$, to its maximum value, at $\theta=\pi-\theta_m$, of

$$A(\pi-\theta_m)=\pi(R-\tfrac{1}{2}b)(R^2-\tfrac{1}{4}b^2)^{1/2}. \tag{3.4}$$

As the projectile passes through the target, the mean value of the projection of the overlap volume onto the XY plane is

$$\bar{A}=\frac{\int_{\theta_m}^{\pi-\theta_m}A(\theta)d\theta}{\int_{\theta_m}^{\pi-\theta_m}d\theta}. \tag{3.5}$$

From this we find $\Delta x_\perp$ by identifying it with the mean radial coordinate of the above mean area,

$$\Delta x_\perp=\frac{\int_0^a r^2dr}{\int_0^a r dr}, \tag{3.6}$$

where a is defined by

$$\bar{A}\equiv\pi a^2. \tag{3.7}$$

This yields[7] for $\Delta x_\perp(b)$

$$\Delta x_\perp(b)=\left|(\tfrac{1}{6}\pi)^{1/2}\right|\tfrac{1}{3}(2R-b), \tag{3.8}$$

and from this, $\Delta p_\perp(b)$ follows immediately from the uncertainty relation

$$\Delta p_\perp(b)=\frac{\hbar}{2}\left(\frac{6}{\pi}\right)^{1/2}\frac{3}{2R-b}. \tag{3.9}$$

When this expression is used in the integral, Eq. (2.6), $\Delta p_\perp(n)$ is found explicitly.

IV. COMPARISON WITH EXPERIMENT

Smith *et al.*[8] present data on proton-proton collisions for a laboratory momentum, $p_L=28.44$ GeV/c, in which the multiplicity dependence of the transverse momentum is reported. We apply our theory to that case, for which the maximum charged secondary multiplicity is $n_M=14$.

Equation (2.2) fixes the values of $b^M(n)$ and $b_m(n)$ which are the limits of integration in Eq. (2.6). Table I tabulates the results.

Using these values of $b^M(n)$ and $b_m(n)$, we calculate the transverse momentum $\Delta p_\perp(n)$ for all even values of $n\leq n_m$. The data of Smith *et al.*[8] give the experimental values of $\Delta p_\perp(n)$ for $n=4$, 6, and 8. Table II lists these theoretical and experimental results. As one sees from the table, theoretical and experimental results agree, on the average, to within 3%.

TABLE I. The values of the impact parameters defining the various multiplicity domains for $n_M=14$.

n	$b^M(n)$ (10^{-13} cm)	$b_m(n)$ (10^{-13} cm)
2	1.600	1.027
4	1.027	0.810
6	0.810	0.634
8	0.634	0.476
10	0.476	0.323
12	0.323	0.168
14	0.168	0

We also calculate the theoretical expressions for $\Delta p_\perp(n)$ at representative higher values of incident beam momentum, p_L, even though experimental studies have not tabulated explicitly the multiplicity dependence of the transverse momentum at these higher beam momenta. We do that at $p_L=102$ GeV/c, where $n_m=18$,[9,10] and at $p_L=405$ GeV/c, where $n_M=26$.[10] Table III gives the results.

The results show a variation in transverse momentum that displays two intuitively satisfying properties that also agree with experimentally observed trends[8]:

(i) For a given beam momentum the transverse momentum decreases with increasing multiplicity.

(ii) For a given multiplicity, the transverse momentum exhibits a slow increase with increasing beam momentum.

V. COMPARISON WITH OTHER MODELS AND DISCUSSION

As we remarked in the Introduction, our model is not presented as a realistic picture of hadron structure. Rather, we have proposed a naively simple model which has parametrized a considerable amount of proton-proton scattering data in some detail, using only one parameter, along with the most fundamental and firm of principles, the

TABLE II. A comparison of theoretical and experimental values of $\Delta p_\perp(n)$ at $p_L=28.44$ GeV/c.

n	$\Delta p_\perp(n)$ (GeV/c) (theory)	$\Delta p_\perp(n)$ (GeV/c) (experiment)
4	0.383	0.383
6	0.395	0.370
8	0.363	0.355
10	0.330	...
12	0.300	...
14	0.274	...

TABLE III. A comparison of $\Delta p_{\perp}(n)$ for three values of the projectile's momentum.

$\Delta p_{\perp}(n)$	$p_L = 28.44$ GeV/c ($n_M = 14$)	$p_L = 102$ GeV/c ($n_M = 18$)	p_L 305 GeV/c ($n_M = 26$)
$\Delta p_{\perp}(4)$	0.383	0.336	0.177
$\Delta p_{\perp}(6)$	0.395	0.403	0.376
$\Delta p_{\perp}(8)$	0.363	0.389	0.404
$\Delta p_{\perp}(10)$	0.330	0.363	0.397
$\Delta p_{\perp}(12)$	0.300	0.337	0.381
$\Delta p_{\perp}(14)$	0.274	0.313	0.363
$\Delta p_{\perp}(16)$		0.290	0.345
$\Delta p_{\perp}(18)$		0.270	0.328
$\Delta p_{\perp}(20)$			0.311
$\Delta p_{\perp}(22)$			0.295
$\Delta p_{\perp}(24)$			0.279
$\Delta p_{\perp}(26)$			0.266

uncertainty relation.

One can gain some perspective on the special features of our model by comparing and contrasting it with other related eikonal or impact-parameter models. Most commonly, one thinks of the eikonal method in describing elastic scatting by an invariant scattering amplitude, in eikonal form

$$M(s,t) = 4\pi i s \int_0^\infty b\,db\,J_0(b\sqrt{-t})(1 - e^{i\chi(b,s)}), \quad (5.1)$$

where χ is the eikonal phase or eikonal function, s and t are the usual Mandelstam variables, and b the impact parameter. To describe particle production in the eikonal framework, the same formalism can be used, with $\chi(b,s)$ an operator.

Several groups of investigators have shown that this eikonal form for the scattering amplitude follows from field-theoretic calculations (usually, if certain simplifying assumptions are made) where incident particles interact through exchange of various entities. For example, Cheng and Wu[11] have studied the problem of the exchange of noninteracting towers in massive quantum electrodynamics, obtaining an elastic scattering amplitude in eikonal form[12]

$$M = \tfrac{1}{2} i m^{-2} s \int d\vec{x}\, e^{i\vec{\Delta}\cdot\vec{x}}(1 - e^{-\chi}). \quad (5.2)$$

In the limit $s \to \infty$

$$\chi \sim b i^{-1} S^a e^{-\mu|\vec{x}|}, \quad (5.3)$$

where b and μ are real constants and

$$S = \left\{ -\frac{(-s)^a}{[\ln(-s)]^2} + \frac{(-u)^a}{[\ln(-u)]^2} \right\}^{1/a}, \quad (5.4)$$

with u the third Mandelstam variable and a a positive constant.

In qualitative terms, their model results in a physical picture of hadron-hadron scattering where at very high energies each particle acts like a Lorentz-flattened sphere (in first order) with an absorptive black core of radius $R = R_0 \ln|S|$ (where, essentially, $S \sim s$) which grows with energy, and (in next-higher orders) with a partially absorptive gray fringe.

Chang and Yan[13] consider the related problem of exchange of noninteracting ladders in ϕ^3 theory to describe elastic and inelastic scattering, and also obtain results in eikonal form, as long as fragmentation processes are neglected. In their case the asymptotic form of the eikonal function (for the special and interesting case of strong coupling) is

$$\chi(s, \vec{b}) = -\frac{i\beta(0)}{8\pi c \mu^2 \ln(s/\mu^2)} \times \exp\left\{ [\alpha(0) - 1] \ln\frac{s}{\mu^2}\left(1 - \frac{b^2}{b_m^{\,2}}\right)\right\}, \quad (5.5)$$

where

$$b_m^{\,2} = \left(\frac{g^2}{24\pi^2\mu^4}\right)\left(\frac{g^2}{16\pi^2\mu^2 - 2}\right) \times [\ln(s/\mu^2)]^2, \quad (5.6)$$

with

$$\alpha(0) = \frac{g^2}{16\pi^2\mu^2} - 1, \quad (5.7)$$

$$\beta(0) = 64\pi^3, \quad (5.8)$$

$$c = \frac{g^2}{96\pi^2\mu^4}, \quad (5.9)$$

and where μ is the pion mass.

One sees that if $b < b_m$, $\chi(s, \vec{b})$ increases as a power of s. If $b > b_m$, $\chi(s, \vec{b})$ behaves like a negative power of s. Thus, the qualitative behavior is that the target behaves like an absorptive disk of radius $b_m \sim \ln s$. (This behavior, as well as that found in the work of Cheng and Wu, is to be contrasted with our model in which the radius of the absorptive disk, R, is energy independent, as explained below.)

Auerbach, Aviv, Blankenbecler and Sugar have discussed a large class of models for which the S matrix satisfies full multiparticle unitarity at high energies.[14] In the earlier versions of their work[15] hadrons were treated in accordance with eikonal principles like particles which neither fragment nor lose an appreciable fraction of their incident momenta. One specific model considered by these authors leads to an eikonal with a behavior

$$\chi(Y,\vec{b}) = -ic(R_0^2Y^2 - b^2)^{3/2}\theta(R_0Y - b), \qquad (5.10)$$

when $c = g^2m^2/24R_0$ and Y is the rapidity.[16] In qualitative terms this corresponds, as in the models of Cheng and Wu, to a black disk with a narrow gray fringe. Asymptotically, the radius of the disc grows as $\ln s$.

Another model described by these authors employs the following eikonal

$$\chi(Y,\vec{b}) = \frac{g^2}{32\pi}\theta(R_0 - b)(\tfrac{1}{2}Y) \times [R_0^2\cos^{-1}(b/R) - b(R_0^2 - b^2)^{1/2}]. \qquad (5.11)$$

In this case the qualitative picture is that of a disk with constant radius, R_0, independent of energy.

Barshay[17] uses a highly intuitive approach in which the eikonal is identified with the overlap integral of hadronic matter of projectile and target, and assumes that this takes on a Gaussian shape:

$$\chi(b) = k\int d^2y\, D(y - b)D(y), \qquad (5.12)$$

where

$$1 - e^{-\chi(b)} = \tfrac{1}{2}ce^{-\lambda b^2}. \qquad (5.13)$$

The distribution of average multiplicity of produced particles in impact parameter space is taken to be proportional to the matter-overlap integral

$$\langle n(b)\rangle \sim e^{-\lambda b^2}. \qquad (5.14)$$

This approach is in spirit closest to that of our own work presented in the present communication.

Several aspects distinguish our model from the above-described and other eikonal models. First of all, our interest has not been to derive eikonal behavior from a more fundamental theory. Also, our interest lies not with elastic scattering, but only with the production of secondaries in a hadronic collision. Our formalism makes no direct use of the scattering amplitude—regardless of whether it is written in eikonal form, and focuses not on energy and impact-parameter dependence of the cross section, but straightaway on the multiplicities and transverse momenta of secondaries.

Our version of the eikonal model consists of the straightforward assumption that the probability of production of secondaries (and hence multiplicity) is proportional to the density overlap function at a given impact parameter [see Eq. (2.2)],

$$n \propto W(b).$$

To arrive at $W(b)$ we use the simplest model of a hadron, a homogeneous hard sphere and, hence, the simplest function for $D(z)$ in the standard eikonal expression,

$$W(b) \sim \int dA\, D_p(z)D_t(z), \qquad (5.15)$$

namely $D(z) = z$.

Indeed, the model used for the hadron, a sphere of radius R, has no explicit energy dependence.[18] In that sense the eikonal appears to be energy independent, and our whole model appears to be energy independent.

Of course, energy dependence does enter, but in a different way. At a given energy we take as given the experimentally determined maximum multiplicity.[19] Then, as shown in the text, transverse momentum depends on multiplicity, because there is associated with each multiplicity a specific amount of overlap and, by the uncertainty relation, a specific transverse momentum. For a different, and, say, a higher energy, the maximum multiplicity is greater. The "radius" of the hadrons is unchanged. Therefore, since more "layers" are to be fitted into the same $2R$, the overlap associated with a given multiplicity (and hence Δx) is less than in the lower-energy case. By the uncertainty principle the transverse momentum is thus higher.

Although we are not aware of any detailed calculation with as straightforward an application of the uncertainty relation as we have carried out, the general idea of the utility of the uncertainty relation to account for the limited transverse momentum is, of course, not novel with the present authors. However, its use in calculating the transverse momentum has been criticized in the past, especially by Hagedorn,[20] as not being able to account for the (slight) energy dependence of the transverse momentum. Our model has been able to overcome this objection and to display such a dependence.

In a more general way, some mention might be made of the curious similarity of some features of our model to the old quantum theory. Specifically, the connection of the continuously varying phase integral, $\int p\,dq$, to the integral quantity, nh, suggests itself for comparison when we connect the continuously varying density overlap function,

$W(b)$, to the multiplicity [Eq. (2.2)].

Two views could be taken toward our calculations: One, that it is another empirical parametrization, and has some virtues of simplicity. On the other hand, since our naive model used only the simplest input, such as a step-function distribution of hadronic matter, it is natural to consider next using more realistic assumptions. However, since the results seem quite good already, one has to wonder about the apparent insensitivity to the distribution function that these quantities display.

At ultrahigh energies (with which the present work does not concern itself) one can no longer consider the hadron constituents as interacting collectively; new production channels are opened as the constituents interact individually. Experiment[21] in this domain clearly shows that much higher values of the transverse momentum are contributing. More detailed recent studies[22] show that the multiplicity distribution of the high-$p_\perp$ emitted particles differs from that of the "normal" low-$p_\perp$ particles ($p_\perp < 0.4$ GeV/c), resulting, indeed, in a higher mean charged multiplicity.

This behavior is qualitatively consistent with our model. The higher transverse momentum is related to the smaller "size" of the constituents. As one enters the individual-constituent domain and particles of high $p_\perp$ are emitted, a new production channel is opened, and the mean multiplicity increases. The predictions of our model in this domain constitute the next application of our approach.

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