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Statistical-model analysis of $e\bar{e}$ annihilation*

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We use a statistical model including both pions and kaons to analyze data on $e\bar{e}$ annihilation. Effects due to the finite size of the pion and kaon mass are included, and both are found to be significant. Good approximations are introduced which do not require computer evaluations of phase-space integrals or infinite sums. Using one parameter we obtain satisfactory fits to charged multiplicities, the π to K ratio, and the shape of one-particle inclusive cross sections. Another parameter, which fixes the scale of the total cross section, gives a satisfactory normalization for the one-particle inclusive cross section and for the four- and six-charged-pion exclusive cross sections when it is chosen in agreement with the quark model.

I. INTRODUCTION AND SUMMARY

When electrons and positrons annihilate at center-of-mass energies of a few GeV, many pions, some kaons, and a few nucleons are produced, as well as $\mu\bar{\mu}$ and $e\bar{e}$ pairs. The recent measurements at the SLAC storage rings[1] have given us particle multiplicities, total cross sections, and one-particle inclusive cross sections for $e\bar{e}$ annihilation. In this paper these observed quantities are fitted by a statistical model[2,3] which is constructed in such a way that only two parameters remain to be determined by the data. Our two-parameter fit provides a good account of how the various experimental quantities are related to each other, and predicts how these quantities should behave at still higher energies.

The relevant data may be summarized as follows: (1) The inclusive cross section $E d\sigma/d^3p$ for hadron energies below 1.2 GeV is well fitted by an exponential

$$E\frac{d\sigma}{d^3p}=Ce^{-aE}, \tag{1.1}$$

where $a^{-1}\cong 0.19$ GeV and C is independent of the particle species (at least for momenta below 0.6 GeV/c). (2) The angular distribution of final-state hadrons is isotropic. This isotropy is true only for the lower-momentum particles, which constitute the majority of the final-state particles observed. Although high-momentum particles are not our concern here, one should be aware that anisotropies are observed for high-momentum pions,[4] which strikingly confirms a prediction that follows if the elementary charged particles are spin $\frac{1}{2}$. A few comments about the relation between statistical models and parton models will be made near the end of this Introduction. (3) The multiplicity of charged particles increases smoothly from about $3\frac{1}{4}$ to $4\frac{1}{4}$ as $\sqrt{s}$ (the total c.m. energy) increases from 3 to 5 GeV. Also the total cross section is measured, and is conveniently expressed in terms of its ratio R to the pointlike $\mu\bar{\mu}$ cross sections, $4\pi\alpha^2/3s$. The data[1] show that $R\simeq 2$ between the resonance region and $\sqrt{s}=3.5$ GeV; between 3.5 and 5 GeV there is an unsettled region presumably due to production of charmed quarks, and above 5 GeV again R is constant, at least until the data run out at 8 GeV, with $R\simeq 5.5$. We shall suppose that R continues to be constant at large energies, i.e., that σ_{total} goes like s^{-1} at large s. This is also suggested by quark-gluon models with asymptotic freedom.[5] Finally, there is an "energy crisis." The fraction of the total energy that goes into charged hadrons is smaller than the $\frac{2}{3}$ that might be expected, decreasing from about 0.6 to 0.5 as $\sqrt{s}$ increases from 3 to 5 GeV.

The first three points above are characteristic of a statistical model. The application to $e\bar{e}$ annihilation has been examined by several persons, including Bjorken and Brodsky[2] and Engels, Schilling, and Satz.[3] Our contributions here are to take into account in a fairly transparent way the masses of the final-state particles, including the mass of the pion, which ought not be neglected to get good results, and to treat the pions and kaons (and such other particles as may be considered) on a footing that does not require separate parameters to describe them. No computer calculations of phase-space integrals are needed.

Our approach follows the work of Bjorken and Brodsky. The probability that a particular channel will be present is assumed to be proportional to the phase space available, suppressed further by the statistical factor,[6]

$$\exp\left(-a\sum_{i=1}^{N} E_i\right), \tag{1.2}$$

where $E_i=(m_i^2+p_i^2)^{1/2}$ is the energy of the ith outgoing particle in the overall center-of-mass system and a is one of the parameters of the theory which will later be seen to be the same as in Eq.

(1.1). If we introduce an unknown proportionality factor a_N, assumed to be independent of energy, the probability that N particles are produced is then proportional to

$$\rho_N(s) = a_N(2\pi)^4 \prod_{i=1}^{N} \int \frac{d^3p_i}{(2\pi)^3 2E_i} e^{-aE_i}\delta^4\left(P - \sum_i p_i\right)$$
$$= a_N \int d^4x\, e^{-i\sqrt{s}\,t} \prod_{i=1}^{N} X_i\,, \qquad (1.3)$$

where $P^2 = s$,

$$X_i(\vec{x}, t) = \int \frac{d^3p_i}{(2\pi)^3 2E_i} e^{-i\vec{p}_i\cdot\vec{x}} e^{-E_i(a-it)}\,. \qquad (1.4)$$

Note that the four-dimensional δ function has been replaced by an integral over space-time. Since the integrals (1.4) cannot be evaluated exactly, Bjorken and Brodsky consider the case where m_i can be neglected, and replace E_i by $p_i = |\vec{p}_i|$. They then evaluate the integrals and obtain an explicit form for the energy dependence of the N-particle cross section

$$\sigma_N = \frac{4\pi\alpha^2}{3s}\rho_N(s)\,. \qquad (1.5)$$

(The details of all of these calculations are presented in Sec. II.) Finally, the coefficients a_N and hence the relative size of the N-particle cross sections are determined by the requirement that the total cross section

$$\sigma_{\text{total}} = \sum_N \sigma_N \qquad (1.6)$$

have the "correct" behavior at infinite energies. They obtain Eq. (1.1) and, for the case when $\sigma_{\text{total}} \to s^{-1}$ at large s, a particle multiplicity

$$\langle N\rangle = \tfrac{1}{2}a\sqrt{s} + 2\,. \qquad (1.7)$$

However, if a is evaluated from the one-particle inclusive data, the predicted multiplicity is too high. At $\sqrt{s} = 5$ GeV, for example, $\langle N_{\text{ch}}\rangle = \frac{2}{3}\langle N\rangle = 11$ is predicted, whereas only 4.3 charge particles are observed on the average.

It is important to consider the mass of even the pion. The mass dependence comes into (1.4) through the quantity E_i, which occurs in both the oscillating exponent and in the denominator. The replacement of E_i by p_i in the exponent is reasonable for small times where the integral tends to be cut off at large values of E, but since the integration over t must give us an energy δ function, the oscillating behavior at large t is crucial. By the stationary-phase approximation, the major contribution at large t comes from the region where

$$\frac{dE_i}{dp_i} = 0\,,$$

which is at threshold where $E_i = m_i$. Hence, in the exponent we replaced E_i by $p_i + \lambda_i$, where λ_i is a constant. The replacement of E_i by p_i in the denominator overemphasizes the small-p_i contribution and it should be sufficient to replace this E_i by some constant ϵ_i, which sets the overall scale of X. These substitutions give

$$X_i(\vec{x}, t) \simeq \int \frac{d^3p_i}{(2\pi)^3 2\epsilon_i} e^{-i\vec{p}_i\cdot\vec{x}} e^{-(p_i+\lambda_i)(a-it)}\,, \qquad (1.8)$$

which can be evaluated. The two parameters λ_i and ϵ_i are determined in Sec. II by comparing (1.8) with the exact X_i for $\vec{x} = 0$. For λ_i one obtains m_i as expected. If there is only one species of particle, the choice of ϵ_i is unimportant because an overall constant can be absorbed into a_N. If there are several species of particles, we shall choose ϵ_i to get the correct ratio of X_i's, which gives $\epsilon_i^{-1} \propto m_i K_1(am_i)e^{a\lambda_i}$, where K_1 is a Bessel function and the constant of proportionality is absorbed into a_N.

If one makes the approximation of neglecting the masses, all the X_i are identical and there is no natural distinction between pion and kaon contributions to the cross section. We can explain these differences through the mass dependences of the X_i. Specializing Eq. (1.3) to the case of n pions and m kaons, where $N = n + m$, we obtain a generalization of the Bjorken-Brodsky results,

$$\rho_{nm}(s) = a_N \frac{N!}{n!\,m!} \int d^4x\, e^{-i\sqrt{s}\,t}(X_\pi)^n (X_K)^m\,, \qquad (1.9)$$

where we have included a statistical factor

$$\frac{N!}{n!\,m!}\,, \qquad (1.10)$$

which counts the number of different channels with n pions and m kaons.

The integral (1.9) can be evaluated using the techniques reviewed in Sec. II. Also, the requisite sums can be done by converting them to integrals and using saddle-point integration methods. The constants a_N are determined by the requirement that the total cross section fall like $1/s$, or that R be constant, for very high energies. Since the λ's and ϵ's are fixed in terms of the particle masses as described above, the theory is left with only two parameters, a and R (∞).

Using the summation technique referred to above, a number of physically interesting quantities are finally determined. We obtain

$$\frac{E\,d\sigma}{d^3p} = \frac{4\pi\alpha^2}{3s} R(\infty) F(s, E) e^{-aE}\,, \qquad (1.11)$$

where

$$F(s,E)=0.608a\sqrt{s}+8.70-2.31aE$$
$$=24.3-12.1E\ , \qquad (1.12)$$

the last being valid for $s=23\ \text{GeV}^2$ and E measured in GeV. Since $R(s)$ is fluctuating at this value of s, we shall use $R\ (\infty)=3\frac{1}{3}$, the value given by the standard colored-quark model with charm. The data and our curve are shown in Fig. 1. The value $a^{-1}=0.19$ GeV was chosen (corresponding to a temperature of 2×10^{12} °K).

With the value of a fixed, the total multiplicity is given by

$$\langle N\rangle=0.244a\sqrt{s}+0.91\ . \qquad (1.13)$$

The reader should note that the effects of including the mass have reduced the slope by a bit over a factor of 2. The charged multiplicity, taken to be $\frac{2}{3}$ of this, is plotted with the data[1,7] in Fig. 2. If the intercept were taken fixed, this slope would be the best possible. However, a larger intercept and a lower slope would fit the data better.

Our cross sections for the four- and six-charged pion exclusive reactions are shown in Figs. 3 and 4. Our curves are gotten from Eq. (2.10). Since this equation is for all types of pions emerging, we have multiplied by the fraction of those events in which all of the emerging pions will be charged. This is $\frac{6}{19}$ and $\frac{20}{141}$ for the four- and six-charged-pion exclusive channels, respectively.

Finally, the fraction of negatively charged particles which are kaons is shown in Fig. 5 as a function of particle momentum for $\sqrt{s}=4.8$ GeV. Equa-

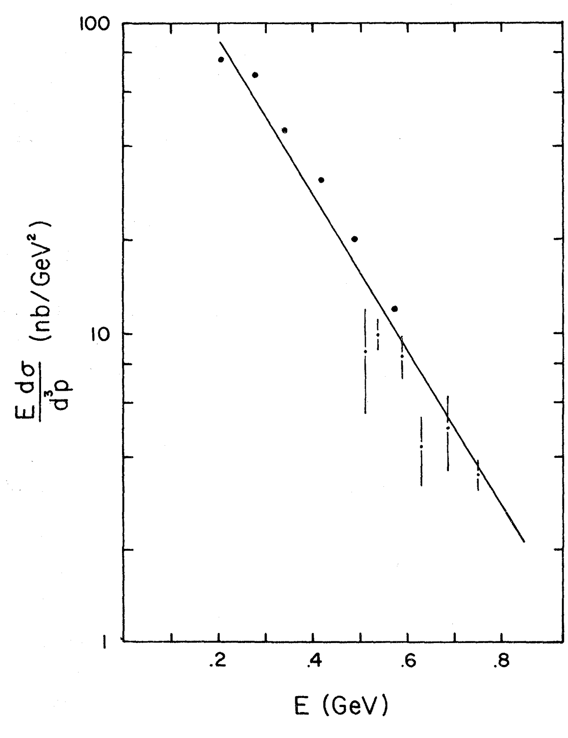

FIG. 1. One-particle inclusive cross section at $\sqrt{s}=4.8$ GeV as a function of particle energy. The dots without error bars are pion data; those with error bars are kaon data. The solid line is the fit given by Eq. (1.12) with $R(\infty)=3\frac{1}{3}$ as given by the colored-quark model with charm. Data are taken from R. Larsen as reported at the Williamsburg conference (Ref. 1).

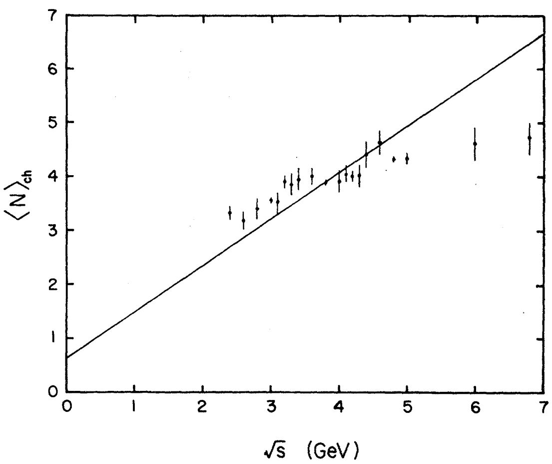

FIG. 2. Charged-particle multiplicities $\langle N\rangle_{ch}$. The solid line is $\frac{2}{3}\langle N\rangle$, where $\langle N\rangle$ is given in Eq. (1.13). Data are taken from J.-E. Augustin (Ref. 1), with the last two points from C. Morehouse, talk at the 1975 Washington APS meeting.

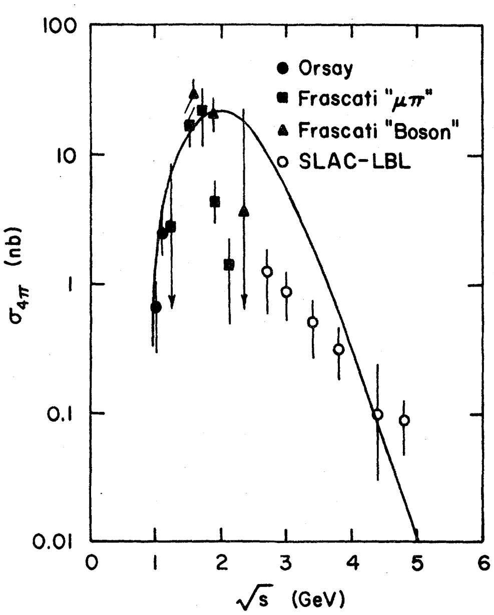

FIG. 3. Four-charged-pion exclusive cross section. Solid line is our fit with $R(\infty)=3\frac{1}{3}$. Data from B. Richter (Ref. 1).

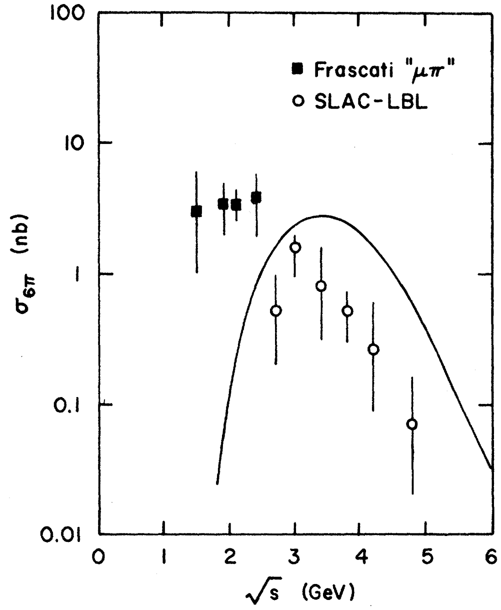

FIG. 4. Six-charged-pion exclusive cross section. Solid line is our fit with $R(\infty)=3\frac{1}{3}$. Data are from B. Richter (Ref. 1).

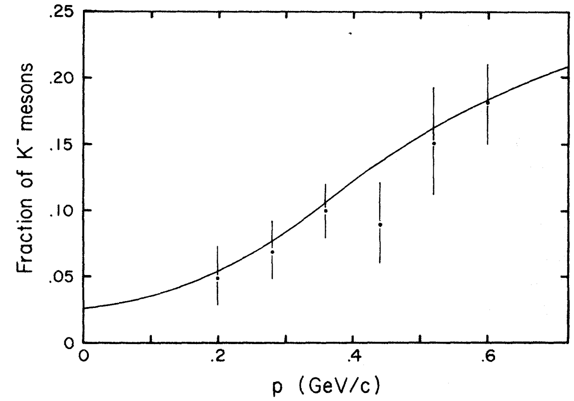

FIG. 5. Fraction of kaons as a function of momenta for $\sqrt{s}=4.8$ GeV. Solid line is our fit. Data are taken from R. Larsen as reported at the Williamsburg Conference (Ref. 1).

tion (1.11) was used, remembering that $\frac{1}{3}$ of the pions and $\frac{1}{4}$ of the kaons are negative.

We offer a few concluding remarks. The statistical model can be viewed as a complement to field-theoretical or parton models. The latter are able to predict total cross sections provided a perturbation expansion is valid, and can do so in some detail, for example predicting quantitatively the increase in hadron cross sections at the threshold for charmed quark production.[8] However, features of the physical observed hadrons, which come from strong-interaction recombinations of quarks and productions of additional quark pairs, cannot be studied perturbatively. It is interesting that the statistical model can describe them well. On the other hand, while the statistical model does study features of the physical hadrons, it cannot give the overall normalization, i.e., the total cross section.

A characteristic of the statistical model is that the multiplicity $\langle N\rangle$ rises linearly with $\sqrt{s}$. This is a significantly more quickly increasing function than the ln s predicted by some models, and it has been a concern that the statistical model may predict too many particles. We have seen that taking account of the pion mass, as well as the masses of heavier particles, reduces sizably the slope of $\langle N\rangle$ vs $\sqrt{s}$ [compare Eq. (1.13) with (1.7)].

We turn finally to the question of the energy deficit in the charged-hadron channels. This is a serious problem, and the possibility that some new process is involved should be kept in mind. On the other hand, conventional physics may be adequate to explain the effect.[9,10] For example, Grammer and Smith[9] have pointed out that hard-photon bremsstrahlung and two-photon processes both increase the energy that appears to go into neutral particles. Neither affects the hadronic physics. For the case of bremmstrahlung from the initial $e\bar{e}$, they calculate that on the average a hard photon will take off 7.5% of the total energy for $\sqrt{s}=5$ GeV and total cross section falling like $1/s$; other processes will add to this. A new process that could affect the energy carried by charged particles is the production of a heavy-lepton pair.[11] It is expected that significantly less than $\frac{2}{3}$ of the decay products of a heavy lepton would be charged. This explanation, of course, can only hold above the threshold for producing heavy leptons, so it cannot be a complete explanation of the "energy crisis," but it can explain the observed sudden worsening of the crisis at $\sqrt{s}=3.6$ GeV. In this paper we have provisionally taken the view that the observed charged hadrons are indeed close to $\frac{2}{3}$ of the total number of hadrons produced.

II. CALCULATIONS AT ASYMPTOTIC ENERGIES

A summary of our calculation and a discussion of the results have already been given in Sec. I. In this section we will fill in the missing steps for the case when the energies are asymptotically large.

The scattering amplitude which describes $e\bar{e}$ annihilation into n pions and m kaons can be written

$$\mathfrak{M}^{n,m}=\frac{e^2}{s}L^{\mu}H_{\mu}^{n,m}, \tag{2.1}$$

where L^{μ} is the lepton current and $H_{\mu}^{n,m}$ is the phenomenological matrix element describing the production of the hadrons by a virtual photon. The total cross section for the production of n pions and m kaons is therefore

$$\sigma_{n,m}=\frac{1}{2s}L^{\mu\nu}\frac{e^4}{s^2}J^{n,m}_{\mu\nu}\,, \tag{2.2}$$

where $P^2=s$ is the square of the total energy and

$$L^{\mu\nu}=\sum_{\text{avg}} L^\mu L^\nu$$
$$=k'^\mu k^\nu+k^\mu k'^\nu-g^{\mu\nu}k\cdot k'\,, \tag{2.3}$$

and k and k' are the electron and positron momenta. For the hadrons we introduce the phenomenological form

$$J^{n,m}_{\mu\nu}\equiv\int\prod_{i=1}^{n}\frac{d^3p_i}{(2\pi)^3 2E^i_\pi}\prod_{j=1}^{m}\frac{d^3p_j}{(2\pi)^3 2E^j_K}(2\pi)^4\delta^4\Big(P-\sum p_i-\sum p_j\Big)H^{n,m}_\mu H^{n,m*}_\nu$$
$$=\frac{1}{6\pi}(P_\mu P_\nu-g_{\mu\nu}P^2)\,\rho_{n,m}(s)\,, \tag{2.4}$$

where our statistical model ansatz for $\rho_{n,m}$ was given in Eq. (1.9). Combining (2.2)–(2.4) gives

$$\sigma_{n,m}=\frac{4\pi\alpha^2}{3s}\rho_{n,m}(s)\,. \tag{2.5}$$

Determination of ϵ and λ

To evaluate ρ as given in Eq. (1.9), it is necessary to evaluate X_π and X_K, where X_i was introduced in Eq. (1.4). Since the integrals cannot be evaluated analytically, we introduce the approximation (1.8) as discussed in Sec. I. To evaluate the parameters λ_i, and ϵ_i, we note that if $\vec{x}=0$, X_i can be computed exactly,

$$X_i(\vec{x}=0,t)=\int\frac{d^3p_i}{(2\pi)^3 2E_i}e^{-E_i(a-it)}$$
$$=\frac{m_i}{(2\pi)^2(a-it)}K_1(m_i(a-it)) \tag{2.6}$$
$$\underset{t\approx\infty}{\approx}\left[\frac{m_i}{4(2\pi)^3(a-it)^3}\right]^{1/2}e^{-m_i(a-it)}\,,$$

where $E_i=(m_i^2+p_i^2)^{1/2}$ and K_1 is the modified Bessel function of the third kind. However, our approximation for X_i gives

$$X_i(\vec{x}=0,t)=\int\frac{d^3p_i}{(2\pi)^3 2\epsilon_i}e^{-(p_i+\lambda_i)(a-it)}$$
$$=\frac{2}{(2\pi)^2(a-it)^3\epsilon_i}e^{-\lambda_i(a-it)}\,. \tag{2.7}$$

There are two features that determine ϵ and λ. The crucial part of the t dependence is the oscillation at $t\to\infty$. Since the cooperative effects of these oscillations from the different X_i give rise to the energy-conserving δ function, the choice $\lambda_i=m_i$ will ensure that this conservation is maintained.

Next we must get the correct relative normalization of the X_i's. The X_i's are largest at $t=0$, and we shall choose ϵ_i to give the correct ratio of the X_i's there, or

$$\frac{\epsilon_\pi}{\epsilon_K}=\frac{m_K K_1(am_K)e^{a\lambda_K}}{m_\pi K_1(am_\pi)e^{a\lambda_\pi}}\,. \tag{2.8}$$

Note that our approximate X_i falls off faster than the exact X_i as $t\to\infty$. This means that our approximation will give an unreliable estimate of the average overall size of X_i, but this overall normalization can be incorporated into the constant a_N [introduced in Eq. (1.9)]. Perhaps it is worth noting that since our approximation does reproduce the relative size of the different X_i's, it should properly describe the *relative* probability that pions or kaons will be produced. It is for this reason that we can assume that our overall normalization constant a_N depends only on the total number of particles and not on n and m separately. Further, the choice of ϵ_π/ϵ_K above will be seen to lead to a self-consistent relation when the one-particle inclusive reaction rates are studied.

Integration and summation

With these choices we now can evaluate ρ_{nm} following the procedure of Brodsky and Bjorken. First, we have

$$X_i(\vec{x},t)=\frac{e^{-\lambda_i(a-it)}}{2\pi^2\epsilon_i}\frac{a-it}{[(a-it)^2+x^2]^2}\,, \tag{2.9}$$

which gives

$$\rho_{n,m}(s) = a_N \frac{N!}{n!m!} \int \frac{d^4x\, e^{-i\sqrt{s}\,t}}{(2\pi^2)^N \epsilon_\pi^n \epsilon_K^m} \frac{e^{-(n\lambda_\pi + m\lambda_K)(a-it)}(a-it)^N}{[(a-it)^2+x^2]^{2N}}$$
$$= a_N \frac{N!}{n!m!}(4\pi)^2 \frac{(4N-4)!}{(2N-1)!(2N-2)!} \frac{e^{-a(n\lambda_\pi+m\lambda_K)}}{(32\pi^2)^N\epsilon_\pi^n\epsilon_K^m} \int_{-\infty}^{\infty} dt \frac{e^{-it(\sqrt{s}-n\lambda_\pi-m\lambda_K)}}{(a-it)^{3N-3}}$$
$$= a_N \frac{N!}{n!m!} \frac{(4N-4)!}{(2N-1)!(2N-2)!} \frac{32\pi^3 e^{-a\sqrt{s}}}{(32\pi^2)^N \epsilon_\pi^n \epsilon_K^m} \frac{(\sqrt{s}-n\lambda_\pi-m\lambda_K)^{3N-4}}{(3N-4)!} \tag{2.10}$$
$$= C_N \frac{N!}{n!m!}\left(\frac{B}{\epsilon_\pi}\right)^n\left(\frac{B}{\epsilon_K}\right)^m \frac{[a(\sqrt{s}-n\lambda_\pi-m\lambda_K)]^{3N-4}}{(3N-4)!} e^{-a\sqrt{s}},$$

where we have substituted

$$a_N \frac{(4N-4)!}{(2N-1)!(2N-2)!} \frac{\pi}{(32\pi^2)^{N-1}} = C_N B^N a^{3N-4}. \tag{2.11}$$

Note that the form of the t integration ensures that $\rho_{nm}=0$ if $\sqrt{s} < n\lambda_\pi + m\lambda_K$ providing an approximate description of the threshold dependence.

Finally, we must sum the ρ_{nm} terms over n and m to determine $R(s)$. As we discussed in Sec. I, we will assume that $R(s) \to$ const as s becomes very large. This means that C_N and B must be chosen so that when $\sqrt{s}$ is very large

$$R(\infty) = \lim_{s\to\infty} \sum_n \sum_m \rho_{nm}(s)$$
$$= \lim_{s\to\infty} e^{-a\sqrt{s}} \sum_n \sum_m C_N \left(\frac{B}{\epsilon_\pi}\right)^n \left(\frac{B}{\epsilon_K}\right)^m \frac{N!}{n!m!} \frac{[a(\sqrt{s}-n\lambda_\pi-m\lambda_K)]}{(3N-4)!}. \tag{2.12}$$

It should be noted that the range of the sum over n and m depends on $\sqrt{s}$ because only when $\sqrt{s}-n\lambda_\pi - m\lambda_K > 0$ are the terms to be included in the sum. Because of strangeness conservation, only even values of m are to be included, but both sums may be taken to start from zero since the factor $(3N-4)!$ in the denominator will automatically ensure that $N \geq 2$.

In preparation for treatment of (2.12), we consider a simpler sum obtained by omitting C_N,

$$S = \mathfrak{N} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left(\frac{B}{\epsilon_\pi}\right)^n \left(\frac{B}{\epsilon_K}\right)^m \frac{N!}{n!m!} \times \frac{[a(\sqrt{s}-n\lambda_\pi-m\lambda_K)]^{3N-4}}{(3N-4)!}, \tag{2.13}$$

where the sum is over all n and even m. At large s an examination of the summand shows that it peaks sharply in n and m. We assume that it is possible to choose $\mathfrak{N}$ and B so that as $\sqrt{s}$ becomes very large,

$$S \to e^{a\sqrt{s}}. \tag{2.14}$$

If this is indeed the case, then $\mathfrak{N}$ and B are implicit functions of a, and we can estimate the average value of $N \equiv \langle N \rangle$ from the identity

$$a\frac{dS}{da} = \left[(3\langle N\rangle - 4) + a\frac{\langle N\rangle}{B}\frac{dB}{da} + \frac{a}{\mathfrak{N}}\frac{d\mathfrak{N}}{da}\right]S. \tag{2.15}$$

Substituting (2.14) gives

$$\langle N \rangle = \xi_1 a\sqrt{s} + \xi_2, \tag{2.16}$$

where ξ_1 and ξ_2 are independent of s. Therefore, as $\sqrt{s}\to\infty$, $\langle N\rangle$ also approaches infinity, and the major contribution to the sum comes from terms in which $n+m$ are large. Furthermore, the statistical factor $N!/m!n!$ ensures that the major contribution comes from terms in which both n and m are individually large. Thus we may approximate the sum by an integral

$$S \simeq \frac{\mathfrak{N}}{2} \int dn\, dm\, e^{g(n,m)}. \tag{2.17}$$

We shall now sketch how to evaluate S. We use the procedure of approximating the exponential by a Gaussian. The work is straightforward, although perhaps we may entice the reader by remarking that some of the cancellations are pretty.

The function $g(n,m)$ may be written out, clearly involving a number of logarithims and factorials; the latter are treated with Stirling's approximation. The function g peaks sharply in the vicinity of some particular values of n and m, which we denote by n_0 and m_0. Expanding g in a Taylor series about this point, keeping terms to second order, and integrating the resulting Gaussian gives

$$S = \frac{\mathfrak{N}}{2} e^{g(n_0,m_0)} \frac{2\pi}{\sqrt{D}}, \tag{2.18}$$

where

$$D = \det g'' = \frac{\partial^2 g}{\partial n_0^2}\frac{\partial^2 g}{\partial m_0^2} - \left(\frac{\partial^2 g}{\partial n_0 \partial m_0}\right)^2 . \tag{2.19}$$

The values of n_0, m_0, and B are determined by requiring that both first derivatives of g vanish and that S is $\exp(a\sqrt{s})$ for large s; i.e., we have three equations, each of which contains $\sqrt{s}$ as a variable.

Drawing from the result (2.16) we assume that, at large $\sqrt{s}$,

$$\begin{aligned} n_0 &= \alpha_\pi Q + \beta_\pi , \\ m_0 &= \alpha_K Q + \beta_K , \end{aligned} \tag{2.20}$$

with

$$Q = a\sqrt{s} . \tag{2.21}$$

Remembering that we are interested in the solution for large Q, we substitute into our three equations. The K/π ratio at infinite Q comes out analytically,

$$\tau_\infty \equiv \frac{\alpha_K}{\alpha_\pi} = \frac{m_K K_1(am_K)}{m_\pi K_1(am_\pi)} = 0.230 \tag{2.22}$$

(all numerical results are given for $a^{-1} = 0.19$ GeV), as does B,

$$\frac{B}{\epsilon_\pi} = \frac{e^{am\pi}}{1+\tau_\infty} . \tag{2.23}$$

The α_i and β_i must be obtained numerically, but this is easy, and

$$\begin{aligned} &\alpha_\pi = 0.199 , \quad \beta_\pi = 0.984 , \\ &\alpha_K = 0.0456 , \quad \beta_K = -0.072 . \end{aligned} \tag{2.24}$$

We then calculate $g(n_0, m_0)$, where we need only the leading and next-to-leading Q dependence, obtaining

$$g(n_0, m_0) = Q - \tfrac{1}{2}\ln[3\alpha_\pi \alpha_K (2\pi Q)^2] , \tag{2.25}$$

where terms of order Q^{-1} have been discarded. Computing D to leading order gives

$$D = \frac{1}{Q^2}\frac{1}{3\alpha_\pi \alpha_K (\alpha_\pi + \alpha_K)^2} . \tag{2.26}$$

Thus (2.18) reduces to

$$S = \mathfrak{N}\frac{\alpha_\pi + \alpha_K}{2} e^Q \tag{2.27}$$

giving

$$\mathfrak{N} = \frac{2}{\alpha_\pi + \alpha_K} = 8.16 . \tag{2.28}$$

We now return to our original discussion of the sum for $R(s)$. By comparing with (2.12), we see that the choice

$$C_N = R(\infty)\mathfrak{N} \tag{2.29}$$

will ensure that as $s\to\infty$, R is constant.

If we wished to require that R behave like a polynomial in s at large s, then C_N would be required to have the corresponding power dependence on N, so that even though we have assumed that R approaches a constant for large energy in this paper, our method can easily be generalized to treat other cases. That is, an N dependence in C_N can be used to give any required $\sqrt{s}$ dependence in R. To sharpen this observation we note that the pth moment of n or m can be easily calculated. Defining

$$\langle n^p \rangle = \frac{\sum_{n,m} n^p \rho_{n,m}(s)}{\sum_{n,m} \rho_{n,m}(s)} \tag{2.30}$$

and evaluating both sums by the stationary-phase method, we obtain

$$\langle n^p \rangle = n_0^p \left[1 + p(p-1)\frac{\partial^2 g/\partial n_0^2}{2n_0^2 D}\right] , \tag{2.31}$$

where we have retained terms of order Q^{-1} relative to the leading term.

Also, for $p = 1$ we obtain the useful result that the average value is equal to the value of n at the peak, n_0. Hence the pion and kaon multiplicities are given by (2.24) and the total multiplicity is

$$\langle N \rangle = 0.244\, a\sqrt{s} + 0.91 . \tag{2.32}$$

The total K to π ratio is τ, which we compute as

$$\tau = \frac{m_0}{n_0} = 0.230\left(1 - \frac{6.51}{a\sqrt{s}}\right) . \tag{2.33}$$

One-particle inclusive cross sections

Next let us consider the one-particle inclusive cross section at large energy. We will consider the pion first. If only one pion is observed, the contribution of the channel with n pions and m kaons present will be as before with one integral left undone and a counting factor n included:

$$\begin{aligned} \frac{E_\pi d\sigma_{n,m}}{d^3 p_\pi} &= \frac{4\pi\alpha^2}{3s} n \frac{N!\, a_N}{m!\, n!} e^{-aE_\pi \frac{1}{2}} \int d^4x\, e^{-iP\cdot x + ip_\pi \cdot x} X_\pi^{n-1}(x,t) X_K^m(x,t) \\ &= \frac{4\pi\alpha^2}{3s}\frac{N!\, a_N}{m!\,(n-1)!}\frac{2}{(32\pi^2\epsilon_\pi)^{n-1}(32\pi^2\epsilon_K)^m}\frac{e^{-a\sqrt{s}}}{(2N-3)!\,(2N-4)!} \\ &\quad \times \left(\frac{d}{d\eta}\right)^{N-1} [(\sqrt{s} + \eta - (n-1)\lambda_\pi - m\lambda_K - E_\pi)^2 - p_\pi^2]^{2N-4} \Big|_{\eta=0} . \end{aligned} \tag{2.34}$$

For a fixed N, we can choose p_π^2 small enough to be neglected, and then the derivatives can be evaluated. This will give a correct answer to leading order in s. However, when the sum is done over N the corrections from the p_π^2 terms to the leading order will be down by a factor of only $a\sqrt{s}$, not $a^2 s$. This is because the term is raised to the power $2N-4$, and the peak value of N itself is increasing like $\sqrt{s}$. The best way to calculate the corrections (which turn out to be important here because of the numerical factors multiplying them) is by a self-consistency argument. We will first get the general form and leading order of the inclusive cross section.

Neglecting p_π^2, evaluating the derivatives, and substituting the previous value for a_N gives

$$\frac{E_\pi\, d\sigma_{n,m}}{d^3 p_\pi} = \frac{4\pi\alpha^2}{3s}\,\frac{BC_N}{\pi}\,\frac{N(2N-1)(2N-2)}{(4N-5)(4N-7)}\,a^3 e^{-a\sqrt{s}}\,\frac{(N-1)!}{(n-1)!\,m!}\,\frac{B^{N-1}}{\epsilon_\pi^{n-1}\epsilon_K^m}\,\frac{[a(\sqrt{s}-(n-1)\lambda_\pi-m\lambda_K-E_\pi)]^{3N-7}}{(3N-7)!}. \tag{2.35}$$

In doing the sum over n and m, we notice that the first group of terms depending on N is slowly varying, and the remainder of the terms are the same as before if we express the summand in terms of $n'=n-1$, and replace $\sqrt{s}$ by $\sqrt{s}-E_\pi$. Using (2.31) to calculate the slowly varying terms in the sum we obtain

$$\begin{aligned}\frac{E_\pi d\sigma}{d^3 p_\pi} &= \frac{4\pi\alpha^2}{3s}\,\frac{B}{8\pi}\,a^3(\langle N\rangle+2.5)e^{-a\sqrt{s}}C_N\sum_{n',m}\frac{N'!}{n'!\,m!}\left(\frac{B}{\epsilon_\pi}\right)^{n'}\left(\frac{B}{\epsilon_K}\right)^{m}\frac{[a(\sqrt{s}-n'\lambda_\pi-m\lambda_K-E_\pi)]^{3N'-4}}{(3N'-4)!}\\ &= \frac{B}{8\pi}a^3(\langle N\rangle+2.5)e^{-aE_\pi}\,\sigma_{\text{total}}\\ &= F(s,E_\pi)e^{-aE_\pi}\,\sigma_{\text{total}}.\end{aligned} \tag{2.36}$$

The value of $\langle N\rangle$ should be evaluated at the energy $\sqrt{s}-E_\pi$.

The dependence of $F(s,E_\pi)$ on $\sqrt{s}$ is of the form $\sqrt{s}$ + constant, where, as discussed before, not all of the constant term has been included. We modify $F(s,E_\pi)$ by adding a constant (which can depend on E_π):

$$F(s,E_\pi) = \frac{Ba^3}{8\pi}[(\alpha_\pi+\alpha_K)a\sqrt{s}-(\alpha_\pi+\alpha_K+c_1)aE_\pi+2.5+\beta_\pi+\beta_K+c_2]\,, \tag{2.37}$$

where c_1 and c_2 will be determined self-consistently. The same formulas are valid for kaons if we let $E_\pi\to E_K$.

To determine the values of c_1 and c_2, and also the absolute normalization of B, we recall that

$$\begin{aligned} n_0\sigma_{\text{total}} &= \int\frac{d^3p_\pi}{E_\pi}F(s,E_\pi)e^{-aE_\pi}\,\sigma_{\text{total}}\,,\\ m_0\sigma_{\text{total}} &= \int\frac{d^3p_K}{E_K}F(s,E_K)e^{-aE_K}\,\sigma_{\text{total}}\,.\end{aligned} \tag{2.38}$$

Taking the leading terms gives

$$\begin{aligned}\alpha_\pi &= \tfrac{1}{2}B(\alpha_\pi+\alpha_K)a^2 m_\pi K_1(am_\pi)\,,\\ \alpha_K &= \tfrac{1}{2}B(\alpha_\pi+\alpha_K)a^2 m_K K_1(am_K)\,.\end{aligned} \tag{2.39}$$

These equations are identical because of our previous result for α_K/α_π, and solving for B gives

$$B=0.429\ \text{GeV}\,. \tag{2.40}$$

In the next order, we obtain

$$\begin{aligned}\beta_\pi = \tfrac{1}{2}Ba^2m_\pi[&-(\alpha_\pi+\alpha_K+c_1)am_\pi K_2(am_\pi)\\ &+(2.5+\beta_1+\beta_2+c_2)K_1(am_\pi)]\end{aligned} \tag{2.41}$$

and a similar equation for β_K. These equations can be solved for c_1 and c_2 to get

$$\begin{aligned} c_1&=0.682\,,\\ c_2&=0.089\,.\end{aligned} \tag{2.42}$$

It might be noted that c_1 is rather large.

This completes our calculations for very high energies. The results have already been collected and compared to data in Sec. I. Some comments regarding the validity of our sums at smaller values of s, where we do not have a large number of kaons, are made in the Appendix.

APPENDIX: CALCULATIONS AT FINITE ENERGIES

In this section we consider calculations at finite energies. The main conclusion is that our approximations are quite good at lower energies, $\sqrt{s}\lesssim 5$ GeV, where there are in fact only a few kaons. The accuracy of our results seems to depend only on having a large number of pions.

The total cross section is related to a sum [see Eq. (2.12)]

$$S=\sum_m S_m\,, \tag{A1}$$

where

$$S_m = \frac{\mathfrak{N}}{m!}\left(\frac{B}{\epsilon_K}\right)^m \sum_n \frac{N!}{n!}\left(\frac{B}{\epsilon_\pi}\right)^n \frac{[a(\sqrt{s} - n\lambda_\pi - m\lambda_K)]^{3N-4}}{(3N-4)!}. \quad \text{(A2)}$$

There are already large numbers of pions for $\sqrt{s} \gtrsim 3$ GeV so that S_m can be summed by familiar means. If we suppose that m is a fixed number not growing with s, then to leading order we obtain the result

$$S_m \simeq \frac{\mathfrak{N}}{m!}\xi_\pi^{\,3}\alpha'_\pi e^{(Q-ma\lambda_K)\xi_\pi}\left(\frac{Q\alpha'_\pi}{\xi_\pi^{\,3}}\frac{B}{\epsilon_K}\right)^m, \quad \text{(A3)}$$

where the sum is dominated by n's near n'_0, where

$$n'_0 = \alpha'_\pi Q + \beta'_\pi \quad \text{(A4)}$$

and

$$\xi_\pi = \frac{3\alpha'_\pi}{1-\alpha'_\pi a\lambda_\pi} = 0.944. \quad \text{(A5)}$$

If this result be sufficiently accurate, we should be able to sum it and recover our old results for S [Eq. (2.27)]. Unfortunately we do not. The problem lies in our result for S_m. It is accurate for fixed m and $\sqrt{s} \to \infty$ (or for m small and $\sqrt{s}$ moderate size), but if we sum over all m, the value of m at which S_m is largest grows with $\sqrt{s}$, and the first correction terms to S_m become necessary. Inserting these corrections gives

$$S_m = \frac{\mathfrak{N}}{m!}\xi_\pi^{\,3}\alpha'_\pi e^{(Q-ma\lambda_K)\xi_\pi}(Q + m\delta_1 + \delta_2)^m \times \left(\frac{\alpha'_\pi}{\xi_\pi^{\,3}}\frac{B}{\epsilon_K}\right)^m. \quad \text{(A6)}$$

The parameters δ_1 and δ_2 can be gotten from an accurate evaluation of S_m, but can be more easily obtained by requiring self-consistency; when the above equation is summed at large $\sqrt{s}$ using the saddle-point integration methods, to give S, the result should be the identical to that given previously in Eq. (2.27). Defining $\xi_K = 1 - \xi_\pi$ gives the identities

$$\frac{\alpha'_\pi}{\xi_\pi^{\,3}}\frac{B}{\epsilon_K} = \xi_K e^{-\delta_1\xi_K + a\lambda_K\xi_\pi}, \qquad \mathfrak{N}\xi_\pi^{\,3}\alpha'_\pi = 2e^{-\delta_2\xi_K}(1-\delta_1\xi_K), \quad \text{(A7)}$$

together with the values

$$\delta_1 = -2.94 = -(a\lambda_K + 0.31), \qquad \delta_2 = 5.00. \quad \text{(A8)}$$

This gives a more transparent result for S_m:

$$e^{-Q}S_m = \frac{2(1-\delta_1\xi_K)}{m!}e^{-(Q+m\delta_1+\delta_2)\xi_K} \times (Q + m\delta_1 + \delta_2)^m \xi_K^m. \quad \text{(A9)}$$

Thus we obtain a formula for $R(s)$ valid at finite energies,

$$R(s) = 2R(\infty)(1-\delta_1\xi_K)e^{-(a\sqrt{s}+\delta_2)\xi_K}\left[1 + \tfrac{1}{2}\eta^2(a\sqrt{s} + 2\delta_1 + \delta_2)^2 + \tfrac{1}{24}\eta^4(a\sqrt{s} + 4\delta_1 + \delta_2)^4 + \cdots\right], \quad \text{(A10)}$$

where

$$\eta = e^{-\delta_1\xi_K}\xi_K = 0.066. \quad \text{(A11)}$$

Evaluation of this formula shows that $R(s) \to R(\infty)$ from above and at $\sqrt{s} = 3$ GeV it is about 10% larger than $R(\infty)$.

The multiplicites at finite energies are similarly insensitive to our improved evaluation. Our results for the total multiplicity agree with the asymptotic results to within 5% in the range 1 GeV $\leq \sqrt{s} \leq 5$ GeV.

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