

Freedom at moderate energies: Masses in color dynamics*

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We discuss the effects of quark and target masses in inclusive lepton-hadron scattering, assuming a color SU(3) gauge theory of strong interactions. Our tools are the renormalization group, suitably extended to include quark masses, and the operator-product expansion. We argue that the best renormalization procedure is to specify all parameters in the Lagrangian (coupling constants and masses) as well as wave-function normalizations at Euclidean momenta of scale M . We use the renormalization group to develop an understanding of the relation between current-algebra and constituent-quark masses. In the operator-product expansion, we use the equations of motion to eliminate operators with \not{D} , D^2 , etc. acting on a quark field. We order the expansion as a power series in the gauge coupling constant as it occurs in the coefficient functions. We predict approximate scaling in a new variable $\xi \neq x$, which depends on Q^2 and the quark and target masses. We also discuss the distribution of antiquarks and heavy quarks in the nucleon.

I. INTRODUCTION

Theoretical understanding of the "scaling" phenomena observed in inclusive lepton-hadron scattering has developed along two parallel but different paths. One approach, called the parton model¹ or more specifically the quark-parton model, sees the deep-inelastic processes as probing the structure of hadrons made up of constituents (partons) which behave as pointlike particles. The other approach, through the operator-product expansion² and the renormalization group,³ connects physical scaling with the scaling behavior of the underlying quantum field theory.⁴ The development of asymptotically free field theories of the strong interactions^{5,6} brings these two paths into contiguity. From the side of field theory, asymptotic freedom is the key to understanding the absence of anomalous dimensions. The short-distance behavior of the theory is almost free, so the coefficients in an operator-product expansion scale canonically up to logarithmic corrections.⁷ From the parton-model point of view, asymptotic freedom of the constituents is attractive because it justifies the impulse approximation. The possibility that the same theory which leads to asymptotic freedom might confine quarks in infrared slavery^{6,8,9} is a bonus. The quarks could be bound permanently into hadrons and still exhibit pointlike behavior when probed at high-momentum transfer.

Still, much about scaling remains mysterious. Why, for instance, do the data "scale" in the Bloom-Gilman variable x' even at Q^2 which are not large compared to m_p ?² How does the existence of the charmed quark affect our scaling predictions? What about even heavier quarks which could exist? Can we calculate the re-

scaling behavior of electroproduction or neutrino scattering above a heavy-quark threshold?

In this paper, we address these questions in the context of asymptotically free color SU(3) gauge models. The answers are both phenomenologically interesting and theoretically instructive. "Scaling" is a misnomer, at least as narrowly defined to have something to do with dimensional analysis or infinite momenta, ignoring masses. It is a much more subtle and general phenomenon than the term suggests. We will find that when the effective coupling constant is small, the contribution of each piece in the current "scales" approximately in an appropriately modified variable, ξ .¹⁰ If our view is correct and such a theory does describe the strong interactions, then the parton modelers have been right all along, and "scaling" reflects directly the pointlike nature or approximate free-field behavior of the quark constituents of hadrons.

In Sec. II, we discuss our general approach, particularly our treatment of the renormalization group. We argue that the "best" technique is to treat all masses as effective coupling constants renormalized at a Euclidean momentum, and we calculate the scale dependence of the gauge coupling and the masses to lowest order. Integrating numerically we obtain an effective coupling constant and effective quark masses. We show how to obtain the quark mass parameters appropriate to current-algebra calculations from the quark masses of a constituent model of hadrons, with results consistent with the standard wisdom in both domains.

In Sec. III, we discuss the operator-product expansion (OPE). We argue that the standard procedure, in which operators of twist greater than two are ignored, is inadequate to analyze

the effects of heavy quarks. As an alternative we propose to eliminate, using the equations of motion, all operators with \not{D} , D^2 , etc. acting on a quark field. We can then analyze the resulting OPE coefficients in a power series in the effective coupling constant.

In Sec. IV, we discuss the lowest-order contribution to the expansion of the OPE coefficients, that is, the free-field OPE. This is the approximation that leads to ξ scaling. Higher orders in the effective coupling constant predict logarithmic violations of ξ scaling, which have been already computed.⁷ We analyze in detail electroproduction involving only light quarks but for Q^2 not large compared to the proton mass. The result can be described as scaling in a variable intermediate between x and x' . This has been previously studied by Nachtmann,¹¹ and we recover his results. We compare the ξ -scaling predictions with recent SLAC data.

In Sec. V, we discuss processes in which a light quark is struck and a heavy quark is produced. For $Q^2 \gg m_p^2$, the result is the same as for only light quarks except that Q^2 is replaced everywhere by $Q^2 + m_H^2$, where m_H^2 is the heavy-quark mass. The structure functions scale in the variable $\xi = (Q^2 + m_H^2)/2p \cdot q$, in agreement with the parton-model result. It is ξ and not x which is the fraction of the proton momentum carried by the struck quark. The importance of these considerations to the interpretation of neutrino-scattering experiments is noted.

In Sec. VI, we analyze processes in which heavy quarks are struck. In this case the elimination of operators with \not{D} or D^2 acting on the heavy-quark field is nontrivial and leads to additional quark-mass dependence of the coefficient functions.

In Sec. VII, we discuss the distribution functions of heavy quarks in the proton. We write down the matrix anomalous dimension of heavy-quark operators. The renormalization-group equations can be integrated to give the moments of the heavy-quark distribution function in terms of their values (and the light quark and gluon moments) at a single Q^2 . This program is carried out for the spin-2 operators using an experimental determination of light-quark and gluon matrix elements and a guess for the heavy-quark matrix elements at low Q^2 . The results are in agreement with a recent study of Witten,¹² who utilizes an expansion in the inverse of the heavy mass. We also discuss the shapes of gluon and heavy-quark distributions as functions of ξ .

In Sec. VIII, an explanation of ξ in parton-model language is offered. The necessary assumptions are shown to correspond to keeping only zeroth order in the effective coupling.

II. QUARK MASSES AND THE RENORMALIZATION GROUP

While our most interesting new results are consequences of free-field theory, where the subtleties of renormalization theory never enter, we have used the techniques of the renormalization group and the operator-product expansion to determine where free-field theory may be a good approximation and how to isolate and consolidate those features of lepton-hadron scattering which involve strong coupling in an essential way. Asymptotic freedom is central in this analysis. It allows self-consistent, perturbative calculations with a small expansion parameter of certain relevant functions. In this program, all quantities must be carefully defined so that approximations can be tested, at least for self-consistency or within perturbation theory. So our first task is to provide definitions of quark masses and coupling constants that are useful when masses are non-negligible. In studying one set of parameters particularly appropriate to lepton-hadron scattering, we will get a bonus of new insights into the relation of approximate hadronic symmetries and constituent-quark masses.

It may or may not make sense to speak of quark states, but there certainly are quark Green's functions. Quarks require mass and wave-function renormalization in a colored-quark-gluon gauge theory. We choose to define these by specifying that the quark propagator, $S(\not{p})$, for a given spacelike $p^2 = -M^2$ agrees with free-field theory:

$$S^{-1}(\not{p})_{p^2 = -M^2} = \not{p} - m. \quad (2.1)$$

This defines a quark mass m which depends implicitly on an arbitrary normalization mass, M . In a theory with several species or flavors of quark, there may be one mass per flavor (degenerate under color transformations.) The renormalized gauge coupling constant g is defined as the value of some three-point function, e.g. the three-gluon vertex, at some specified spacelike momenta, all of scale M . The effects of quark masses are included in this vertex function, so g will have an implicit dependence on both m and M .

Renormalizability implies that changes in M with appropriate changes in g and m leave the vertex functions of the theory, Γ^n , unchanged, that is to say, that the total derivative of the Γ^n with respect to M vanishes: $M(d/dM)\Gamma^n = 0$. In terms of the explicit dependences, this is the renormalization-group equation. In a Landau gauge,

$$M \frac{d}{dM} \Gamma^n = \left[M \frac{\partial}{\partial M} + \beta_g \left(g, \frac{m}{M} \right) \frac{\partial}{\partial g} + \gamma_m \left(g, \frac{m}{M} \right) m \frac{\partial}{\partial m} + \gamma^n \left(g, \frac{m}{M} \right) \right] \Gamma^n = 0, \quad (2.2a)$$

where

$$\begin{aligned}\beta_g\left(g, \frac{m}{M}\right) &= M \frac{dg}{dM}, \\ \gamma_m\left(g, \frac{m}{M}\right) &= \frac{M}{m} \frac{dm}{dM},\end{aligned}\quad (2.2b)$$

and γ^n is the sum of the anomalous dimensions

$$\begin{aligned}\beta_g &= -\frac{g^3}{16\pi^2} \left\{ 11 - \frac{2}{3} \sum_{\text{quarks}} \left[1 - 6 \frac{m_i^2}{M^2} + \frac{12m_i^4/M^4}{(1+4m_i^2/M^2)^{1/2}} \ln \frac{(1+4m_i^2/M^2)^{1/2} + 1}{(1+4m_i^2/M^2)^{1/2} - 1} \right] \right\} \\ &\approx -\frac{g^3}{16\pi^2} \left[11 - \frac{2}{3} \sum_{\text{quarks}} \left(\frac{1}{1+5m_i^2/M^2} \right) \right],\end{aligned}\quad (2.3a)$$

$$\begin{aligned}\gamma_{m_i} &= -8 \frac{g^2}{16\pi^2} \left[1 - \frac{m_i^2}{M^2} \ln \left(1 + \frac{M^2}{m_i^2} \right) \right] \\ &\approx -\frac{g^2}{2\pi^2} \left(\frac{1}{1+2m_i^2/M^2} \right),\end{aligned}\quad (2.3b)$$

where the approximate forms are useful interpolating formulas which have the correct limits for m^2/M^2 small and large and are good to a few percent for intermediate m^2/M^2 .

For M such that g is small, we can use Eq. (2.3) to integrate numerically Eq. (2.2b) to exhibit the M dependence of g and m . Each differential equation requires a boundary condition—one for g and one for each quark mass. These are the free parameters of the theory and must therefore be specified by experiment.

We can refine the notion that in the charmonium¹³ interpretation of the J , ψ' , etc., the charmed quark mass $m_{\phi'}$ is 1.5–2 GeV; this will provide a definition of what is a heavy quark and give a boundary condition for heavy-quark mass functions. In computing the smeared e^+e^- cross section,¹⁴ let M be the center-of-mass energy. For M such that g is small, the smeared cross section will approximately scale until there is a threshold for a new quantum number. The location of this “threshold” is twice the new-quark mass, as measured at M , which is also the energy of the threshold. So $2m_{\phi'}(M \cong 3 \text{ GeV}) \cong 3 \text{ GeV}$, or more generally when $2m(M) = M$ there is a threshold in the smeared cross section for the quantum number of that quark, if the quark is heavy. Our definition of “heavy quark” is a quark for which $g(M)$ is small at $M = 2m_{\text{heavy}}(M)$; a “light quark” is one for which $g(M)$ is large at $M = 2m_{\text{light}}(M)$.

An analogous condition determines m_λ , the strange-quark mass, from the mass of the ϕ , but there is a larger experimental uncertainty

(or logarithmic M derivatives of the wave-function renormalization constants) of the n fields appearing in Γ^n . Here m stands for all quark masses, with the appropriate sum understood in Eq. (2.2a). The functions β_g and γ_m can be inferred order by order in g by inserting quark and gluon two- and three-point functions into Eq. (2.2a). To lowest order for an SU(3) gauge theory

here, for although we interpret precocious electroproduction scaling as evidence for the validity of perturbative ideas at $M \sim 1 \text{ GeV}$, it is difficult to determine the strangeness threshold in the smeared e^+e^- cross section for two reasons: (1) The electric charge squared of the λ quark is $\frac{1}{4}$ that of the ϕ' or ϕ , and (2) somewhere not far below 1 GeV the coupling constant becomes large. However, we maintain that $2m_\lambda(1 \text{ GeV}) \sim 1 \text{ GeV}$ is a reasonable estimate.

The threshold for light quarks is a strong-coupling problem, so we seek a different input. For $M \gg m_i$ (for some flavors i), the $m_i(M) \rightarrow 0$ for increasing M , as long as g stays small. However, as is evident from Eqs. (2.2b) and (2.3b), their ratios go to finite numbers. Furthermore, it is easy to convince oneself that these ratios go to finite numbers. Furthermore, it is easy to convince oneself that these ratios are equal to the ratios of bare masses, i.e.,

$$\lim_{M \gg m_{\lambda, \phi}} \frac{m_\lambda(M)}{m_\phi(M)} = \frac{m_{\lambda_0}}{m_{\phi_0}}. \quad (2.4)$$

It is the ratios of bare quark masses which determine the approximate symmetries of the hadronic Hamiltonian and enter into the partial conservation of axial-vector current (PCAC), current-algebra analysis. For instance, it is argued that the ratios of bare quark masses are equal to the ratios of masses squared of the appropriate pseudoscalar mesons.¹⁵ We assume roughly degenerate ϕ and λ quark masses and conclude that $m_{\phi, \lambda} \approx \frac{1}{20} m_\lambda$ for large M .

Finally, we need a boundary condition on g which determines the size of the logarithmic violations of scaling in lepton-hadron scattering^{7,16,17} and e^+e^- annihilation^{18,14} and the violations of the Callan-Gross relation.¹⁷ A definitive analysis is not yet available, but most guesses suggest that for $1 < M < 4$ GeV, $0.1 \lesssim g^2/4\pi \lesssim 1.0$.

In Fig. 1 we show the resulting functions for the following input parameters: At $M=3$ GeV, $m_{\phi'} = 1.5$ GeV, $m_\lambda = 0.4$ GeV, $m_\phi = m_\pi = m_\lambda/20$, and $g^2/4\pi = 0.5$. The results are similar for any other reasonable boundary conditions. The most striking feature of these curves is the intersection of $m_{\phi,\pi}$ with $2m(M) = M$. The intersection of a heavy-quark mass with this diagonal line crudely gives the mass of the appropriate lowest-lying vector meson. Using PCAC and m_π and m_K , we imposed the condition that $m_\lambda(M) \approx 20m_{\phi,\pi}(M)$ for large M . The theory then implies that $m_\lambda(M=2m_\lambda) \approx 1.7m_{\phi,\pi}(M=2m_{\phi,\pi})$. It is not justifiable to claim a calculation of $m_\phi \approx 1.7m_\rho$ because the coupling constant is becoming large near m_ρ , but that is clearly the qualitative message: Current-algebra mass ratios and constituent-mass ratios are related to each other through the dynamics, e.g. Eq. (2.2b), and the latter are necessarily much closer than the former. Clearly the constituent masses [the intersection with $2m(M) = M$] of the light quarks are very g -dependent. For the range of plausible g , we find typically that $1.3 \lesssim m_\lambda/m_{\phi,\pi} \lesssim 2.0$ (in any case much less than 20).

We note that charm PCAC is a very poor approximation. Given $m_{\phi'} \sim 1.5$ GeV, we can go to large M and read off the relevant ratio. The PCAC formulas would then imply that the charmed pseudoscalar would have a mass of about 1 GeV, while from experiment and any other kind of theoretical estimate one arrives at least 1.5 or closer to 2 GeV. We attribute the failure to the fact that $m_{\phi'}$ is not small compared to typical hadron masses.

A final observation from Fig. 1 is that the light quarks are really very light when measured on any scale above 1 GeV. So it is improbable that the light-quark masses will be distinguishable from zero in inclusive lepton-hadron scattering, except perhaps in quantities which would vanish in their absence.

Anticipating our detailed analysis, we wish to point out that our definition of quark mass has a simple physical interpretation in parton language. First remember that from a field-theoretic standpoint the parton model must be generalized to allow a weak momentum-transfer (Q^2) dependence. The observed parton distribution functions will depend on what wavelength

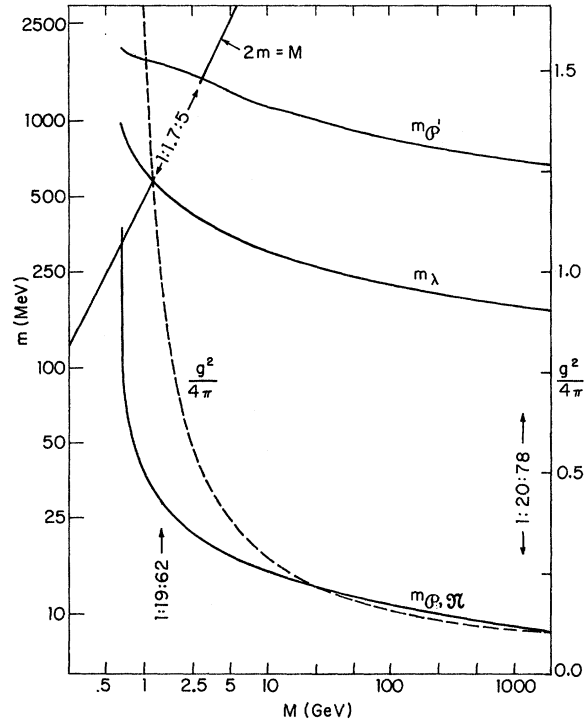


FIG. 1. The M dependence of quark masses and the gauge coupling constant.

photon is used to measure them. The parton mass, too, will depend on the Q^2 used to probe it. And for $M^2 = Q^2$, our quark mass $m(M)$ is the appropriate parton mass. The reason $m(M)$ is a property of the quark propagator for spacelike momentum (near Q^2) rather than near the mass shell is that in the field-theoretic analysis, the quark masses enter only in the coefficient functions of the operator-product expansion, which are functions of the single spacelike kinematic variable Q^2 .

The outstanding virtue of our definitions of g and m from a field theory point of view is that the Appelquist-Carazzone theorem¹⁹ is manifest: For some $m(M) \gg M$, to leading order in M^2/m^2 the effects of the heavy fields are negligible in Green's functions with external momenta of scale M , and the theory can be described by a renormalizable Lagrangian containing light fields only. For $m(M) \ll M$, the light fields can be approximated by $m=0$. This is in contrast to the several other renormalization prescriptions that have been suggested in the literature for which the coupling constant does not reflect the full mass dependence. With other prescriptions potentially large logarithms of mass factors may enter the perturbation expansion in g and thus confuse the issue of what is reliably calculable and what is

not. Defining g independent of all m 's, e.g. by the massless theory, is ideal for the study of $M \gg m$ (see Ref. 20) and our prescription differs only by $O(m^2/M^2)$; however, Green's functions with external momenta $p_i = O(M) \ll m$ will contain factors of $\ln(m^2/M^2)$ if the massless theory is used to prescribe normalization conditions. If m dependence is included in the definition of g but m is defined on its own scale, independent of M , e.g. as the location of the pole in perturbation theory, then Green's functions with external $p_i = O(M) \gg m$ will contain factors of $\ln(p_i \cdot p_j/m^2)$. With our definitions, functions of spacelike momenta of scale M contain no large logarithms for any m .

III. THE OPERATOR-PRODUCT EXPANSION

In previous analyses, target and constituent masses were ignored in deriving the Q^2 dependence of hadron structure functions for large Q^2 .⁷ Violations from scaling arose from computing the asymptotic effects of a nonzero $g(Q)$ and may be thought of as interaction corrections to the impulse approximation. These predictions are valid as asymptotic statements independent of the size of $g(m_p)$ because $g(Q)$ goes to zero as Q goes to infinity. However, experiment suggests that $Q^2 \sim 1 \text{ GeV}^2$ is in some sense already asymptotic despite the fact that m_p^2/Q^2 is not negligible. We can now include all relevant mass effects in the same sort of analysis by expanding in both $g(m_p)$ and $g(Q)$ but keeping all orders of the various m^2/Q^2 's. We regard precocious scaling as qualitative evidence for the validity of $g(m_p)$ as an expansion parameter. Confirmation of the utility of this approach can come only from the success of our detailed predictions. Preliminary results as well as numerical determinations of g by other methods are very encouraging.

By relevant mass effects we mean those that enter into the Q^2 dependence. We still require a structure function at a single Q^2 as experimental input and predict only the Q^2 evolution. This input function certainly depends implicitly on the several mass parameters, but it enters our analysis as a function of a single explicit kinematic variable, e.g. $F(\xi)$. We will show how the shape of $F(\xi)$ also affects Q^2 dependence. But given $F(\xi)$ we can compute all Q^2 and mass dependence. In previous organizations of the problem, if one wanted to take account of m^2/Q^2 corrections, a new experimentally determined func-

tion would have to be introduced for each higher power of m^2/Q^2 considered. This renders the method useless in any situation where some m^2/Q^2 is not small, e.g. precocious scaling or neutrino production of heavy quantum numbers. Even for all $m^2/Q^2 \ll 1$, an unambiguous prescription for extracting these functions from experiment has never been proposed. With our organization, new input functions must be introduced for each successive power of $g(m_p)$ that is included. This may not sound so different in abstraction, but it allows us to study additional phenomena and give a straightforward prescription for carrying the analysis to higher order.

It remains an outstanding challenge to derive the $F(\xi)$ from the fundamental masses and coupling constant of the theory. We first sketch the derivation, suppressing all indices and variables that are inessential to the basic argument.

The structure functions are related to the absorptive part of the forward target matrix element of the time-ordered product of the two relevant hadronic currents. Studying the time-ordered product allows us to circumvent questions of the nature of specific final states; taking the absorptive part is left to last.⁴ The current product is expanded in a complete set of local operators whose coefficients depend on the currents' four-momenta, $\pm q$:

$$\begin{aligned} \langle p | iT(J(x)J(0)) | p \rangle e^{iq \cdot x} d^4x \\ = \sum_{n,i} c_{n,i}(q, M) \langle p | O_i^n | p \rangle. \end{aligned} \quad (3.1)$$

The index n is the spin of the operator O_i^n , i runs over all other labels (about which we will say more), and p is the target momentum. All the operators O_i^n are renormalized at a Euclidean momentum of scale M . In the present analysis, the $\langle p | O_i^n | p \rangle$ must be inferred from experiment, but they are q -independent; all of the q dependence resides in the $c_{n,i}$. By applying $M \partial/\partial M$ to Eq. (3.1), one can derive a renormalization-group equation for the $c_{n,i}$:

$$\left[\left(M \frac{\partial}{\partial M} + \beta_g \frac{\partial}{\partial g} + \gamma_m m \frac{\partial}{\partial m} \right) \delta_{ij} + \gamma_{ij}^n \right] c_{n,j} = 0, \quad (3.2)$$

where γ_{ij}^n is minus the transpose of the matrix anomalous dimension of the set of operators of spin n , allowing for the phenomenon of mixing.

The q^2 dependence of $c_{n,i}$ can be obtained from its g and m dependence by integrating Eq. (3.2):

$$c_{n,i}(q, g(M), m(M), M) = \sum_j c_{n,j}(q, g(Q), m(Q), Q) \left[\exp T \int_M^Q \gamma^n \left(g(M'), \frac{m(M')}{M'} \right) \frac{dM'}{M'} \right]_{ij}. \quad (3.3)$$

We use the convention $Q \equiv (-q^2)^{1/2}$. The T in the exponential signifies that the matrices $\underline{\gamma}^n$ are ordered (in M') in the integral. The necessity of ordering the integral arises because even to lowest order in g the eigenvectors of $\underline{\gamma}^n$ are M' dependent.

We will not be able to express our final answers in closed, analytic form. However, from the standpoint of deviations from naive scaling, a reexpansion of Eq. (3.3) allows us to state the answer precisely and compare the relative importance of various contributions. While a reexpansion may make the whole renormalization-group analysis appear extraneous and even silly, it is invaluable as a guide to what should be computed in perturbation theory.

Imagine measuring the structure functions for some Q_0^2 such that $g(Q_0)$ is small. We can express the predictions for some Q^2 in a power series in $g_0 \equiv g(Q_0)$ as long as Q^2 is comparable to Q_0^2 , i.e., $g_0^2 \ln(Q/Q_0)$ is also small. In this situation,

$$g(Q) = g_0 + \beta_g \left(g_0, \frac{m_0}{Q_0} \right) \ln \frac{Q}{Q_0} + O(g_0^5), \quad (3.4a)$$

$$m(Q) = m_0 + m_0 \gamma_m \left(g_0, \frac{m_0}{Q_0} \right) \ln \frac{Q}{Q_0} + O(g_0^4), \quad (3.4b)$$

$$c_{n,i} = \sum_j c_{n,i,j}(q, g(Q), m(Q), Q) \times \left[\delta_{ij} + \gamma_{ij}^n \left(g_0, \frac{m_0}{Q_0} \right) \ln \frac{Q}{Q_0} \right]. \quad (3.4c)$$

The factor $c_{n,i}$ in Eq. (3.4c) is the coefficient function computed in perturbation theory, using Q as the renormalization scale, $g(Q)$ as the coupling, and $m(Q)$ as quark masses; these last two can then be expanded as in Eqs. (3.4a) and (3.4b). If in perturbation theory (suppressing $n, i,$ and j)

$$c(q) = A(q, m) + g^2 B(q, m, M) + O(g^4) \quad (3.5a)$$

and

$$\underline{\gamma} = g^2 \underline{D} \left(\frac{m}{M} \right) + O(g^4), \quad (3.5b)$$

then Eq. (3.4c) becomes

$$c(q) = A(q, m_0) \underline{1} + g_0^2 \ln \frac{Q}{Q_0} A(q_0, m_0) \underline{D} \left(\frac{m_0}{Q_0} \right) + \gamma_m \left(g_0, \frac{m_0}{Q_0} \right) \ln \left(\frac{Q}{Q_0} \right) m_0 \frac{dA}{dm_0} \underline{1} + g_0^2 B(q, m_0, Q) \underline{1} + O(g_0^4). \quad (3.6)$$

The c 's are specific, known Lorentz tensors constructed out of the four-momentum q times dimensionless functions of all the variables. To

make contact with previous analyses note that for $m_0 \approx 0$ the first term in Eq. (3.6) is a constant times the appropriate tensor; the second term contains the logarithmic violations of scaling which in this form are small because g_0^2 is small (since $\ln Q/Q_0$ has a strong Q^2 dependence for $Q^2 \approx Q_0^2$); the third term is zero; and the fourth term is g_0^2 times the standard tensor in q times a constant because a dimensionless function of $q^2/Q^2 (= -1)$ is independent of Q^2 . In the presence of some non-negligible m_0 , all of these terms may contribute to observable deviations from naive scaling. If g_0 is very small, the first term contains the largest deviations; if g_0^2 is small but still larger than m_0/Q_0 , then the second term is most important; if m_0/Q_0 is of order unity, then the last three terms may be comparable but all down by a factor of g_0^2 with respect to the first term.

To derive any detailed, practical information from Eq. (3.1), the operators O_i^n must be organized somehow according to their relative importance. If mass effects are not to be ignored, then the standard approach using only dimensional analysis is inadequate. In that approach, the leading operators were those that gave terms depending only on q^2 and $p \cdot q$ in the absence of all masses (up to logarithmic corrections), and successive sets of operators were those that necessarily brought in successive powers of m^2/Q^2 , for some mass m . For this organization to be consistent, even if all masses are negligible compared to q^2 and $p \cdot q$, the relative sizes of the $\langle p | O_i^n | p \rangle$ must be consistent with dimensional analysis, i.e., some characteristic number times the appropriate power of the target mass m_p , and not orders of magnitude too large or small. We will first sharpen these estimates to include the possibility of large quark masses. With these improved estimates we show how the parameter $g(Q)$ can be used to organize the operators for any Q such that $g(Q)$ is small.

If all operators are normalized on the scale of the target mass, $M \sim m_p$, and ψ stands for a particular quark field, then the above-mentioned dimensional analysis suggests for example that $\langle p | \psi \not{D} \psi | p \rangle$, where D_μ is the gauge covariant derivative, and $m_p \langle p | \bar{\psi} \psi | p \rangle$ are of comparable magnitudes. However, these renormalized operators are related by some variant of the field equations of motion. Although these equations may be fairly complex,²¹ there nevertheless exist true relations between $\bar{\psi} \not{D} \psi$ and $m \bar{\psi} \psi$, where m is the renormalized quark mass. If m is not of order m_p , then the dimensional analysis estimate must be wrong. This is particularly troublesome if $m \gg m_p$ because the effects of operators

with more and more factors of \not{D} , D^2 , etc., will be proportional to powers of m/Q , which may not be small.

We propose the following alternative. We may simply drop all operators containing powers of \not{D} , D^2 , etc., acting on a quark field from the operator-product expansion. All these operators are related by the equations of motion to operators without \not{D} 's, and so the set of operators is no less complete (in the sense of spanning operator products) without them. Dropping them induces additional m dependence in the coefficient functions of the remaining operators, which are calculable.

For matrix elements of the remaining operators we believe that the dimensional analysis estimates are reliable as order-of-magnitude upper bounds. We say upper bound because there are certain obvious possible suppressing factors; for instance, a heavy-quark bilinear operator will be much smaller in a light target than the analogous light-quark operator. But the matrix elements of operators with more fields are bounded in order of magnitude by the appropriate additional power of m_p . (This is certainly true in low orders of perturbation theory.)

While we will not investigate $Q^2 \ll m_p^2$, we may be interested in $Q^2 \sim m_p^2$, and hence these multi-field operators are not negligible by dimensional analysis. However, now consider their total contribution. For $g=0$, the expansion of the product of two quark currents contains only the appropriate quark bilinears; all more complicated operators have zero coefficients. In perturbation theory, operators with more and more fields will contribute only in higher and higher order. In leading order, only quark bilinears appear, and in subsequent orders there is a systematic addition of more and more complex operators.

The key question is: According to which coupling constant are we ordering the operator-product expansion? In keeping with our philosophy of the dimensional-analysis estimates of matrix elements using m_p as a renormalization point, $g(m_p)$ is the relevant expansion parameter. If $g(m_p)$ is sufficiently small that an expansion in $g(m_p)^2/4\pi$ makes sense, then the same analysis which explains approximate scaling in inclusive electroproduction at $Q^2 > 20 \text{ GeV}^2$ also explains precocious scaling in the same phenomenon at $Q^2 \approx 1 \text{ GeV}^2$.

To see that $g(m_p)$ is the relevant parameter consider the following. When shuffling operators to put them all in a form in which the equations of motion apply [e.g., $\bar{\psi} \cdots (i\not{D} - m)\psi = 0$], one makes use of the identity $[D_\mu, D_\nu] = -igF_{\mu\nu}$. If all operators are renormalized at m_p , then the g that appears will be $g(m_p)$. So by applying our

prescription to the OPE one generates, in addition to a single quark bilinear operator for each spin, operators with two quark fields and any number of gluon field strengths, each with a factor of $g(m_p)$.

Note that the contribution of these multifield operators to the forward Compton amplitude decreases with increasing Q^2 not because $g(Q)$ is a decreasing function [for $g(m_p)$ is fixed] but by the old dimensional-analysis argument, i.e., their contributions vanish like powers of m_p^2/Q^2 for $Q^2 \gg m_p^2$.

Since we cannot yet compute the proton's structure from first principles, we allow it to be arbitrary and then set it at one Q^2 from experiment. In the OPE language, we allow the proton matrix elements of local operators to be what they are. However, to make any progress at all, we assume from the start that no matrix elements are anomalously large. This assumption is made in all applications of the OPE. This applies to the operators exclusive of any explicit small factors. Hence we assume that

$$F^{\mu_1\alpha} D^{\mu_3} \cdots D^{\mu_n} F_{\alpha}^{\mu_2}$$

and

$$(1/m_p) \bar{\psi} F^{\mu_1\alpha} \sigma_{\alpha}^{\mu_2} D^{\mu_3} \cdots D^{\mu_n} \psi$$

have proton matrix elements that are not much larger (though perhaps much smaller than those of $\bar{\psi} \gamma^{\mu_1} D^{\mu_2} \cdots D^{\mu_n} \psi$ when renormalized at m_p). So to zeroth order in $g(m_p)$, the product of two currents involves no operators containing gluon field strengths.

We finally note how to compute the relevant target-mass dependence. The operators of the operator-product expansion are arranged to be of definite spin, i.e., traceless and symmetric. There is no loss of generality because the trace terms are just $g_{\mu\nu}$'s, which go into coefficient functions, times operators of lower spin. In previous analyses the issue of traces never arose because traces and trace subtractions will induce powers of m_p^2/Q^2 , which is precisely why we now must consider them. In fact, it is the trace subtractions that contain all of the relevant target-mass dependence (in the sense of directly entering the Q^2 dependence as discussed earlier). The trace subtractions are completely determined by the spin of each operator. For instance, if we work only to zeroth order in $g(m_p)$, there is only one quark bilinear for each spin. (All others were dropped using the equations of motion.) Measuring the structure functions at a given Q^2 is equivalent to measuring the target matrix elements of each operator, which is just a number times a unique tensor in p , the target four-mo-

mentum, and $m_p^2 g_{\mu\nu}$. Knowing the q dependence of the coefficient functions, we can then compute the structure functions for larger Q^2 , including the m_p dependence. (This is essentially the Nachtmann¹¹ analysis.)

IV. PRECOCIOUS SCALING

In this section, we outline the derivation of ξ scaling and discuss in detail the target mass dependence of electroproduction off light quarks. The starting point is an operator-product expansion of the forward Compton amplitude

$$\begin{aligned} -g_{\mu\nu}W_1 + \frac{\hat{p}_\mu \hat{p}_\nu}{m_p^2} W_2 \\ = \frac{1}{\pi} \text{Im} \int e^{i\alpha x} d^4x \langle p | iT(J_\mu(x)J_\nu(0)) | p \rangle \\ = \frac{1}{\pi} \text{Im} \sum_{n,i} c_{n,i}(q) \langle p | O_n^i | p \rangle. \end{aligned} \quad (4.1)$$

We can choose the local operators O^n to have definite twist (dimension minus spin). The coefficients $c_{n,i}$ are calculable in a power series in $g(Q)$. If g is small, it makes sense to consider only the first term in this expansion as first approximation for the $c_{n,i}$. This is the approximation which leads to ξ scaling.¹⁰ The lowest-order terms are just what we would obtain in a free-field OPE.

We can organize the free-field OPE so that the only relevant operators which appear are the twist-two operators, bilinear in the quark fields, traceless and symmetric in tensor indices. The target matrix element of each such operator, O_n^j , is completely specified by one unknown parameter, A_n^j :

$$\langle p | O_n^j | p \rangle = A_n^j \Pi^{\mu_1 \dots \mu_n}, \quad (4.2)$$

where $\Pi^{\mu_1 \dots \mu_n} [\equiv p^{\mu_1} \dots p^{\mu_n} - \text{terms involving } g^{\mu_i \mu_j}]$ is the unique traceless, symmetric rank- n tensor which can be formed with the target momentum p . The index j runs over the quarks which appear in the current J_μ . The A_n^j are the moments of a function which turns out to be the j -quark distribution function of the parton language. The structure functions in the ξ -scaling approximation involve calculable functions of Q^2 , quark mass, and target mass, and the quark distribution functions of the single variable ξ , where

$$\xi = \frac{Q'^2}{2m_p\nu} \frac{2}{1 + (1 + Q^2/\nu^2)^{1/2}}, \quad (4.3)$$

with

$$\begin{aligned} 2Q'^2 = Q^2 + m_F^2 - m_I^2 \\ + [Q^4 + 2Q^2(m_F^2 + m_I^2) + (m_F^2 - m_I^2)^2]^{1/2}. \end{aligned}$$

The struck-quark mass is m_I , the produced-quark mass is m_F , and the target mass is m_p .

The explicit calculation proceeds as follows: (1) Organize the free-field OPE as described above. If there are heavy-quark fields in the current, this step involves using the equations of motion to eliminate all operators with twist not equal to two, thereby putting all nontrivial quark-mass dependence in the $c_n(q)$. (2) Collect the terms in the OPE proportional to a given power of $(p \cdot q)$ and relate the coefficient to the appropriate moment of a structure function, in the manner of Christ, Hasslacher, and Mueller.⁴ Because of Eq. (4.2), each moment gets a contribution from an infinite number of operators, so that the moments have a complicated target-mass dependence. (3) Use the inverse Mellin transform to invert the moments and exhibit structure functions.

Nachtmann¹¹ has given a general analysis which applies to steps (2) and (3), and furthermore one can use his line of approach when incorporating quark masses. He observed that one must translate the usual power-series expansions into an expansion of operators of definite spin, i.e., traceless, and then invert the resulting moment statements. This is analogous to relating a plane-wave expansion to a spherical-wave expansion, and so he solves the problem elegantly using the theory of representations of the Lorentz group. For simple-minded souls like ourselves, we offer an alternate analysis which is particular to the situation at hand. We isolate and exhibit each origin of mass dependence; these are collected and then summed. The erudite reader may recognize at several steps specific examples of general group-theoretic results or may simply elect to skip over the derivation.

For pedagogical purposes we will discuss in detail two limiting cases which simplify the calculation in different ways. In this section we will deal only with light quarks, $m_I \cong m_F \cong 0$. In this limit, step (1) is trivial. The higher-twist operators can be eliminated without any effect on the coefficients of the twist-two operators. In subsequent sections, we will deal with quarks of arbitrary mass, but restrict ourselves to $Q^2 \gg m_p^2$. In this limit step (2) is trivial because only the first term in $\Pi^{\mu_1 \dots \mu_n}$ is important. Combining steps (1) and (2) to obtain the general result presents no theoretical difficulties.

We now discuss electroproduction off light quarks. If the current is $J^\mu = \bar{\psi} \gamma^\mu \psi$, where ψ is a free-quark field, the operator-product expansion is

$$\int e^{i\alpha x} \langle p | iT(J^\mu(x)J^\nu(0)) | p \rangle d^4x = \sum_{k=1}^{\infty} (-g^{\mu\nu}q_{\mu_1}q_{\mu_2} + g_{\mu_1}^{\mu}q^{\nu}q_{\mu_2} + q^{\mu}q_{\mu_1}g_{\mu_2}^{\nu} + g_{\mu_1}^{\mu}g_{\mu_2}^{\nu}Q^2)q_{\mu_3} \cdots q_{\mu_{2k}} \frac{2^{2k}}{Q^{4k}} \langle p | P^{\mu_1 \cdots \mu_{2k}} | p \rangle + \cdots \quad (4.4)$$

The P operators are the symmetric bilinears

$$P^{\mu_1 \cdots \mu_n} = i^{n-1} \bar{\psi} \gamma^{\mu_1} \partial^{\mu_2} \cdots \partial^{\mu_n} \psi \text{ (symmetrized)}. \quad (4.5)$$

The terms not written explicitly in Eq. (4.4) involve operators whose spin-averaged matrix elements vanish, for example antisymmetric tensors.

Because the quarks are light, we can replace the P operators by *traceless* symmetric operators O ,

$$O^{\mu_1 \cdots \mu_n} = P^{\mu_1 \cdots \mu_n} - \text{traces}. \quad (4.6)$$

All the traces involve \not{x} or ∂^2 acting on a light-quark field, so $O \simeq P$. Then Eq. (4.4) becomes

$$\int e^{i\alpha x} \langle p | iT(J^\mu(x)J^\nu(0)) | p \rangle d^4x = \sum_{k=1}^{\infty} (-g^{\mu\nu}q_{\mu_1}q_{\mu_2} + g_{\mu_1}^{\mu}q^{\nu}q_{\mu_2} + q^{\mu}q_{\mu_1}g_{\mu_2}^{\nu} + g_{\mu_1}^{\mu}g_{\mu_2}^{\nu}Q^2)q_{\mu_3} \cdots q_{\mu_{2k}} \frac{2^{2k}}{Q^{4k}} A_{2k} \Pi^{\mu_1 \cdots \mu_{2k}}. \quad (4.7)$$

To complete step (2) of our procedure, we need to know Π explicitly. Straightforward combinatorics yields

$$\Pi^{\mu_1 \cdots \mu_{2k}} = \sum_{j=0}^k (-1)^j \frac{(2k-j)!}{2^j (2k)!} g \cdots g p \cdots p. \quad (4.8)$$

The j th term in Eq. (4.8) is a coefficient times the symmetric sum of the $(2k)!/[2^j!(2k-2j)!]$ terms with j $g^{\mu_i \mu_j}$'s and $(2k-2j)$ p^{μ_i} 's.

We can now extract the moments of the structure functions. Look first at W_2 . The coefficient of $p^\mu p^\nu$ in Eq. (4.7) is

$$Q^2 \sum_{k=1}^{\infty} \frac{2^{2k}}{Q^{4k}} \sum_{j=0}^{k-1} (-1)^j \frac{(2k-j)!}{2^j (2k)!} \frac{(2k-2)!}{2^j j! (2k-2j-2)!} (q^2 p^2)^j (p \cdot q)^{2k-2j-2} A_{2k}. \quad (4.9)$$

Changing summation variables to j and $l = k - j - 1$ and rearranging gives

$$\frac{4}{Q^2} \sum_{l=0}^{\infty} \frac{1}{x^{2l}} \sum_{j=0}^{\infty} \left(\frac{p^2}{Q^2} \right)^j \frac{(2l+j+2)! (2l+2j)!}{(2l+2j+2)! j! (2l)!} A_{2l+2j+2}. \quad (4.10)$$

The coefficient of x^{-n} in Eq. (4.10) is related to the $(n-2)$ th moment of $\nu W_2/m_p$:

$$\int_0^1 dx x^{n-2} \nu W_2(Q^2, x)/m_p = \sum_{j=0}^{\infty} \left(\frac{p^2}{Q^2} \right)^j \frac{(n+j)!}{j! (n-2)!} \frac{A_{n+2j}}{(n+2j)(n+2j-1)}. \quad (4.11)$$

This completes part (2). As promised, each moment is a sum of contributions from an infinite number of operators with increasing powers of p^2/Q^2 , and $p^2 = m_p^2$. When $Q^2 \gg m_p^2$, only the $j=0$ term in Eq. (4.11) is important, and it reduces to the standard scaling result. When we invert the moments, the $j \neq 0$ terms will conspire to give ξ scaling.

We invert the moments with the inverse Mellin

transform:

$$\frac{\nu W_2(Q^2, x)}{m_p} = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dn x^{-n+1} \sum_{j=0}^{\infty} \left(\frac{m_p^2}{Q^2} \right)^j \frac{1}{j!} \frac{\Gamma(n+j+1)}{\Gamma(n-1)} \times \frac{A_{n+2j}}{(n+2j)(n+2j-1)}. \quad (4.12)$$

To evaluate this integral, we first write the A_n 's as the moments of a quark distribution function F ,

$$A_n = \int_0^1 dy y^n F(y). \quad (4.13)$$

Then

$$\frac{A_{n+2j}}{(n+2j)(n+2j-1)} = \int_0^1 dy y^{n+2j-2} G(y), \quad (4.14)$$

where

$$G(y) = \int_y^1 dy' \int_{y'}^1 dy'' F(y''). \quad (4.15)$$

Inserting Eq. (4.14) into Eq. (4.12) and freely changing orders of summation and integrations, we

can do the sum over j to obtain

$$\frac{1}{2\pi i} \int_0^1 \frac{G(y)}{y^2} dy \int_{-i\infty}^{i\infty} dn x^{-n+1} y^n \frac{n(n-1)}{(1-y^2 m_p^2/Q^2)^{n+1}}. \quad (4.16)$$

Now doing the n integration gives

$$x^2 \frac{\partial^2}{\partial x^2} \int_0^1 \frac{G(y)}{y^2} dy \frac{x}{1-y^2 m_p^2/Q^2} \delta(\ln y - \ln x - \ln(1-y^2 m_p^2/Q^2)) = x^2 \frac{\partial^2}{\partial x^2} \left[\frac{x^2 G(\xi)}{\xi^2 (1+4x^2 m_p^2/Q^2)^{1/2}} \right], \quad (4.17)$$

where

$$\begin{aligned} \xi &= \frac{2x}{1+(1+4x^2 m_p^2/Q^2)^{1/2}} \\ &= \frac{Q^2}{2m_p \nu} \frac{2}{1+(1+Q^2/\nu^2)^{1/2}}, \end{aligned} \quad (4.18)$$

which is identical to Eq. (4.3) with $m_I = m_F = 0$. Evaluating derivatives and reexpressing Eq. (4.17) in terms of F using Eq. (4.15) gives the final result:

$$\begin{aligned} \nu W_2(Q^2, x)/m_p &= \frac{x^2}{(1+4x^2 m_p^2/Q^2)^{3/2}} F(\xi) + 6 \frac{m_p^2}{Q^2} \frac{x^3}{(1+4x^2 m_p^2/Q^2)^2} \int_\xi^1 d\xi' F(\xi') \\ &\quad + 12 \frac{m_p^4}{Q^4} \frac{x^4}{(1+4x^2 m_p^2/Q^2)^{5/2}} \int_\xi^1 d\xi' \int_{\xi'}^1 d\xi'' F(\xi''). \end{aligned} \quad (4.19)$$

Precisely analogous treatment of the coefficient of $g^{\mu\nu}$ in Eq. (4.7) yields

$$\begin{aligned} W_1(Q^2, x) &= \frac{x}{2(1+4m_p^2/Q^2)^{1/2}} F(\xi) + \frac{m_p^2}{Q^2} \frac{x^2}{(1+4x^2 m_p^2/Q^2)} \int_\xi^1 d\xi' F(\xi') \\ &\quad + 2 \frac{m_p^4}{Q^4} \frac{x^3}{(1+4x^2 m_p^2/Q^2)^{3/2}} \int_\xi^1 d\xi' \int_{\xi'}^1 d\xi'' F(\xi''). \end{aligned} \quad (4.20)$$

These results are slightly simpler in terms of W_L and W_T :

$$\begin{aligned} 2x(2W_T - W_L) &= 6xW_1 - (1+4x^2 m_p^2/Q^2) \nu W_2/m_p \\ &= \frac{2x^2}{(1+4x^2 m_p^2/Q^2)^{1/2}} F(\xi), \end{aligned} \quad (4.21a)$$

$$\begin{aligned} 2xW_L &= (1+4x^2 m_p^2/Q^2) \nu W_2/m_p - 2xW_1 \\ &= 4 \left[\frac{m_p^2}{Q^2} \frac{x^3}{1+4x^2 m_p^2/Q^2} \int_\xi^1 d\xi' F(\xi') + 2 \frac{m_p^4}{Q^4} \frac{x^4}{(1+4x^2 m_p^2/Q^2)^{3/2}} \int_\xi^1 d\xi' \int_{\xi'}^1 d\xi'' F(\xi'') \right]. \end{aligned} \quad (4.21b)$$

We can also exhibit the ξ -scaling predictions for neutrino scattering off light quarks producing light quarks. For pure $V \pm A$ currents, W_1 and W_2 are given by Eqs. (4.20) and (4.19) as in electroproduction. The interference term W_3 is

$$\nu W_3/m_p = \frac{x}{2(1+4x^2 m_p^2/Q^2)} F(\xi) + 2 \frac{m_p^2}{Q^2} \frac{x^2}{(1+4x^2 m_p^2/Q^2)^{3/2}} \int_\xi^1 d\xi' F(\xi'). \quad (4.22)$$

It is slightly misleading to call this effect a target-mass dependence. In terms of the variables Q^2 and ν , which depend only on lepton momenta, the ξ variable is²²

$$\begin{aligned} \xi &= \frac{Q^2/m_p \nu}{1+(1+Q^2/\nu^2)^{1/2}} \\ &= \frac{\nu}{m_p} [(1+Q^2/\nu^2)^{1/2} - 1]. \end{aligned} \quad (4.23)$$

It depends on the target mass only as an overall multiplicative constant, which does not affect scaling in ξ .

We feel that the ξ -scaling described in Eq. (4.21) gives some insight into precocious scaling. The variable x' approaches ξ for large x and Q^2 since

$$\begin{aligned} x' &= x - x^2 m_p^2/Q^2 + O(m_p^4/Q^4), \\ \xi &= x - x^3 m_p^2/Q^2 + O(m_p^4/Q^4). \end{aligned} \quad (4.24)$$

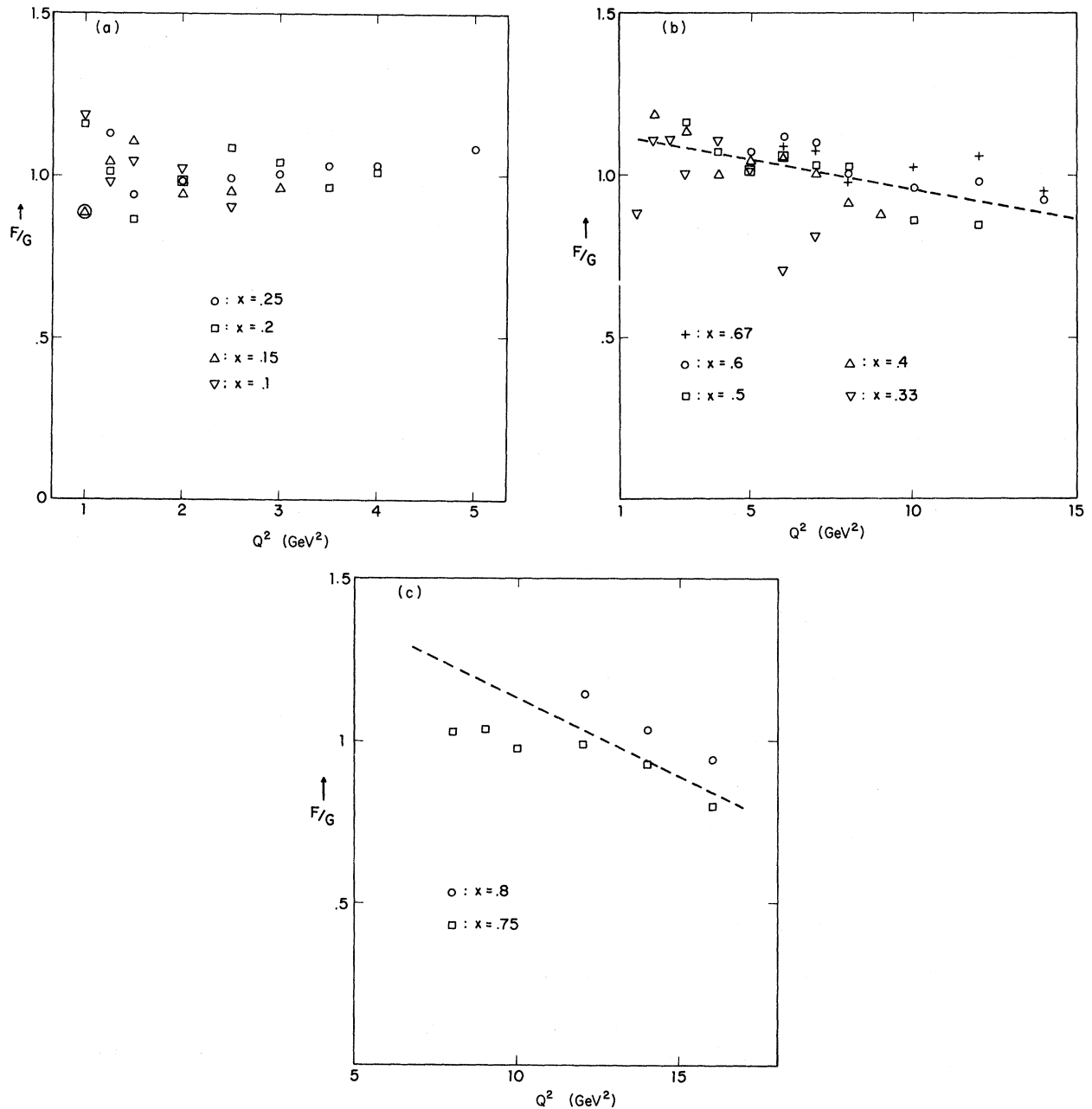


FIG. 2. Experimental values of F/G versus Q^2 . Dotted lines in *b* and *c* are linear eyeball fits to data.

At large x , where the structure functions are rapidly varying, x' scaling is not so different from ξ scaling.

Figure 2 shows a comparison of ξ -scaling predictions with recent SLAC data.²³ The structure functions are quoted at fixed x for various Q^2 values. This provides a direct test of naive scaling, but not of ξ scaling since the same value of x for different Q^2 corresponds to different ξ values. To get a test of ξ scaling, we have calcu-

lated the experimental values of $F(\xi)$ using Eq. (4.21a). We found a smooth function of ξ which fitted the experimental $F(\xi)$ reasonably well. It is

$$G(\xi) = \frac{3}{\xi} (1 - \xi)^{3.5}. \quad (4.25)$$

In Fig. 2, we have plotted $F(\xi)/G(\xi)$ as a function of Q^2 . If ξ scaling were exact, this ratio would be 1, independent of Q^2 .

We have plotted F/G for three different ranges

of x (and therefore of ξ). For small x [Fig. 2(a)], while there is considerable scatter, there is no obvious systematic dependence of F/G on Q^2 . The quoted experimental errors are not shown, but are all of the order of 10%. In this region ξ scaling, naive scaling, and x' scaling are all more or less consistent.

For intermediate and large x [Figs. 2(b) and 2(c)] there seems to be a systematic dependence of F/G on Q^2 . The ratio decreases as Q^2 increases. This is what we expect from the logarithmic effects of higher-order terms in g . Loosely speaking, their effect is to peak the structure functions more sharply at very small ξ as Q^2 increases. For fixed ξ (not small), the structure functions should decrease as Q^2 increases. In this region $G(\xi)$ can be identified approximately with the "scaling" function $F(\xi)$ at some average $Q^2 = Q_0^2$, so Fig. 2 can be interpreted as a plot of $F(\xi, Q^2)/F(\xi, Q_0^2)$ vs Q^2 . A detailed analysis of the logarithmic corrections shows that the slope of this plot should become more negative as ξ increases.¹⁶ This is consistent with Figs. 2(b) and 2(c) where the slope for large x appears to be more negative than that for intermediate x .

Note that for intermediate x , ξ scaling is better than naive scaling, but x' scaling is better still. This is an accident. The logarithmic effects of higher orders in g happen to go in the same direction as the additional m_p^2/Q^2 dependence in x' scaling. But x' scaling is not the point. The right way to extract the logarithmic dependence of the higher-order terms and check the detailed predictions of asymptotic freedom is to examine the deviations from ξ scaling. In a forthcoming paper we include the effects of the first logarithmic corrections in a detailed analysis of SLAC data and find good agreement with theory using a small value of the coupling constant, i.e., $g^2/4\pi^2$ (1 GeV) ≈ 0.3 , over the range $1 \lesssim Q^2 \lesssim 15$ GeV². This agrees with an earlier determination of g^{14} and supports the whole ξ -scaling picture which rests on an expansion in $g^2/4\pi^2$ (1 GeV).

Another test of the validity of our expansion is $R = \sigma_L/\sigma_T$. Equation (4.21) gives a prediction for R to zeroth order in $g(m_p)$. To this we may add a contribution unambiguously calculable to order g^2 (studied in Ref. 17), which goes to zero only logarithmically as Q^2 goes to infinity. The discrepancy between the data and this prediction is a measure of the error we make by dropping operators whose contributions to electroproduction are $O(g(m_p)m_p^2/Q^2)$. In fact a measurement of such a difference would serve to determine the matrix elements of the first set of operators we have ignored. However, the current status of the data is such that the above-described prediction [with *no*

free parameters, i.e., with $g^2/4\pi^2$ (1 GeV) = 0.3] gives as good a fit to R as any of the experimentalists' one- or two-parameter fits.

The function $G(\xi)$ was introduced purely for convenience in illustrating the logarithmic violation of ξ scaling. Its overall shape should not be taken as the distribution function for any value of Q^2 because the range of Q^2 for which experimental points exist is very different at small and large x . With better data, we could extract $F(\xi)$ for various values of Q^2 .

The parton language is a useful mnemonic for the form of ξ in Eq. (4.18). If a massless quark carries a fraction ξ of the proton momentum and is kicked onto its mass shell by the collision, then

$$(\xi p + q)^2 = 0 = \xi^2 m_p^2 + 2\xi p \cdot q - Q^2.$$

The positive solution of this quadratic equation is Eq. (4.18).

V. RESCALING

In this section we discuss currents involving heavy-quark fields for $Q^2 \gg m_p^2$. Such currents necessarily lead to violations of scaling associated with thresholds for production of states containing the heavy quarks. We will try to determine the nature of these scaling violations and the details of the rescaling process in which scaling is recovered at energies well above threshold. In this section we discuss processes in which a light quark is struck and a heavy quark is produced. Analysis of heavy struck quarks will be done in Sec. VI. The first case is trivial to analyze theoretically, but the results are quite interesting phenomenologically. The second is quite complicated and interesting theoretically, but in practice is probably a small effect because heavy quarks are rare in the proton.

We now consider the light struck quark. The process we have in mind can occur in neutrino physics when a piece of the hadronic weak current has the form $\bar{\psi}_H \gamma^\mu (1 \pm \gamma_5) \psi_L$ where $\psi_{H(L)}$ is a heavy- (light-) quark field. For example, the charm-changing $\Delta S = 0$ part of the standard weak current (a piece proportional to the sine of the Cabibbo angle) has this form where the heavy quark is the charmed quark. So does the $\Delta S = 1$ charm-changing current if the strange quark is regarded as light. The light nonstrange quarks may also be connected to other heavy quarks through right-handed currents, and even more complicated situations are imaginable. The free-field OPE for the product of such a current with its Hermitian conjugate involves two kinds of operators: bilinears in the light-quark field, corresponding to a light struck quark, and bilinears in the heavy-quark

field, corresponding to a heavy struck quark. We will discuss the former first, assuming that the proton matrix elements of the heavy-quark operators are small.

The calculation of the structure functions differs from the analogous calculation for a current involving two light-quark fields in only one respect. The OPE derives from the expansion of a heavy-quark propagator rather than a light-quark propagator. Wherever Q^2 appears in the light-quark OPE, there is an analogous term from the expansion of the heavy-quark propagator with Q^2 replaced by $Q^2 + m_H^2$, where m_H is the heavy-quark mass. The rest of the calculation is the same, so since the light-quark result for $Q^2 \gg m_p^2$ is $2xW_1 = \nu W_2/m_p = \pm 2x\nu W_3/m_p = F_2(x)$, the heavy-quark result is

$$2\xi W_1 = \nu W_2/m_p = \pm 2\xi W_3/m_p = F_2(\xi), \quad (5.1)$$

where

$$\xi = \frac{Q^2 + m_H^2}{2p \cdot q} = \frac{Q^2 + m_H^2}{2m_p \nu}. \quad (5.2)$$

The distribution function F_2 is the same function which appears in the light-quark result, because it is related to the same matrix elements of the light-quark operators.

The result in Eqs. (5.1) and (5.2) is exactly what we expect from the parton picture. If the struck light quark carries a fraction ξ of the proton momentum, to produce a heavy quark on its mass shell we must have $(\xi p + q)^2 = m_H^2$. For $Q^2 \gg m_p^2$ this is $2\xi p \cdot q - Q^2 = m_H^2$, which gives Eq. (5.2).

Clearly we could include the effect of the target mass using the techniques of Sec. IV. In practice, the target-mass effects in neutrino scattering are small except at very small y , so we will ignore them.

To see what this result means for inclusive

$$\frac{d^2\sigma_H}{dx dy} = \frac{G^2 m_p E}{\pi} \left[\left(1 - y + \frac{1}{2}y^2\right) - \frac{y E_{\text{th}}}{2\xi E} \pm \left(y - \frac{1}{2}y^2\right) \mp \left(1 - \frac{1}{2}y\right) \frac{E_{\text{th}}}{\xi E} \right] F_2(\xi), \quad (5.6)$$

where the \pm sign depends on the helicity matching of the lepton and hadron currents, as for light quarks. A detailed analysis of presently available data using Eq. (5.6) has been done by Barnett.²⁴

VI. HEAVY STRUCK QUARKS

In this section, we describe the predictions of ξ scaling for processes in which a heavy quark is struck. In this case, step (1) of our program becomes nontrivial.

neutrino-hadron scattering, we can write Eq. (5.2) in terms of x , y , and the neutrino energy E as

$$\xi = x + \frac{m_H^2}{2m_p E y}. \quad (5.3)$$

From the form Eq. (5.2) and the constraint $\xi \leq 1$ [which must be satisfied for $F_2(\xi)$ to be nonzero] we obtain

$$2p \cdot q \geq Q^2 + m_H^2 \quad (5.4a)$$

or

$$x \leq \frac{Q^2}{Q^2 + m_H^2}. \quad (5.4b)$$

This is simply the statement that there is a threshold. The total mass of the hadronic state must be greater than m_H . The effect of the heavy quark appears first at small x . But this does not mean that the struck quark in the production process carries a small fraction of the proton momentum. On the contrary, ξ , which is the momentum fraction of the struck quark, is bounded away from zero. Because $x > 0$, Eq. (5.3) implies that

$$\xi > \frac{m_H^2}{2m_p E y}. \quad (5.5)$$

This relation is very significant. It means that the effective threshold in neutrino scattering is larger than one might naively expect. The quark distribution functions are large only at small ξ , and to probe small ξ , E must be large enough so that Eq. (5.5) is satisfied. The naive threshold energy is $E_{\text{th}} = m_H^2/(2m_p)$. Loosely speaking, Eq. (5.5) implies that the effective threshold is $E_{\text{th}}/\langle \xi \rangle$. This also shows that "sea" quarks with smaller ξ are less effective than valence quarks for heavy-quark production.

Explicitly, the differential cross section for production of a heavy quark off a light quark in neutrino (or antineutrino) scattering is

The free-field OPE of a product of currents is simple in terms of the operators $P^{\mu_1 \dots \mu_n}$ of Eq. (4.5). For example, for a heavy-quark contribution to the electromagnetic current, $J^\mu = \bar{\psi} \gamma^\mu \psi$, the OPE is given by Eq. (4.4) just as for light quarks. There is no quark-mass dependence from the heavy-quark propagator because the struck quark and the produced quark have the same mass. In general, suppose $J^\mu = \bar{\psi}_1 \gamma^\mu \psi_2$ and look at the terms in the free-field OPE of $J^\mu J^{\nu \dagger}$ involving bilinears in ψ_I . This corresponds to a process in which the

I quark is struck and the F quark is produced. The two-body operator piece of the time-ordered product is

$$\int e^{i q x} d^4 x i T(J_\mu(x) J_\nu^\dagger(0)) = \int \frac{d^4 p}{(2\pi)^4} \bar{\psi}_I(-p) \gamma_\mu \frac{\not{p} + \not{q} + m_F}{(p+q)^2 - m_F^2} \gamma_\nu \psi_I(p) + \dots, \tag{6.1}$$

where \dots involves bilinears in ψ_F . The OPE is obtained by appropriately expanding the F -quark propagator. The denominator is $p^2 + 2p \cdot q - Q^2 - m_F^2$. Because the I quarks are free fields (in this approximation) we can replace p^2 by m_I^2 and expand as follows:

$$\frac{1}{m_I^2 + 2p \cdot q - Q^2 - m_F^2} = - \sum_{n=0}^{\infty} \frac{(2p \cdot q)^n}{(Q^2 + m_F^2 - m_I^2)^{n+1}}. \tag{6.2}$$

This gives an OPE in terms of the P operators (constructed with ψ_I) with coefficients proportional to inverse powers of $(Q^2 + m_F^2 - m_I^2)$. Here we will discuss only electroproduction in detail.

The OPE, Eq. (4.4), for heavy-quark currents

is not useful as it stands because the proton matrix element of each of the $P^{\mu_1 \dots \mu_n}$ operators involves, in general, many unknown constants. To extract as much information as possible, we re-express the OPE in terms of the traceless operators $O^{\mu_1 \dots \mu_n}$ using the following formula:

$$P^{\mu_1 \dots \mu_{2k}} = \sum_{j=0}^k g \dots g \frac{(2k-2j+1)!}{2^j (2k-j+1)!} m^{2j} O^{2k-2j}. \tag{6.3}$$

The j th term in this summation is a symmetric sum of the $2k!/[2^j(2k-2j)!]$ distinct combinations of j $g^{\mu_i \mu_k}$'s and an O operator with $2k-2j$ indices. Equation (6.3) is easily derived by noting that

$$g_{\mu_1 \mu_2} P^{\mu_1 \mu_2 \dots \mu_n} = m^2 P^{\mu_3 \dots \mu_n},$$

where m is the quark mass. The P and O operators with no Lorentz indices are defined as follows: $P = O = (1/m) \bar{\psi} \psi$. In deriving Eq. (6.3), we have used the fact that $-\partial^2$ acting on ψ (free in this approximation) is m^2 , so we have carried through step (1) of our program.

Putting Eq. (6.3) into Eq. (4.4), we obtain

$$\int e^{i q x} d^4 x \langle p | i T(J^\mu(x) J^\nu(0)) | p \rangle = \sum_{k=1}^{\infty} \sum_{j=0}^k \left\{ (-g^{\mu\nu} + q^\mu q^\nu / q^2) q_{\mu_1} q_{\mu_2} (2k-2j+1) \left[\frac{(2k)!}{2^{2j} j! (2k-j+1)!} + \frac{(2k-2)! j}{2^{2j} j! (2k-j+1)!} \right] \right. \\ \left. + Q^2 (g_{\mu_1}^\mu - q^\mu q_{\mu_1} / q^2) (g_{\mu_2}^\nu - q^\nu q_{\mu_2} / q^2) \frac{(2k-2j+1)(2k-2j)(2k-2j-1)(2k-2)!}{2^{2j} j! (2k-j+1)!} \right\} \\ \times \frac{m^{2j} (-Q^2)^j 2^{2k}}{Q^{4k}} q_{\mu_3} \dots q_{\mu_{2k-2j}} \langle p | O^{\mu_1 \dots \mu_{2k-2j}} | p \rangle. \tag{6.4}$$

We can let the sum over k in Eq. (6.4) run to $k=0$ without affecting the positive moments of the structure functions. In going from the T product to moments of the observed structure functions one encounters contour integrals in the complex $\omega (= 2p \cdot q / Q^2)$ plane.⁴ For simplicity, we have ignored the contributions from large arcs as $|\omega| \rightarrow \infty$, a limit governed by Regge behavior, i.e., Q^2 fixed, $|\nu| \rightarrow \infty$, and we have done formal continuations in the angular momentum plane. Strictly speaking, our results apply to the structure functions only once the leading Regge behavior as $x \rightarrow 0$ is subtracted away. It is an open, experimental question whether these subtractions are necessary.

Then with $l = k - j$, we can rewrite the right-hand side as

$$\sum_{l=0}^{\infty} \left\{ (-g^{\mu\nu} + q^\mu q^\nu / q^2) q_{\mu_1} q_{\mu_2} (2l+1) \left[X_{2l} \left(\frac{m^2}{Q^2} \right) + 2Y_{2l} \left(\frac{m^2}{Q^2} \right) \right] \right. \\ \left. + Q^2 (g_{\mu_1}^\mu + q^\mu q_{\mu_1} / q^2) (g_{\mu_2}^\nu + q^\nu q_{\mu_2} / q^2) (2l+1) 2l(2l-1) Z_{2l} \left(\frac{m^2}{Q^2} \right) \right\} \\ \times \frac{2^{2l}}{Q^{4l}} q_{\mu_3} \dots q_{\mu_{2l}} \langle p | O^{\mu_1 \dots \mu_{2l}} | p \rangle, \tag{6.5}$$

where the functions X_n , Y_n , and Z_n are

$$\begin{aligned}
 X_n(x) &= \sum_{j=0}^{\infty} (-x)^j \frac{(n+2j)!}{j!(n+j+1)!}, \\
 Y_n(x) &= \sum_{j=0}^{\infty} (-x)^j \frac{(n+2j-2)!j}{j!(n+j+1)!}, \\
 Z_n(x) &= \sum_{j=0}^{\infty} (-x)^j \frac{(n+2j-2)!}{j!(n+j+1)!}.
 \end{aligned}
 \tag{6.6}$$

To evaluate the functions X , Y , and Z explicitly consider the following function of two variables:

$$\begin{aligned}
 \varphi(x, y) &= \sum_{k=0}^{\infty} y^{2k} \left[1 - \left(\frac{x}{y^2} \right)^{2k+1} \right] X_{2k}(x) \\
 &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} y^{2k} (-x)^j \frac{(2k+2j)!}{j!(2k+j+1)!} + \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} y^{-2k-2} (-x)^{2k+j+1} \frac{(2k+2j)!}{j!(2k+j+1)!} \\
 &= \sum_{l=k+j=0}^{\infty} y^{2l} \sum_{m=j=0}^l \left(\frac{-x}{y^2} \right)^m \frac{(2l)!}{m!(2l+1-m)!} + \sum_{l=k+j=0}^{\infty} y^{2l} \sum_{\substack{m=2k+j+1 \\ =l+1}}^{2l+1} \left(\frac{-x}{y^2} \right)^m \frac{(2l)!}{(2l+1-m)!m!} \\
 &= \sum_{l=0}^{\infty} \frac{y^{2l}}{2l+1} \left(1 - \frac{x}{y^2} \right)^{2l+1} \\
 &= \frac{1}{2y} \ln \frac{1+y-x/y}{1-y+x/y} \\
 &= \frac{1}{2y} \ln \left\{ - \frac{[y + \frac{1}{2} + (\frac{1}{4} + x)^{1/2}][y + \frac{1}{2} - (\frac{1}{4} + x)^{1/2}]}{[y - \frac{1}{2} + (\frac{1}{4} + x)^{1/2}][y - \frac{1}{2} - (\frac{1}{4} + x)^{1/2}]} \right\}.
 \end{aligned}
 \tag{6.7}$$

In the complex y plane for $x > 0$, φ is an analytic function with three cuts on the real axis from $-\infty$ to $-\frac{1}{2} - (\frac{1}{4} + x)^{1/2}$, from $\frac{1}{2} - (\frac{1}{4} + x)^{1/2}$ to $-\frac{1}{2} + (\frac{1}{4} + x)$, and from $\frac{1}{2} + (\frac{1}{4} + x)^{1/2}$ to ∞ . We can reconstruct the function X_{2k} by evaluating a contour integral counterclockwise around the finite cut:

$$\begin{aligned}
 X_{2k}(x) &= - \frac{1}{2\pi i} \int_c dy \left(\frac{y}{x} \right)^{2k+1} \varphi(x, y) \\
 &= \frac{1}{(2k+1) \left[\frac{1}{2} + (\frac{1}{4} + x)^{1/2} \right]^{2k+1}}.
 \end{aligned}
 \tag{6.8}$$

An analogous calculation for the odd-index X functions gives the general result

$$X_n(x) = \frac{1}{(n+1) \left[\frac{1}{2} + (\frac{1}{4} + x)^{1/2} \right]^{n+1}}.
 \tag{6.9}$$

We can then directly calculate Y and Z in terms of X :

$$\begin{aligned}
 Y_n &= - \frac{n+1}{(n+2)(n+3)} X_n \\
 &\quad + \frac{n(3n+5)}{(n+1)(n+2)(n+3)} X_{n-1} \\
 &\quad - \frac{2(n-1)}{(n+1)(n+3)} X_{n-2}, \\
 Z_n &= \frac{1}{(n+2)(n+3)} X_n \\
 &\quad - \frac{2(2n+3)}{(n+1)(n+2)(n+3)} X_{n-1} \\
 &\quad + \frac{4}{(n+1)(n+3)} X_{n-2}.
 \end{aligned}
 \tag{6.10}$$

Returning to the OPE, Eq. (6.5), we can identify the moments of the structure functions in the usual way:

$$\int_0^1 x^{2l-1} dx W_1(Q^2, x) = \frac{2l+1}{2} \left[X_{2l} \frac{m^2}{Q^2} + 2Y_{2l} \left(\frac{m^2}{Q^2} \right) \right] A_{2l}, \quad (6.11)$$

$$\int_0^1 x^{2l-2} dx \nu W_2(Q^2, x)/m_p = (2l+1)2l(2l-1) Z_{2l} \left(\frac{m^2}{Q^2} \right) A_{2l}.$$

A_{2l} is related to the proton matrix element of $O^{\mu_1 \dots \mu_{2l}}$ as in Eq. (4.2), and we have assumed that $Q^2 \gg m_p^2$ and have ignored target-mass effects for simplicity.

To see the meaning of Eq. (6.11), consider the contribution to W_1 from the functions X , call it W_x . Inserting Eq. (6.8) we get

$$\int_0^1 x^{2l-1} dx W_x(Q^2, x) = \frac{1}{2} \frac{A_{2l}}{\left[\frac{1}{2} + \left(\frac{1}{4} + m^2/Q^2 \right)^{1/2} \right]^{2l+1}}. \quad (6.12)$$

If $F(X)$ is the distribution with moments A_n , as in Eq. (4.13), then Eq. (6.12) is obviously satisfied for

$$W_x(Q^2, x) = \frac{1}{2} x F(\xi), \quad (6.13)$$

$$\frac{n+1}{2} \frac{Q^2 + m_F^2 + m_I^2}{Q^2} \left(\frac{Q^2}{Q^2 + m_F^2 - m_I^2} \right)^{n+1} X_n \left(\frac{m_I^2 Q^2}{(Q^2 + m_F^2 - m_I^2)^2} \right) A_n. \quad (6.15)$$

An important technical point arises for $m_I > m_F$, because the argument of the function X_n goes to infinity at $Q^2 = m_I^2 - m_F^2$. Analyticity in Q^2 requires that for $Q^2 < m_I^2 - m_F^2$ one must be on the second sheet of the analytic function X_n , that is, change the sign of the square root in Eq. (6.9). Then Eq. (6.15) becomes for all Q^2

$$\frac{1}{2} \frac{Q^2 + m_F^2 + m_I^2}{Q^2} \left(\frac{Q^2}{Q'^2} \right)^{n+1} A_n, \quad (6.16)$$

where

$$2Q'^2 = Q^2 + m_F^2 - m_I^2 + [Q^4 + 2Q^2(m_F^2 + m_I^2) + (m_F^2 - m_I^2)^2]^{1/2}. \quad (6.17)$$

The piece of the structure function which gives Eq. (6.16) is therefore

$$\frac{1}{2} \frac{Q^2 + m_F^2 + m_I^2}{Q^2} x F(\xi), \quad (6.18)$$

where

$$\xi = xQ'^2/Q^2 = \frac{Q'^2}{2m_p \nu}. \quad (6.19)$$

As for electroproduction, the general result is complicated, involving integrals of the function F .

where

$$\xi = x \left[\frac{1}{2} + \left(\frac{1}{4} + m^2/Q^2 \right)^{1/2} \right]. \quad (6.14)$$

The other terms are similar but more complicated because the extra n dependence in Eq. (6.10) leads to integrals of F when the moments are inverted. But once again, the result has the basic form of ξ scaling. The variable ξ in Eq. (6.14) is the same as in Eq. (4.3) for $m_F = m_I = m$ and $Q^2 \gg m_p^2$.

We will not dwell on details of these contributions to the structure functions because the target matrix elements of the heavy-quark operators will probably be too small for the details to be observable. One general comment is important. The heavy-quark distributions are presumably strongly decreasing functions of ξ (see Sec. VII). For fixed x , as Q^2 increases, ξ in Eq. (6.14) decreases, and therefore a heavy-quark contribution to the electroproduction structure functions [such as Eq. (6.13)] increases. Except at very small x , the logarithmic corrections to scaling have the opposite effect.¹⁶

For $m_I \neq m_F$ in neutrino scattering the results are similar. For example, there is a contribution to the $(n-1)$ st moment of W_1 of the form (for $Q^2 \gg m_p^2$):

Clearly the methods of this section can be combined with those of Sec. IV to get expressions valid for Q^2 not large compared to the proton mass. The results always involve calculable functions of Q^2 and the quark and target masses multiplying the function $F(\xi)$ (and various integrals of F) where ξ is given by Eq. (4.3).

VII. HEAVY-QUARK DISTRIBUTION FUNCTIONS

As defined by our renormalization-group equation, Eq. (2.2), anomalous dimensions are now generally mass-dependent. The anomalous dimensions of the leading operators in the expansion of two currents enter into the Q^2 dependence to $O(g^2(Q^2))$ as indicated in Eq. (3.6) for the coefficient functions. In Sec. III, we analyzed lepton-hadron scattering in terms of operators renormalized at a fixed mass, roughly on the scale of the target mass, and Q^2 -dependent coefficient functions which satisfy renormalization-group equations. There is an alternate but completely equivalent way of viewing the analysis. We can shift the logarithmic Q^2 dependence onto the matrix

elements by renormalizing the operators at the variable scale Q . From this standpoint to leading order in $g(Q^2)$, the scattering process measures the target matrix elements of bilinears of the struck quark as defined at Q . Their dependence on Q is determined by the operators' anomalous dimensions.

The calculation of the relevant anomalous di-

mensions is straightforward and proceeds in direct analogy to the massless case.⁷ To the first order in which quark bilinears are renormalized, they also mix with gluon bilinears, and hence the gluon renormalizations must also be computed. We quote the results in parametric form, which is adequate for extracting some important consequences. The relevant operators are

$$O_0^n = \frac{i^{n-2}}{2n!} (F^{\alpha\mu_1} D^{\mu_2} \dots D^{\mu_{n-1}} F_{\alpha}{}^{\mu_n} + \text{permutations of vector indices} - \text{traces}), \quad (7.1)$$

$$O_j^n = \frac{i^{n-1}}{n!} (\bar{\psi}_j \gamma^{\mu_1} D^{\mu_2} \dots D^{\mu_n} \psi_j + \text{permutations} - \text{traces}),$$

where the D 's are the relevant gauge-covariant derivatives and j can stand for any quark flavor; color sums are understood. We use the notation

$$c_1 \delta_{ab} = f_{acd} f_{bcd}, \quad c_2 \delta^{ab} = \text{tr}(T^a T^b), \quad c_3 I = T^a T^a, \quad (7.2)$$

where f_{abc} are the structure constants of the gauge group, and T^a are the representation matrices normalized to $[T_a, T_b] = i f_{abc} T_c$. For color SU(3), $c_1 = 3$, $c_2 = \frac{1}{2}$ per quark, and $c_3 = \frac{4}{3}$. We find to $O(g^2)$

$$\gamma_{jj}^n = \frac{g^2}{16\pi^2} c_3 \int_0^1 d\alpha \left\{ 2 \left\{ [2n\alpha^n - (3n-2)\alpha^{n-1}] \frac{\alpha M^2}{\alpha M^2 + m_j^2} - \frac{n\alpha^n (1-\alpha) M^2 m_j^2}{(\alpha M^2 + m_j^2)^2} \right\} \right. \\ \left. - 4 \left\{ \frac{\alpha M^2}{\alpha M^2 + m_j^2} \sum_{k=2}^n [(k-1)\alpha^{k-2} - (k-2)\alpha^{k-1}] + \left| \frac{m_j^2 M^2 (1-\alpha)}{(\alpha M^2 + m_j^2)^2} \sum_{k=2}^n \alpha^{k-1} \right| \right\} \right\},$$

$$\gamma_{jj'}^n = 0 \text{ for } j \neq j'$$

$$\gamma_{0j}^n = \frac{g^2}{16\pi^2} (-2c_3) \int_0^1 d\alpha \left\{ \frac{\alpha M^2}{\alpha M^2 + m_j^2} [2(1-\alpha)^{n-2} - 2(n-1)\alpha(1-\alpha)^{n-2} + n\alpha(1-\alpha)^{n-1}] \right. \\ \left. + \frac{2M^2 m_j^2}{(\alpha M^2 + m_j^2)^2} \alpha(1-\alpha)^{n-1} \right\},$$

$$\gamma_{j0}^n = \frac{g^2}{16\pi^2} (-16c_2) \int_0^1 d\alpha \left\{ \frac{\alpha(1-\alpha)M^2}{\alpha(1-\alpha)M^2 + m_j^2} [(1-\alpha)\alpha^{n-1} + 2(1-\alpha)\alpha^n + n(1-\alpha)(2\alpha-1)\alpha^{n-1}] \right. \\ \left. + \frac{\alpha(1-\alpha)M^2 m_j^2}{[\alpha(1-\alpha)M^2 + m_j^2]^2} 4(1-\alpha)\alpha^n \right\}, \quad (7.3)$$

$$\gamma_{00}^n \cong -\frac{g^2}{16\pi^2} \left\{ 2c_1 \left[\frac{1}{3} - \frac{4}{n(n-1)} - \frac{4}{(n+1)(n+2)} + 4 \sum_{k=2}^n \frac{1}{k} \right] + \frac{8}{3} \sum_i c_2 \frac{1}{1+5m_i^2/M^2} \right\}.$$

In γ_{00}^n the factor $(1+5m_i^2/M^2)^{-1}$ is the interpolating form for the same expression as in Eq. (2.3a).

The integrated form of the renormalization-group equation for matrix elements of operators renormalized at $M(O_\alpha^n(M))$ is

$$\langle p | O_\alpha^n(M) | p \rangle = \left[\exp T \int_{M_0}^M \gamma^N \left(g(M'), \frac{m(M')}{M'} \right) \frac{dM'}{M'} \right]_{\alpha\beta} \langle p | O_\beta^n(M_0) | p \rangle. \quad (7.4)$$

For orientation, let us first discuss a numerical integration for $n=2$. To proceed we require initial conditions. We are not in a position yet to compute them, but we can make some plausible guesses guided by general principles, experiment, and Eq. (7.3). If a given quark is very heavy, m_i

$\gg m_{\text{proton}}$, then it is likely that its proton matrix elements renormalized on the scale of m_p will be very small. This should follow from simple quantum mechanics as well as the theorem of Appelquist and Carazzone^{10,12} regarding the effective decoupling of heavy fields. Also, experiments sug-

gest a low upper bound on the distribution functions of all heavy quarks at $Q^2 \leq 15 \text{ GeV}^2$. Furthermore, if these matrix elements are negligibly small, they will remain so until $M \sim m$ because they are enhanced by a mixing with gluon operators, through γ_{j0}^n , which is proportional to $g^2(M)M^2/m^2$ for $m^2 \gg M^2$. We propose extending these ideas to the heavy quarks, λ and \mathcal{O}' , of flavor SU(4) though even the \mathcal{O}' mass is not much larger than m_{proton} . For justification we sight the upper bounds from ν and e scattering and the phenomena associated with Zweig's rules which suggest the suppression of mixing of heavy quarks. In particular, we set $\langle p | O_\lambda^2(m_p) | p \rangle = \langle p | O_{\mathcal{O}'}^2(m_p) | p \rangle = 0$.

For the light quarks and gluons note that

$$-2O_0^2 + \sum_j O_j^2 = \theta_{\mu\nu} - \frac{1}{4} g_{\mu\nu} \theta_\lambda^2,$$

where $\theta_{\mu\nu}$ is the total, symmetric energy-momentum tensor. With a standard normalization $\langle p | \theta_{\mu\nu} | p \rangle = 2p_\mu p_\nu$, independent of scale. The trace term, θ_λ^2 , is the sum of quark mass operators and is negligible because m_p is much greater than the light-quark masses (see Fig. 1) and

$$\langle p | \bar{\psi}_{\text{heavy}} \psi_{\text{heavy}}(m_{\text{proton}}) | p \rangle \approx 0.$$

We define $\langle p | -2O_0^2 | p \rangle = \theta_{\text{gluon}} \Pi^2$, $\langle p | O_j^2 | p \rangle = \theta_j \Pi^2$, where Π^2 is the traceless rank-2 tensor of Eq. (4.2). They are the contributions of each field to the total energy-momentum. Then $\theta_{\mathcal{O}} + \theta_{\mathcal{O}'} + \theta_{\text{gluon}} \approx 2$ at $m_{\mathcal{O}}$. Finally we use the experimental input from Gargamelle and SLAC that $\theta_{\text{gluon}} \approx \theta_{\mathcal{O}} + \theta_{\mathcal{O}'}$ somewhere in the appropriate region, say at 3 GeV. In Fig. 3 we display the results of a numerical integration using the same coupling and mass parameters as in Fig. 1.

We note that $\theta_{\mathcal{O}}$ increases more slowly than θ_λ

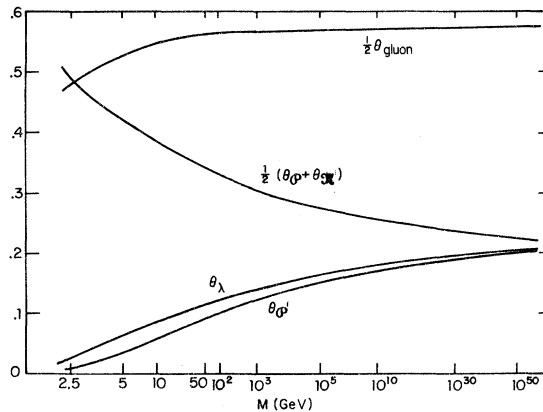


FIG. 3. Contributions to the total energy-momentum from various quarks and from gluons versus M . θ_{quark} includes both quark and antiquark contributions. Each light antiquark ($\bar{\mathcal{O}}$ or $\bar{\mathcal{O}'}$) contributes roughly $\frac{1}{2} \theta_\lambda$.

because $m_{\mathcal{O}'} > m_\lambda$. Both $\theta_{\mathcal{O}}$ and θ_λ should be detectable in experiments at Fermilab. We display the results for M up to 10^{50} GeV , where asymptotic SU(4) sets in, although weak, electromagnetic, and gravitational interactions, irrelevant at presently accessible scales, will totally alter these predictions well before 10^{50} GeV . We wish to emphasize the prediction of interesting phenomena at presently available energies which are far below the asymptotic region.

We do not continue to $n=4$ operators because there is no analog of the conserved energy-momentum tensor which essentially allowed an experimental determination of θ_{gluon} . Without a deeper theoretical understanding, plausible guesses may differ by factors of 20 or more.

However, we can draw some conclusions about the shapes of distribution functions for $\xi \rightarrow 1$ which is equivalent to the behavior of the $\langle p | O^n | p \rangle$ for large n because $\langle P | O_j^n | P \rangle$ is essentially $\int F_i(\xi) \xi^{n-2} d\xi$. The shapes of heavy-quark distributions, for M such that they are relatively small, are determined by the gluon distribution function. As a means of guessing the latter, we propose the following: If for $\xi \rightarrow 1$ the gluon function is much smaller than the light-quark function (in the parton language, if the amplitude for finding a hard gluon is negligible), then the gluon shape is determined by mixing from the light quarks. Since $\gamma_{0j} \propto 1/n$ for large n , if the light-quark distribution function vanishes like $(1-\xi)^a$ as $\xi \rightarrow 1$ (e.g., $3 \leq a \leq 4$), then the gluon function goes like $(1-\xi)^{a+1}$, so that high moments of $F_{\text{gluon}}(\xi)$ will drop by $1/n$ relative to moments of $F_{\mathcal{O},\mathcal{O}'}(\xi)$.

Now consider γ_{j0}^n for large n in the two limits $m \ll M$ and $m \gg M$, $\gamma_{j0}^n \propto 1/n$ for $m \ll M$ and $\gamma_{j0}^n \propto 2M^2/n^2 m^2$ for $m \gg M$. These determine how moments of the heavy-quark distributions, $F_H(\xi)$, behave relative to $(1-\xi)^{a+1}$, the gluon shape. We conclude that $F_H = K(1-\xi)^{a+2}$ for $m \ll M$ and $F_H = K[2M^2/(a+3)m^2](1-\xi)^{a+3}$ for $m \gg M$ with some single constant K . These can be combined into a single interpolating formula:

$$F_H \propto \frac{(1-\xi)^{a+2}}{1 + [(\alpha+3)/2](m^2/M^2)1/(1-\xi)}. \quad (7.5)$$

The above estimates are for $\xi \rightarrow 1$ and $n \rightarrow \infty$. In practice, they may be valid for $\xi > \frac{1}{2}$ or $\frac{1}{3}$. One crude way of getting a guess of the absolute magnitude is to continue these forms to $\xi=0$ and use the $n=2$ or $\int F_i(\xi) d\xi$ information of Fig. 3. A better estimate will require a specific model of the target and its gluon distribution.

We can use the techniques of this section to make some plausible guesses of the size and shape of $\bar{\mathcal{O}}$ and $\bar{\mathcal{O}'}$ distributions in the proton. While these have nothing to do with heavy quarks, they share with

the heavy-quark distributions the property that they are small at low M , so their M dependence is determined primarily by the light-quark and gluon distributions.

Until now, we have sloughed over the distinction between quark and antiquark distributions. Actually, the matrix element A_n^j of Eq. (4.2) for even n is the appropriate moment of the sum of j -quark and j -antiquark distribution functions, while for odd n , it is a moment of the difference between the j -quark and j -antiquark distribution functions. Thus θ_j , for example, is a sum of j -quark and j -antiquark contributions. To calculate the difference between j -quark and j -antiquark contributions to the energy-momentum matrix element, we would need to know the matrix elements of O_j^n for odd n and analytically continue in n to $n=2$. However, if we assume that antiquark distributions are small at small M , we can take the measured value of $\theta_\phi + \theta_{\bar{\phi}}$ at small M as the initial value both for the sum and the difference of quark and antiquark contributions. We can then calculate their values at larger M using Eq. (7.4) and solve for the antiquark contribution. The point is that the γ 's in Eq. (7.4) are different for the sum and difference. For the sum, which is related to a matrix element of $O_\phi^2 + O_{\bar{\phi}}^2$, the appropriate γ 's are given by Eq. (7.3) for $n=2$. But the difference is the continuation in m to $m=2$ of the matrix elements of $O_\phi^m + O_{\bar{\phi}}^m$ for m odd, and for m odd there are no gluon operators. The only γ 's are γ_{jj}^m , so the difference evolves in M according to Eq. (7.4) with $\gamma = \gamma_{jj}^2$.

The results for the antiquark contribution to the energy-momentum (or equivalently the integral of the antiquark distribution functions) can be read off Fig. 3. For λ and ϕ' , the quark-antiquark difference is taken to be virtually zero at $M=1$ GeV and remains so for all M . Hence the λ and $\bar{\lambda}$ contributions to θ_λ are equal, and similarly ϕ' and $\bar{\phi}'$ each contribute one half of $\theta_{\phi'}$. The light-antiquark distributions are chosen to vanish for $M=1$ GeV and are virtually identical to the $\bar{\lambda}$ contribution ($=\frac{1}{2}\theta_\lambda$) for $M \gtrsim 2$ GeV.

We can similarly discuss the tail of the antiquark distribution near $\xi=1$. If, as assumed above, the light- (valence-) quark distribution goes as $(1-\xi)^a$, so that the gluon distribution goes as $(1-\xi)^{a+1}$, then the light-antiquark distribution goes as $(1-\xi)^{a+2}$.

Witten¹² offers an alternative analysis of these problems, based on a generalization of the Appelquist-Carazzone theorem applied to the expansion of products of heavy operators. We agree with his results for $n=2$ operators, and our discussion of distribution tails is consistent with his analysis of other low moments.

VIII. ξ AND THE PARTON MODEL

We offer a discussion of ξ scaling in the language of the parton model as a handy mnemonic and as a less technical way of explaining the nature of the approximation and expansion scheme proposed in Sec. III. Although ξ is a totally frame-independent quantity, it is convenient to write the initial proton four-momentum as $p=(p^0, 0, 0, p^3)$ and the virtual photon momentum as $q=(q^0, 0, 0, q^3)$. Let p_I be the initial momentum of the struck quark and $p_F = p_I + q$ be the final momentum. If ξ is defined as the natural light-cone variable

$$\xi \equiv \frac{p_I^0 + p_I^3}{p^0 + p^3}, \quad (8.1)$$

then it is a simple exercise to derive the full formula for ξ given in Eq. (4.3) under the three assumptions

$$\begin{aligned} p_F^2 &= m_F^2, \\ p_I^2 &= m_I^2, \\ p_T^2 &= 0, \end{aligned} \quad (8.2)$$

where p_T is the part of p_I transverse to p and q , i.e., $p_I = (p_I^0, p_I^1, p_T^2, p_I^3)$.²⁵

Scaling in ξ was derived in Secs. III and IV by retaining only those terms which are zeroth order in $g(m_p)$, the effective coupling evaluated at the scale of the proton mass. Putting the initial and final quarks on shell is the momentum-space analog of saying that the quarks can travel long distance relative to the wavelength $1/Q$ without appreciable interaction. For $Q^2 \gtrsim m_p^2$ and $g(m_p)$ negligible, this is certainly a reasonable approximation. The third assumption, $p_T^2=0$, is a consistent part of the same approximation. It is the finite size of the proton which necessitates $p_T \neq 0$. Inasmuch as the proton is large and quarks travel far as measured by Q^2 , $p_T^2 \approx 0$. The size of the proton, the extent to which the quarks are off shell, and the size of p_T are all measures of g . Our field-theoretic techniques offer a systematic expansion procedure in $g(m_p)$.

From careful analysis of e^+e^- annihilation for $1 \leq E_{\text{c.m.}} \leq 7$ GeV,¹⁴ it has been estimated that $g^2/4\pi^2 (1 \text{ GeV}) \approx 0.3$. This can be translated into a fundamental mass scale, Λ , using

$$\frac{g^2}{4\pi} (M) \approx \frac{12\pi}{25} \frac{1}{\ln M^2/\Lambda^2} \quad (8.3)$$

for $M^2 \gg \Lambda^2$. Then $\Lambda \approx 500$ MeV. Our own analysis of SLAC electroproduction data for $1 < Q^2 < 15$ GeV² agrees with this value (and will be discussed elsewhere.) The quality of the ξ -scaling limit is determined by the validity of the assertion that $\Lambda^2 \ll m_p^2$. We are arguing that this is the only coherent explanation of precocious scaling yet

proposed, and that the basic assumption is supported by the e^+e^- analysis and by the smallness of scaling violations at higher Q^2 .

From hadron-hadron interactions one surmises that the quark distribution in p_T is sharply cut off above $p_T^2 \sim 0.1 \text{ GeV}^2$. Hence, ignoring p_T is qualitatively the same sort of approximation as ignoring Λ . The relation is not accidental, and we can give a more precise discussion than the one above relating $p_T \neq 0$ to the binding of the proton into a finite size which is set roughly by $1/\Lambda$ which we gather from experiment to be rather larger than $1/m_p$.

In the operator-product expansion to next order in $g(m_p)$, there appear operators such as $\psi F_{\mu\nu} \sigma^{\mu\nu} D^{\lambda_1} \cdots D^{\lambda_n} \psi$. If we wish to include these operators, their matrix elements must be deduced from experiments because with present technology we can no sooner compute them than the proton structure functions themselves. These matrix elements are the moments of a function which can be interpreted as a quark distribution in transverse momentum. In $O(g^0)$, quark bilinear

operators essentially count quarks. In these new three-body operators, the initial and final quarks need not have the same momentum, and hence their proton matrix elements are measures of the transverse momentum distribution. To $O(g^0)$, we neglected these effects, and indeed $\langle p_T \rangle / m_p \lesssim 0.3$. To include these effects, we will have to extract a second distribution function from the data in addition to our $F(\xi)$.

The quark-mass dependence of ξ is of no great importance unless the process involves quarks of mass much greater than Λ . Then the ξ -scaling limit is certainly of phenomenological importance.

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¹R. P. Feynman, in *High Energy Collisions*, Proceedings of the Third International Conference on High Energy Collisions, Stony Brook, 1969, edited by C. N. Yang *et al.* (Gordon and Breach, New York, 1969); J. D. Bjorken and E. A. Paschos, *Phys. Rev.* **185**, 1975 (1969).

²K. Wilson, *Phys. Rev.* **179**, 1499 (1969).

³K. Wilson, *Phys. Rev. D* **3**, 1818 (1971).

⁴N. Christ, B. Hasslacher, and A. H. Mueller, *Phys. Rev. D* **6**, 3543 (1972).

⁵H. D. Politzer, *Phys. Rev. Lett.* **30**, 1346 (1973).

⁶D. Gross and F. Wilczek, *Phys. Rev. Lett.* **30**, 1343 (1973).

⁷H. Georgi and H. D. Politzer, *Phys. Rev. D* **9**, 416 (1974); D. Gross and F. Wilczek, *ibid.* **8**, 3622 (1973); **9**, 980 (1974).

⁸S. Weinberg, *Phys. Rev. Lett.* **31**, 494 (1973).

⁹K. Wilson, *Phys. Rev. D* **10**, 2445 (1974); J. Kogut and L. Susskind, *ibid.* **9**, 3501 (1974).

¹⁰H. Georgi and H. D. Politzer, *Phys. Rev. Lett.* **36**, 1281 (1976).

¹¹O. Nachtmann, *Nucl. Phys.* **B63**, 237 (1973).

¹²E. Witten, Princeton report, 1975 (unpublished).

¹³T. Appelquist and H. D. Politzer, *Phys. Rev. Lett.* **34**, 43 (1975); *Phys. Rev. D* **5**, 1404 (1975); A. De Rújula and S. L. Glashow, *Phys. Rev. Lett.* **34**, 46 (1975); **34**, 1252 (1975).

¹⁴E. Poggio, H. Quinn, and S. Weinberg, *Phys. Rev. D* **13**, 1958 (1976); A. De Rújula and H. Georgi, *ibid.* **13**, 1296 (1976).

¹⁵S. L. Glashow and S. Weinberg, *Phys. Rev. Lett.* **20**, 224 (1968); M. Gell-Mann, R. J. Oakes, and B. Renner, *Phys. Rev.* **175**, 2195 (1968).

¹⁶A. De Rújula, H. Georgi, and H. D. Politzer, *Phys. Rev. D* **10**, 2141 (1974).

¹⁷A. Zee, F. Wilczek, and S. B. Treiman, *Phys. Rev. D* **10**, 2881 (1974).

¹⁸T. Appelquist and H. Georgi, *Phys. Rev. D* **8**, 4000 (1973); A. Zee, *ibid.* **8**, 4038 (1973).

¹⁹T. Appelquist and J. Carazzone, *Phys. Rev. D* **11**, 2856 (1975).

²⁰S. Weinberg, *Phys. Rev. D* **8**, 3497 (1973).

²¹W. Zimmermann, in *Lectures on Elementary Particles and Quantum Field Theory*, edited by S. Deser *et al.* (MIT Press, Cambridge, 1970), p. 395.

²²Others have suggested scaling variables which resemble our ξ in some kinematic regions; see F. Gürsey and S. Orfanidis, *Nuovo Cimento* **11A**, 225 (1972); O. W. Greenberg and D. Bhaumik, *Phys. Rev. D* **4**, 2048 (1971); G. Domokos, *ibid.* **4**, 3708 (1971); M. Briedenback and J. Kuti (unpublished); also related in H. J. Schnitzer, *Phys. Rev. D* **4**, 1429 (1971).

²³E. M. Riordan *et al.*, Report No. SLAC-PUB-1634, 1975 (unpublished).

²⁴M. Barnett (private communication).

²⁵This derivation was pointed out to us separately by G. C. Fox and P. Frampton.