PHYSICAL REVIEW D VOLUME 14, NUMBER 7 1 OCTOBER 1976

Calculation of the $K_{\mu 4}$ decay rate

T. Becherrawy and N. F. Nasrallah
Faculty of Science, Lebanese University, Hadet, Beirut, Lebanon
(Received 18 November 1975)

The $K_{\mu 4}$ decay rate is calculated. We show that the variation of the K_{l4} form factors between the soft-pion point and the physical point accounts for a large amount of the discrepancy between the current-algebra theoretical value and experiment.

The form factors for the decays K_{l4} were first calculated by Callan and Treiman[1] from current-algebra and soft-pion techniques. Their result for the form factor F_3 differed considerably depending on which of the final-state pions was taken to be soft. Weinberg[2,3] later explained this rapid variation of F_3 by taking a nearby K pole explicitly into account. The rates for the decays K_{e4} and $K_{\mu 4}$ calculated from the values of the form factors obtained by the authors of Refs. 1 and 2 show some discrepancy with the experimentally measured ones. Some time ago Schilcher and one of us[4] showed that the discrepancy between theory and experiment in the K_{e4} case could be accounted for by the variation of the form factors F_1 and F_2 between the soft-pion limit and the physical point.

The amplitudes used for extrapolation by the authors of Ref. 3 differed slightly from the previously measured ones, the choice being motivated by the use of the collinear-dispersion-relation method of Fubini and Furlan.[5] These amplitudes are simply related to the K_{l4} form factors and to the K-π scattering amplitudes which appear on equal footing, and it was argued by the authors that their different choice of smooth functions constituted a better approximation, for it minimizes the contribution of the continuum which represents corrections to the soft-pion limit by damping it strongly in the low and intermediate energies of the π-π system.

A calculation of the $K_{\mu 4}$ decay rate, to which F_3 contributes, was not attempted in Ref. 4 as no measurement of the rate was available at the time and as some controversy subsisted about the factor ξ of K_{l3} decay which appeared in the final expressions. This note presents a calculation of the $K_{\mu 4}$ decay rate.

The K_{l4} form factors are defined in the following way:

$$\frac{i}{f_\pi m^2}\int dy\, e^{iq_b y}(m^2-q_b{}^2)\langle \pi^a(q_a)|TD_b(y)A_\lambda^{K^-}(0)|K^+(p)\rangle = \frac{i}{\sqrt{2}\,M}\left[F_1^b(q_++q_-)_\lambda+F_2^b(q_+-q_-)_\lambda+F_3^b(p-q_+-q_-)_\lambda\right], \quad (1)$$

where m and M denote the masses of the π and K meons, respectively, $a,b=+,-$ or $a,b=-,+$, $D=\partial^\mu A_\mu$ is the divergence of the axial-vector current with the quantum numbers of the π, $\langle 0|D|\pi\rangle = f_\pi m^2$ with $f_\pi = 94$ MeV, $A_\lambda^{K^-}$ is the axial-vector current with the quantum numbers of the K^-, and the F_i's are functions of the invariants.

Instead of the amplitude defined by Eq. (1) we choose for extrapolation the related one with the K pole removed:

$$M_\lambda^b = i\,\frac{M^2-q_l{}^2}{M^2}\int dy\, e^{iq_b y}(m^2-q_b{}^2)\times\langle\pi^a|T\phi_b(y)A_\lambda^{K^-}(0)|K^+\rangle, \quad (2)$$

with

$$q_l=(p-q_+-q_-),\quad \phi=D/f_\pi m^2\ .$$

We work in the special configuration with all particles at rest and use the collinear parametrization

$$q_b = xq_a = x\left(\frac{m}{M}\right)P\ , \quad (3)$$

using the notation

$$\tilde{F}_i(x)=\frac{M^2-q_l{}^2}{M^2}F_i(x)\ . \quad (4)$$

$\tilde{F}_i(x=1)=\{[M^2-(M-2m)^2]/M^2\}F_i(x=1)$ are form factors evaluated at the physical point where all particles are at rest.

Standard soft-pion techniques and the use of collinear dispersion relations in Ref. 4 yielded the following sum rules[6,7]:

$$\tfrac{1}{2}[\tilde{F}_1(1)+\tilde{F}_2(1)]=\frac{M^2-(M-m)^2}{M^2}\,\frac{-2Mf_+}{f_\pi}+\frac{1}{\pi}\int\frac{dx}{x(x^2-1)}\,\mathrm{Abs}[\tilde{F}_1^-(x)+\tilde{F}_2^-(x)]\ , \tag{5a}$$

$$\tfrac{1}{2}[\tilde{F}_1(1)-\tilde{F}_2(1)=\frac{1}{\pi}\int\frac{dx}{x(x^2-1)}\,\mathrm{Abs}[\tilde{F}_1^+(x)-\tilde{F}_2^+(x)] \tag{5b}$$

and

$$\frac{1}{2}\left[\tilde{F}_3(1)+\frac{\sqrt{2}\,f_k}{M}\,T_{\mathrm{th}}(K^+\pi^+\to K^+\pi^+)\right]=\frac{M^2-(M-m)^2}{M^2}\,\frac{-M(f_++f_-)}{f_\pi}+\frac{1}{\pi}\int\frac{dx}{x(x^2-1)}\,\mathrm{Abs}\tilde{F}_3^-(x)\ , \tag{6a}$$

$$\frac{1}{2}\left[\tilde{F}_3(1)+\frac{\sqrt{2}\,f_k}{M}\,T_{\mathrm{th}}(K^+\pi^-\to K^+\pi^-)\right]=\frac{1}{\pi}\int\frac{dx}{x(x^2-1)}\,\mathrm{Abs}\tilde{F}_3^+(x)\ , \tag{6b}$$

with the absorptive parts separated from

$$\begin{aligned}\mathrm{Abs}M^b(x)=\frac{(2\pi)^4}{2}\Big[&\sum_n\langle 0|j_b|n,K^+\rangle\langle n,\pi^a|J_{K^-}|0\rangle\,\delta(p_n+p-q_b)+\sum_m\langle 0|j_b|m\rangle\langle m,\pi^a|J_{K^-}|K^+\rangle\,\delta(p_m-q_b)\\&+\sum_l\langle\pi^a|j_b|l,K^+\rangle\langle l|J_{K^-}|0\rangle\,\delta(p_l+p-q_a-q_b)+\sum_k\langle\pi^a|j_b|k\rangle_c\langle k|J_{K^-}|K^+\rangle_c\delta(p_k-q_a-q_b)\\&-\sum_{k'}\langle 0|J_{K^-}|k',K^+\rangle\langle k',\pi^a|j_b|0\rangle\,\delta(p_{k'}+q_a+q_b)\\&-\sum_{l'}\langle 0|J_{K^-}|l'\rangle\langle l',\pi^a|j_b|K^+\rangle\,\delta(p_{l'}+q_a+q_b-p)\\&-\sum_{m'}\langle 0|J_{K^-}|m',K^+\rangle\langle m'|j_b|0\rangle\,\delta(p_{m'}-q_b)-\sum_{n'}\langle\pi^a|J_{K^-}|n'\rangle_c\,\langle n'|j_b|K^+\rangle_c\delta(p_n+q_b-p)\Big],\end{aligned} \tag{7}$$

where the index c denotes the connected part of a matrix element and where

$$j_b=(\Box+m^2)\phi\ ,$$

$$J_{K^-}=\frac{1}{M^2}\,(\Box+M^2)A_0^{K^-}\ ,$$

and where $f_\pm$ are the usual K_{l3} form factors

$$\langle\pi^0(q)|V_\lambda^{K^-}|K^+(p)\rangle=-\frac{1}{\sqrt{2}}\,[f_+(p+q)_\lambda+f_-(p-q)_\lambda]$$

evaluated at $(p-q)^2=(M-m)^2$.

Equations (5) and (6) yield the on-mass-shell values of the form factors. As was argued in Ref. 4, our choice of a smooth function minimizes the contribution of the continuum to $(\tilde{F}_1\pm\tilde{F}_2)$ by strongly damping the contribution of the π-π system in the range $2m\le(p_K^2)^{1/2}\le 12m$. We thus have from Eqs. (5) and upon neglect of the continuum[8]

$$F_1(1)=F_2(1)=2\,\frac{M^2-(M-m)^2}{M^2-(M-2m)^2}\,\frac{-Mf_+}{f_\pi}=1.20\,\frac{-Mf_+}{f_\pi}$$

$$=5.30 \tag{8}$$

the factor 1.20 representing the correction to the soft-pion value.

For F_3 we have Eqs. (6) at our disposal. We recall that the K pole, which did not affect $(\tilde{F}_1\pm\tilde{F}_2)$, contributes here to the continuum. The contribution of the states m,m',l,l' (0^- states other than the π and K themselves) are the "PCAC (partial conservation of axial-vector current) correction" terms and no reliable way of taking them into account is known. The states $|n'\rangle=|K,\pi\rangle$ contribute at threshold; this contribution can be calculated and amounts only to a few percent. The contribution of the states n,n' can be estimated by saturation with a "κ" resonance and also amounts to only a few percent. The most sizeable contribution is expected to be that of the k,k' states (0^+) and estimates can be tried saturating with an "ϵ" resonance. These estimates are very uncertain, however, owing to the coupling of these states to the $K\bar{K}$ system. We shall thus try to give no numerical estimate of the contribution of the continuum to Eqs. (6), and we shall neglect it bearing in mind that this may introduce an error of $\sim(30$–$40)\%$ in the evaluation of $\tilde{F}_3(1)$. We thus write

$$\frac{1}{2}\left[\tilde{F}_3(1)+\frac{\sqrt{2}\,f_k}{M}\,T_{\mathrm{th}}(K^+\pi^+\to K^+\pi^+)\right]=\frac{M^2-(M-m)^2}{M^2}\,\frac{-M(f_++f_-)}{f_\pi} \tag{9}$$

and

$$\frac{1}{2}\left[\tilde{F}_3(1)+\frac{\sqrt{2}\,f_k}{M}\,T_{\mathrm{th}}(K^+\pi^-\to K^+\pi^-)\right]=0\ . \tag{10}$$

We next use current algebra and collinear dispersion relations to calculate $T_{\rm th}(K^+\pi^\pm \to K^+\pi^\pm)$. Let

$$T^{ij}(x) = i(m^2 - q^2)^2 \times \int dy\, e^{iqy} \langle K^+(p) | T\phi^j(y)\phi^i(0) | K^+(p)\rangle_c \quad (11)$$

and

$$q = px \ .$$

Also let

$$T^{ij}(x) = T^+(x)\delta^{ij} + i\,\epsilon_{ij3}\, x\, T^-(x) \ ,$$

where $T^\pm(x)$ are even functions of x. Standard soft-pion techniques yield

$$T^+(0) = \Sigma = -\frac{i}{f_\pi^2} \langle K^+(p) | [Q_A, D]_{\rm e.t.} | K^+(p)\rangle \ ,$$

which can be estimated from the Gell-Mann–Oakes–Renner[9] model of chiral symmetry breaking to be

$$T^+(0) = \frac{1}{2}\frac{m^2}{f_\pi^2}$$

and

$$T^-(0) = -\frac{M^2}{f_\pi^2} \ .$$

Also

$$T_{\rm th}(K^+\pi^+ \to K^+\pi^+) = T^{+-}(m/M) = T^+(m/M) + \frac{m}{M} T^-(m/M)$$

and

$$T_{\rm th}(K^+\pi^- \to K^+\pi^-) = T^{+-}(-m/M) = T^+(m/M) - \frac{m}{M} T^-(m/M) \ .$$

The asymptotic behavior of $T^\pm(x)$ is usually inferred from the existence of the equal-time commutators $[\phi, \phi]_{\rm e.t.}$ and $[\phi, \dot{\phi}]_{\rm e.t.}$. Equivalently the Wilson[7] expansion for the operator product $T\phi(y)\,\phi(0)$ can be used, which leads again to the same conclusions (unless $2 < d < 3$, which we exclude), and then

$$T^\pm\left(\frac{m}{M}\right) = T^\pm(0) + \frac{2}{\pi}\left(\frac{m}{M}\right)^2 \int_0^\infty \frac{dx}{x(x^2 - m^2/M^2)} \,{\rm Abs}\,T^\pm(x) \ .$$

Consistently with the approximations which led to Eqs. (9) and (10), we neglect the contributions of the continua (again, threshold and the K contribute only a few percent) and get

$$T_{\rm th}(K^+\pi^+ \to K^+\pi^+) = -6.70 \ ,$$
$$T_{\rm th}(K^+\pi^- \to K^+\pi^-) = 8.90 \ .$$

Equations (17) and (18) now yield two values for $F_3(1)$:

$$F_3(1) = 2.80 \quad \text{and} \quad F_3(1) = 3.70 \ , \quad (13)$$

which are to be compared with the value of Weinberg

$$F_3(1) = \frac{M f_K}{2\sqrt{2}\, f_\pi^2} = 2.35 \ . \quad (14)$$

The transition matrix elements for the K_{l4} decays,

$$M = \frac{G}{\sqrt{2}} \sin\theta\, \bar{u}(\nu)(1-\gamma_5)\gamma_\mu u(-l)\langle \pi, \pi | A^\mu | K\rangle \ ,$$

are used to calculate the decay rates. F_1 and F_2 are taken to be constant over the whole spectrum, whereas the variation of F_3 due to the K pole is taken into account (in our notation $\tilde{F}_3$ is taken to be constant on the mass shell). Integration over phase space yields

$$\Gamma(K_{e4}) = \sin^2\theta[1600 F_1^2(1) + 310 F_2^2(1)]$$

and

$$\Gamma(K_{\mu 4}) = \sin^2\theta[180 F_1^2(1) + 25 F_2^2(1) + 71 {\rm Re} F_1(1) F_3^*(1) + 13 F_3^2(2)] \ .$$

As we have taken $f_+(0) = 1/\sqrt{2}$, the value we use for the Cabibbo angle is the one obtained from the K_{l3} decays[8]

$$\sin\theta = 0.215 \ .$$

The values obtained for the form factors give them

$$\Gamma(K_{e4}) = 2.60 \times 10^3 \ {\rm sec}^{-1} \quad \text{and} \quad \Gamma(K_{\mu 4}) = 0.33 \times 10^3 \ {\rm sec}^{-1} \ , \quad (15)$$

which are to be compared with the values obtained from the uncorrected soft-pion theorems

$$\Gamma(K_{e4}) = 1.5 \times 10^3 \ {\rm sec}^{-1} \ , \quad \Gamma(K_{\mu 4}) = 0.18 \times 10^3 \ {\rm sec}^{-1} \ , \quad (16)$$

and the experimentally measured ones[10]

$$\Gamma(K_{e4}) = (3.0 \pm 0.2) \times 10^3 \ {\rm sec}^{-1} \ ,$$
$$\Gamma(K_{\mu 4}) = (0.7 \pm 0.3) \times 10^3 \ {\rm sec}^{-1} \ .$$

We finally note that a large value of the π-π, s-wave $I = 0$ scattering length would decrease the theoretical values (15) and (16), as was shown by Cabibbo and Maksymowicz.[12]

Part of this work was done while the authors were visiting the International Centre for Theoretical Physics, Trieste. They would like to thank Professor Abdus Salam and Professor P. Budini and the International Atomic Energy Agency and UNESCO for hospitality at the ICTP and also the National Council for Scientific Research of Lebanon for a travel grant.

---

[1]C. G. Callan and S. B. Treiman, Phys. Rev. Lett. 16, 153 (1966).

[2]S. Weinberg, Phys. Rev. Lett. 17, 336 (1966).

[3]L. M. Chounet, J. M. Gaillard, and M. K. Gaillard, Phys. Rep. 4C, 199 (1972); this review article gives a list of references to related work.

[4]N. F. Nasrallah and K. Schilcher, Phys. Rev. D 2, 2698 (1970).

[5]S. Fubini and G. Furlan, Ann. Phys. (N.Y.) 48, 322 (1968).

[6]The validity of the sum rules depended on the existence of the equal-time commutators $[Q_A^{K^-}, \phi]_{\text{e.t.}}$ and $[\phi, \phi]_{\text{e.t.}}$ Equivalently, the asymptotic behavior of the amplitudes can be obtained from the Wilson (Ref. 7) expansion for the operator product $T\phi(y)A_\lambda^{K^-}(0)$ at short distances. Sum rules (5) and (6) are invalidated if $2 < d < 3$ (here d is the scale dimension of ϕ).

[7]K. Wilson, Phys. Rev. 179, 1499 (1969).

[8]We have $f_+ = f_+(0)[1 + (\lambda^+/m^2)(M-m)^2]$, with $f_+(0) = 1/\sqrt{2}$ [the SU(3) symmetry value] and $\lambda^+ = 0.03$, also $f_- \simeq 0$. See G. Donaldson, D. Fryberger, D. Hitlin, J. Liu, B. Meyer, R. Piccioni, A. Rothenberg, D. Uggla, S. Wojcicki, and D. Dorfan, Phys. Rev. Lett. 31, 337 (1973).

[9]M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. 175, 2195 (1968).

[10]Particle Data Group, Phys. Lett. 50B, 1 (1974).

[11]Our two values for $F_3(1)$ give very close values for $\Gamma(K_{\mu 4})$.

[12]N. Cabibbo and A. Maksymowicz, Phys. Rev. 137, B438 (1965); 168, 1926(F) (1968).