Global ground state of ϕ^6 theory in three dimensions*

Paul K. Townsend

Department of Physics, Brandeis University, Waltham, Massachusetts 02154

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We show that the effective potential of three-dimensional ϕ^6 theory to leading order in the $1/N$ expansion may have a minimum at $\phi^2 = 0$ even though the classical potential has a maximum. When this happens the effective potential is double-valued near $\phi^2 = 0$. We discuss some interesting consequences of this but show that provided the ϕ^6 coupling is of normal strength the global minimum of the effective potential remains at nonzero ϕ^2. This confirms earlier results on the spontaneous breaking of the O(N) symmetry of this theory.

I. INTRODUCTION

Investigations into the nature of the ground state of quantum field theories generally involve a perturbative evaluation of the effective potential of those fields which may develop vacuum expectation values. However, in a simple loop expansion we cannot consider the possibility that the true ground state is not to be found from perturbations about the tree-approximation ground state. Recently, several authors have shown how this question can be investigated in O(N)-symmetric $\lambda\phi^4$ theory through the $1/N$ expansion.[1] This was previously thought to be an inconsistent expansion in four dimensions because of the presence of tachyons in the Green's functions.[2] This criticism is not correct because the effective potential, as a real function of ϕ^2, is actually double-valued, with the global minimum occurring at $\phi^2=0$ on a branch not previously considered. The O(N)-symmetric Green's functions constructed about this global minimum are free of tachyons.[1]

In this paper we pursue these issues in another renormalizable scalar field theory with an O(N) internal symmetry, ϕ^6 theory in three dimensions, for which the Lagrangian is

$$\mathcal{L}(\phi) = \tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}m_0{}^2\phi^2 - \frac{\lambda_0\phi^4}{4!N} - \frac{\eta_0\phi^6}{6!N^2}\,. \qquad (1.1)$$

The effective potential for this theory to leading order in $1/N$ was derived in Ref. 3, hereafter referred to as (1). Renormalized results were given only for the derivatives of the effective potential, but the renormalized potential itself can be obtained as easily and is

$$\frac{1}{N}V(\phi) = \tfrac{1}{2}m_0{}^2\left(\frac{\phi^2}{N}\right) - \frac{\lambda\chi^2}{4!} - \frac{\eta\chi^3}{3\times 5!} + \frac{\eta\chi^2}{2\times 5!}\left(\frac{\phi^2}{N}\right) + \frac{\lambda}{12}\chi\left(\frac{\phi^2}{N}\right) - \frac{\hbar}{4\pi}\left(m^2 + \frac{\lambda\chi}{6} + \frac{\eta\chi^2}{5!}\right)^{3/2}, \qquad (1.2)$$

in which we have chosen to use the composite field notation.[5,3] $\chi(\phi)$ is given implicitly as a function of ϕ^2 through the "gap equation"

$$\chi = \frac{\phi^2}{N} - \frac{\hbar}{4\pi}\left(m^2 + \frac{\lambda\chi}{6} + \frac{\eta\chi^2}{5!}\right)^{1/2}. \qquad (1.3)$$

Notice that Eq. (1.2) does reduce to the correct classical potential as $\hbar \to 0$ because, to zeroth order in $\hbar$, $\chi = \phi^2/N$. If Eq. (1.2) is differentiated and use is made of the gap equation we obtain

$$\frac{\partial V(\phi)}{\partial\phi_a} = \left(m^2 + \frac{\lambda\chi}{6} + \frac{\eta\chi^2}{5!}\right)\phi_a\,. \qquad (1.4)$$

To find the minima of $V(\phi)$ we set $\partial V(\phi)/\partial\phi_a$ to zero. In (1) we studied the consequences of the choice

$$m^2 + \frac{\lambda\chi}{6} + \frac{\eta\chi^2}{5!} = 0. \qquad (1.5)$$

With the gap equation for $\chi(\phi)$ we see that this is equivalent to

$$m^2 + \frac{\lambda}{6}\left(\frac{\phi^2}{N}\right) + \frac{\eta}{5!}\left(\frac{\phi^2}{N}\right)^2 = 0, \qquad (1.6)$$

which is the tree-approximation result. What we did *not* consider in (1) was the probability that the other choice,

$$\phi = 0, \qquad (1.7)$$

might provide the global minimum of the effective potential and prevent the spontaneous breaking of the O(N) symmetry, as does happen for four-dimensional $\lambda\phi^4$ theory.[1] At first sight this would seem unlikely because it was shown in (1) that the propagators constructed about a broken-symmetry minimum of $V(\phi)$ are free of tachyons, in contrast with four-dimensional $\lambda\phi^4$ theory. But it would also appear from the gap equation, (1.3), that there might be more than one real branch of $\chi(\phi)$ and hence, through Eq. (1.4), of $V(\phi)$. We will show in the next section that $V(\phi)$ may indeed be double-valued, and that it is therefore important to check whether the global minimum of the effective potential can occur on one of these branches. While we do find some interesting nonperturbative results paralleling those of four-dimensional $\lambda\phi^4$ theory, we also find that the conclusions of (1) concerning spontaneous symmetry breaking remain valid provided the ϕ^6 coupling is not too large.

II. THE GAP EQUATION

We must first find all real solutions to the gap equation, (1.3). By squaring we obtain

$$\left(\chi - \frac{\phi^2}{N}\right)^2 = \left(\frac{\hbar}{4\pi}\right)^2 \left(m^2 + \frac{\lambda\chi}{6} + \frac{\eta\chi^2}{5!}\right). \tag{2.1}$$

This is a quadratic equation with the solutions

$$(1-\zeta)\chi = \frac{\phi^2}{N} + \left(\frac{10\lambda}{\eta}\right)\zeta \pm \zeta^{1/2}\left\{\left(\frac{\phi^2}{N}\right)^2 + \frac{20\lambda}{\eta}\left(\frac{\phi^2}{N}\right) + \frac{5!\,m^2}{\eta} + \left[\left(\frac{10\lambda}{\eta}\right)^2 - \frac{5!\,m^2}{\eta}\right]\zeta\right\}^{1/2}, \tag{2.2}$$

where ζ is the constant

$$\zeta = \frac{\hbar^2\eta}{5!(4\pi)^2}\,. \tag{2.3}$$

The most important fact about the solutions of Eq. (2.2) is that not all are necessarily allowed because in squaring the gap equation we may have introduced spurious solutions. In fact, a solution of Eq. (2.2) is "allowed" only if

$$\chi - \frac{\phi^2}{N} \leq 0 \tag{2.4}$$

because the sign of the square root in Eq. (1.3) is unambiguous. This is a reflection of the fact that

$$\int_0^k \frac{dk}{k^2+M^2} = \frac{\pi}{2} M^{-1/2} \tag{2.5}$$

is unambiguously positive for real M.

We have seen that $\chi - \phi^2/N$ vanishes at the nonzero extrema of the effective potential, and therefore we will plot $\chi - \phi^2/N$ for various choices of m^2, λ, and η. To this end we make several useful observations. Firstly, the large-ϕ^2 behavior of $\chi(\phi)$ is found to be

$$\chi \underset{\phi^2\to\infty}{\sim} \frac{\phi^2}{N}\left[\frac{1\pm\zeta^{1/2}}{(1-\zeta^{1/2})(1+\zeta^{1/2})}\right]. \tag{2.6}$$

For $\zeta<1$ we must choose the minus sign, in order to satisfy Eq. (6.21). This gives us

$$\chi \underset{\phi^2/N\to\infty}{\sim} \frac{\phi^2}{N}\left(\frac{1}{1+\zeta^{1/2}}\right). \tag{2.7}$$

If $\zeta>1$ we may also choose the plus sign, for which

$$\chi \underset{\phi^2/N\to\infty}{\sim} -\frac{\phi^2}{N}\left(\frac{1}{\zeta^{1/2}-1}\right). \tag{2.8}$$

In this case there are two allowed branches of $\chi(\phi)$ for large ϕ^2, but Eq. (2.8) clearly shows the singular nature of the theory at $\zeta=1$. For this reason we will consider only

$$\zeta \ll 1 \tag{2.9}$$

in the following.

Our second observation is that

$$\left(\frac{\phi^2}{N}\right)^2 + \frac{20\lambda}{\eta}\left(\frac{\phi^2}{N}\right) + \frac{5!\,m^2}{\eta} = \frac{2\times 5!}{\eta}U'\left(\frac{\phi^2}{N}\right), \tag{2.10}$$

where the left-hand side appears in Eq. (2.2) and U' is $\partial U/\partial(\phi^2/N)$ with $U(\phi^2/N)$ the classical potential. Note that the minimum of U' occurs at

$$\frac{\phi^2}{N} + \frac{10\lambda}{\eta} = 0, \tag{2.11}$$

with a minimum value of

$$-\left[\left(\frac{10\lambda}{\eta}\right)^2 - \frac{5!\,m^2}{\eta}\right]. \tag{2.12}$$

There are two distinct regions in parameter space to consider. Firstly, if

$$\left(\frac{10\lambda}{\eta}\right)^2 - \frac{5!\,m^2}{\eta} < 0 \tag{2.13}$$

$\chi(\phi)$ and hence $V(\phi)$ are real everywhere. It is not difficult to show in this case that only one branch of $\chi(\phi)$ and $V(\phi)$ is allowed. We plot $\chi - \phi^2/N$ and $V(\phi)$ for this case in Figs. 1(a) and 1(b), respectively. If, on the other hand,

$$\left(\frac{10\lambda}{\eta}\right)^2 - \frac{5!\,m^2}{\eta} > 0, \tag{2.14}$$

there is a nonzero minimum of $U(\phi^2/N)$, and $V(\phi)$ is complex for roughly those values of ϕ^2 for which U' is negative. For example, if $m^2>0$ and $\lambda<0$, there is a nonzero maximum and a nonzero minimum for positive ϕ^2 and an allowed real branch of $\chi(\phi)$ and $V(\phi)$ near $\phi^2=0$. We plot $\chi - \phi^2/N$ and $V(\phi)$ for this case in Figs. 2(a) and 2(b).

If $m^2<0$, U' will be negative from $\phi^2=0$ to the nonzero minimum. We might therefore expect $V(\phi)$ to be complex in this entire region. But in fact $V(\phi)$ may be real close to $\phi^2=0$. The precise condition is that

$$\frac{5!\,m^2}{\eta} + \left[\left(\frac{10\lambda}{\eta}\right)^2 - \frac{5!\,m^2}{\eta}\right]\zeta > 1, \tag{2.15}$$

that is,

$$\frac{-6m^2\eta}{5\lambda^2}\left(\frac{1-\zeta}{\zeta}\right) < 1. \tag{2.16}$$

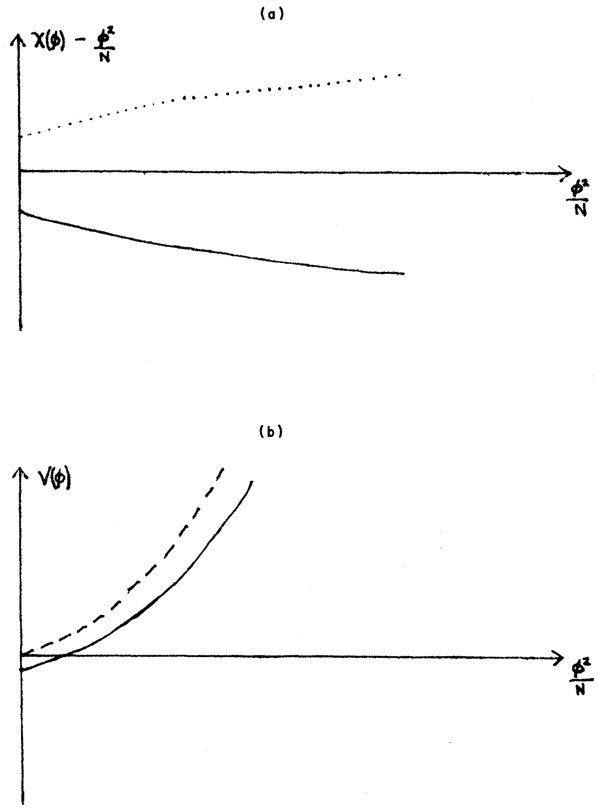

FIG. 1. The symmetric case; no nonzero minimum of $V(\phi)$. Both $\chi(\phi)$ and $V(\phi)$ are single-valued everywhere. (a) $\chi(\phi)-\phi^2/N$; the dotted curve is the spurious solution. (b) $V(\phi)$; the energy is lowered slightly by the radiative corrections. The dashed curve is the classical potential.

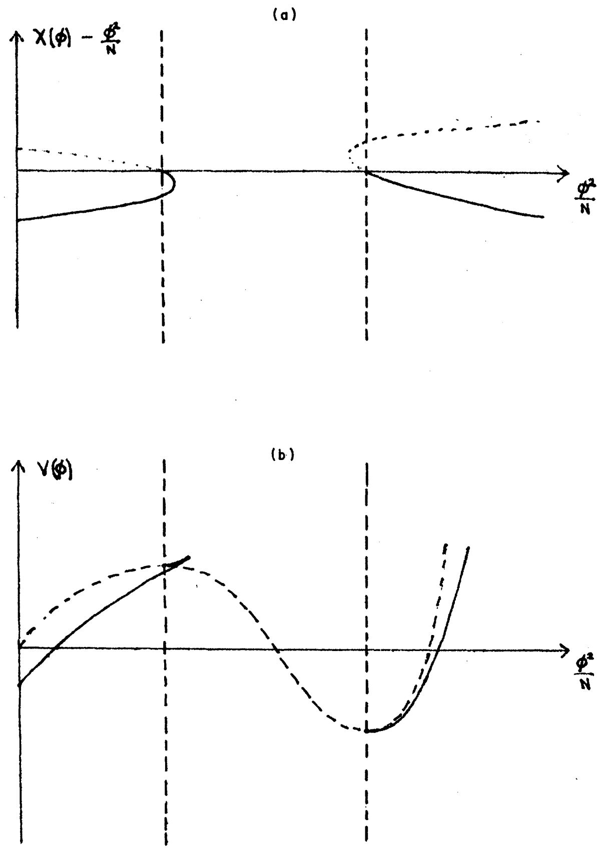

FIG. 2. Symmetry breaking for $m^2>0$, $\lambda<0$. The vertical dashed lines indicate the positions of the nonzero maximum and minimum. Both $\chi(\phi)$ and $V(\phi)$ are complex within these two lines. (a) $\chi(\phi)-\phi^2/N$; the dashed curve is the spurious solution. (b) $V(\phi)$; notice that $V(\phi)$ is double-valued close to the maximum. The dashed curve is the classical potential.

This can be satisfied even for $\zeta \ll 1$ if we choose m^2, λ, and η appropriately. We plot $\chi-\phi^2/N$ and $V(\phi)$ in this eventuality for $\lambda>0$ in Figs. 3(a) and 3(b), and for $\lambda<0$ in Figs. 4(a) and 4(b). We see that if $\lambda>0$ neither branch is allowed and if $\lambda<0$ both are allowed, so that we need consider *only* $\lambda<0$.

To see the significance of a real branch of $V(\phi)$ near $\phi^2=0$ differentiate Eq. (1.4):

$$\frac{\partial^2 V(\phi)}{\partial\phi_a\,\partial\phi_b}=\left[m^2+\frac{\lambda\chi(0)}{6}+\frac{\eta\chi^2(0)}{5!}\right]\delta_{ab}=\left[\frac{4\pi}{\hbar}\chi(0)\right]^2. \qquad (2.17)$$

This shows that $V(\phi)$ is *necessarily a minimum at* $\phi^2=0$ *on any real branch*. It is therefore surprising that such a real branch can exist for $m^2<0$ because *this means that the radiative corrections to leading order in* $1/N$ *have turned a maximum of the classical potential into a minimum of the effective potential*. This phenomenon is just that found for $\lambda\phi^4$ theory. In addition, we argue in Appendix B that the *upper* branch near $\phi^2=0$ will become complex to second order in $1/N$ owing to the presence of tachyons in the Green's functions constructed about a point on this branch. The crucial question for ϕ^6 theory is whether the minimum of $V(\phi)$ at $\phi^2=0$ on the *lower* branch is sufficiently low to prevent the occurrence of spontaneous symmetry breaking. We address this question in the next section.

III. THE GLOBAL GROUND STATE

To find the value of $V(\phi)$ in the *broken-symmetry* minimum we set $\chi=\phi^2/N$ in Eq. (1.2), which becomes

$$\frac{1}{N}\,V(\phi)\Big|_{\phi=\phi_{\min}}=\frac{m^2}{2}\left(\frac{\phi^2}{N}\right)+\frac{\lambda}{4!}\left(\frac{\phi^2}{N}\right)^2+\frac{\eta}{6!}\left(\frac{\phi^2}{N}\right)^3, \qquad (3.1)$$

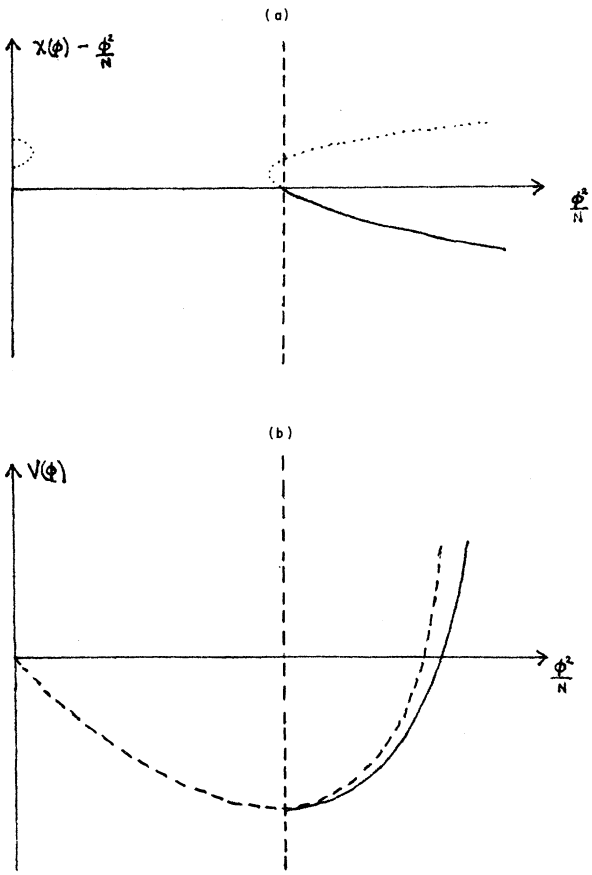

FIG. 3. Symmetry breaking for $m^2 < 0$, $\lambda > 0$. There is no allowed real branch at $\phi^2 = 0$, and $\chi(\phi)$ and $V(\phi)$ are complex to the left of the minimum indicated by the vertical dashed line. (a) $\chi(\phi) - \phi^2/N$; the dotted curve is the spurious solution. (b) $V(\phi)$; the only minimum is at nonzero ϕ^2. The dashed curve is the classical potential.

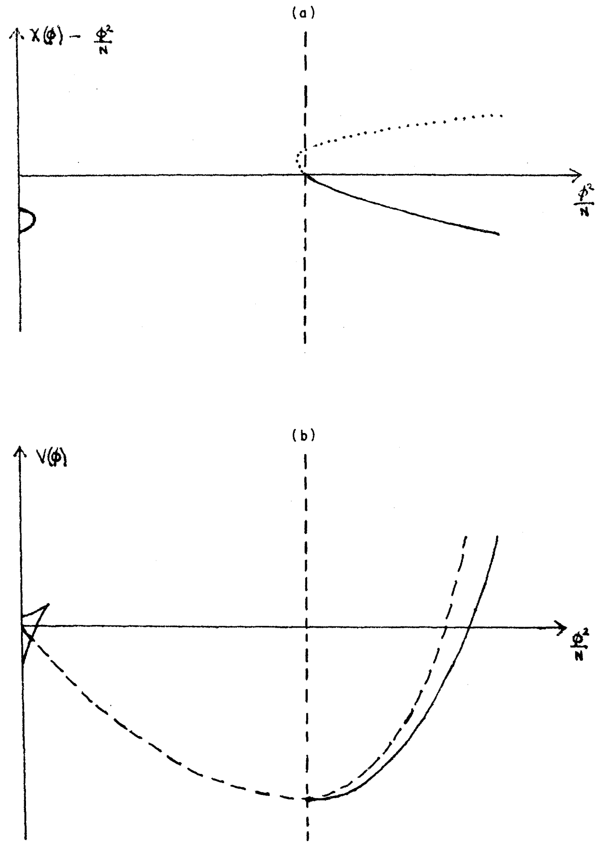

FIG. 4. Symmetry breaking for $m^2 < 0$, $\lambda < 0$, with $K \gg 1$. The vertical dashed line indicates the position of the minimum. $\chi(\phi)$ and $V(\phi)$ are real and double-valued close to $\phi^2 = 0$. (a) $\chi(\phi) - \phi^2/N$; the dotted curve is the spurious solution. (b) $V(\phi)$; there are two minima of $V(\phi)$ at $\phi^2 = 0$, but the nonzero minimum will be the global one provided $\zeta \ll 1$. The upper branch at $\phi^2 = 0$ becomes complex to second order in $1/N$. The lower branch remains real and plunges to $-\infty$ as $\zeta \to 1$.

where ${\phi_{\min}}^2$ is determined from Eq. (1.6). The value of $V(\phi)$ in a *symmetric* minimum is found by setting $\phi^2 = 0$ in Eq. (1.2), which becomes

$$\frac{1}{N} V(0) = -\frac{\lambda \chi^2(0)}{4!} + \frac{\eta \chi^3(0)}{3 \times 5!} \left(\frac{1-\zeta}{\zeta} \right). \qquad (3.2)$$

To compare the values of these two energies we need to solve for $\chi(0)$. From Eq. (2.2) we can write

$$\chi(0) = \frac{10\lambda}{\eta} \left(\frac{\zeta}{1-\zeta} \right) [1 \pm K^{-1/2}(1+K)^{1/2}], \quad m^2 > 0$$
$$= \frac{10\lambda}{\eta} \left(\frac{\zeta}{1-\zeta} \right) [1 \pm K^{-1/2}(K-1)^{1/2}], \quad m^2 < 0 \qquad (3.3)$$

where K is defined by

$$K = \left| \frac{5\lambda^2}{6m^2\eta} \left(\frac{\zeta}{1-\zeta} \right) \right|. \qquad (3.4)$$

There are three ranges of values of K to be considered. These are the following.

1. $K \ll 1$. In this case

$$\chi(0) \simeq -\left[\frac{5!\, m^2}{\eta} \left(\frac{\zeta}{1-\zeta} \right) \right]^{1/2}, \quad m^2 > 0$$

and (3.5)

$\chi(0)$ is complex for $m^2 < 0$.

2. $K \sim 1$. In this case

$$\chi(0) \sim \frac{20\lambda}{\eta} \left(\frac{\zeta}{1-\zeta} \right), \quad m^2 > 0 \qquad (3.6a)$$

and

$$\left.\begin{aligned}\chi(0) &\gtrsim \frac{10\lambda}{\eta}\left(\frac{\zeta}{1-\zeta}\right)\\ &\lesssim \frac{10\lambda}{\eta}\left(\frac{\zeta}{1-\zeta}\right)\end{aligned}\right\}, m^2<0. \tag{3.6b}$$

3. $K \gg 1$. In this case

$$\begin{aligned}\chi(0) &\simeq 0, \quad m^2<0\\ &\simeq \frac{20\lambda}{\eta}\left(\frac{\zeta}{1-\zeta}\right), \quad m^2 \gtrless 0.\end{aligned} \tag{3.7}$$

In the tree approximation $V(0)$ vanishes (with our choice of overall constant). We would expect radiative corrections to alter $V(0)$ slightly and hence alter the precise conditions under which the broken-symmetry minimum of $V(\phi)$ obtains the lower value. We are not attempting to find these conditions here but merely to determine whether the minimum on the lower branch of $V(0)$ at $\phi^2=0$ is so low that it seriously alters the previous conclusions of (1) regarding the regions in parameter space for which the O(N) symmetry is broken. If

$$\chi(0) < \frac{15\lambda}{\eta}\left(\frac{\zeta}{1-\zeta}\right) \tag{3.8}$$

then $V(0)$ is positive. Since this is greater than the tree-approximation value we discount any solution for $\chi(0)$ satisfying the inequality (3.8). There are therefore only two candidates for the global minimum at $\phi^2=0$. These are

$$\chi(0) = -\left[\frac{5!\,m^2}{\eta}\left(\frac{\zeta}{1-\zeta}\right)\right]^{1/2}, \quad K \ll 1 \tag{3.9a}$$

and

$$\chi(0) \simeq \frac{20\lambda}{\eta}\left(\frac{\zeta}{1-\zeta}\right), \quad K\sim 1, \ K \gg 1. \tag{3.9b}$$

We leave the tedious comparison of $V(\phi_{\min})$ with $V(0)$ for these two cases to Appendix A and simply state the result that, provided $\zeta \ll 1$, the global minimum of the effective potential will remain at nonzero ϕ^2. The condition on the coupling constant implied by the restriction on ζ is ($\hbar=1$)

$$\eta \ll (4\pi)^2\, 5! \simeq 4800. \tag{3.10}$$

IV. CONCLUSIONS

We have used the $1/N$ expansion to obtain essentially nonperturbative information about the effective potential and ground state of three-dimensional ϕ^6 theory. In particular, even though the classical potential has a maximum at $\phi^2=0$, the infinite number of graphs in the leading term of the $1/N$ expansion can turn this into a minimum of the effective potential. When this happens the effective potential is double-valued near $\phi^2=0$ and the upper branch becomes complex in the next-to-leading order in $1/N$ owing to the presence of tachyons in the Green's functions constructed about this branch. But in spite of the unexpected minimum on the *lower* branch at $\phi^2=0$ the global minimum of the effective potential remains at nonzero ϕ^2 provided the ϕ^6 coupling is much less than the critical value

$$\eta = (4\pi)^2\, 5!\,. \tag{4.1}$$

The theory is singular at this critical value. One example of this is that the lower branch of the effective potential at $\phi^2=0$, when it exists, drops to $-\infty$ as the critical value of η is reached. While this would ensure the impossibility of symmetry breaking for values of η close to the critical value, it is not clear how far our results may be trusted for such large ϕ^6 couplings, and we choose to restrict η to values much less than that of (4.1), that is, to couplings of "normal" strength. With this restriction we confirm the results of Ref. 3 for the spontaneous breaking of the O(N) symmetry for three-dimensional ϕ^6 theory.

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APPENDIX A

We have found two candidates for a global minimum of the effective potential at $\phi^2=0$, with values of $\chi(0)$ given by Eqs. (3.9a) and (3.9b). To find the energy at $\phi^2=0$ we must substitute these values of $\chi(0)$ into Eq. (3.2). We obtain the following results for the two cases.

1. $K \ll 1$, $m^2>0$. $\chi(0)$ is given by Eq. (3.9a), and the corresponding energy is

$$\frac{1}{N}V(0) \simeq m^2\left(\frac{10}{3}\right)\left[\frac{6}{5\eta}\left(\frac{\zeta}{1-\zeta}\right)\right]^{1/2}; \tag{A1}$$

the approximation is valid for $K \ll 1$.

2. $K\sim 1$, $K \gg 1$. $\chi(0)$ is given by Eq. (3.9b) and the corresponding energy is

$$\frac{1}{N}V(0) \simeq \frac{50}{9}\left(\frac{\zeta}{1-\zeta}\right)^2\frac{\lambda^3}{\eta^2}. \tag{A2}$$

Equations (A1) and (A2) must be compared to $V(\phi_{\min})$ of Eq. (3.1). ${\phi_{\min}}^2$ is the solution of Eq. (1.6) and is

$$\frac{{\phi_{\min}}^2}{N} = -\frac{10\lambda}{\eta} + \left[\left(\frac{10\lambda}{\eta}\right)^2 - \frac{5!\,m^2}{\eta}\right]^{1/2}, \tag{A3}$$

which can be written for $\lambda<0$ as

$$\frac{{\phi_{\min}}^2}{N} = -\frac{10\lambda}{\eta}\left[1 + K'^{-1/2}(K'-1)^{1/2}\right], \tag{A4}$$

where K' is the quantity

$$K' = \left| \frac{5\lambda^2}{6m^2\eta} \right| . \tag{A5}$$

Because ϕ_{min}^2 is complex if $K'<1$ there are only two regions of K' to be considered. These are the following.

1. $K' \sim 1$. From Eq. (A4) we find

$$\frac{\phi_{\text{min}}^2}{N} \sim -\frac{10\lambda}{\eta} \tag{A6}$$

and all the terms of Eq. (3.1) are of the same order of magnitude. Using $K' \sim 1$ in the form $\lambda \sim m\eta^{1/2}$ we find

$$\frac{1}{N} V(\phi_{\text{min}}) \sim \frac{m^3}{\eta^{1/2}} . \tag{A7}$$

2. $K' \gg 1$. From Eq. (A4) we find

$$\frac{\phi_{\text{min}}^2}{N} \sim -\frac{20\lambda}{\eta} \tag{A8}$$

and we may neglect the $m^2(\phi^2/N)$ term in Eq. (3.1) for $K' \gg 1$. We obtain

$$\frac{1}{N} V(\phi_{\text{min}}) \sim \frac{5\lambda^3}{\eta^2}\left(\frac{50}{9}\right) \gg \frac{m^3}{\eta^{1/2}} . \tag{A9}$$

To compare $V(0)$ with $V(\phi_{\text{min}})$ note that $K \ll 1$ implies no restriction on K' (for $\zeta \ll 1$), whereas $K \sim 1$ or $K \gg 1$ implies that $K' \gg 1$. Using the above results we find the following.

1. For $K \ll 1$ we have

$$\left| \frac{V(\phi_{\text{min}})}{V(0)} \right| \gtrsim \left(\frac{1-\zeta}{\zeta}\right)^{1/2} . \tag{A10}$$

2. For $K \sim 1$, $K \gg 1$ we have

$$\left| \frac{V(\phi_{\text{min}})}{V(0)} \right| \simeq \frac{1}{5}\left(\frac{1-\zeta}{\zeta}\right)^{2} . \tag{A11}$$

Equations (A10) and (A11) imply that $V(\phi_{\text{min}}) < V(0)$ if $\zeta \ll 1$, and therefore that the global minimum of the effective potential remains at nonzero ϕ^2. Notice that as ζ approaches unity the lower branch of $V(\phi)$ at $\phi^2 = 0$ drops to $-\infty$. In this case the global minimum would certainly occur at $\phi^2 = 0$, but because of the singular nature of the theory at $\zeta = 1$ (in this approximation) it is not clear that we can trust our results for such large ϕ^6 coupling. We therefore restrict the results to $\zeta \ll 1$.

APPENDIX B

We have seen that even for small ϕ^6 coupling, $\zeta \ll 1$, there may be two real branches of the effective potential, $V(\phi)$, near $\phi^2 = 0$ for certain choices of the parameters, namely $\lambda < 0$ and $K > 1$. It was shown by Root[2] for four dimensional $\lambda\phi^4$ theory that the upper branch of the effective potential becomes complex to second order in $1/N$ due to the presence of the tachyon in the Green's functions constructed about this branch. Abbott, Kang, and Schnitzer showed[1] that the tachyon on the upper branch is caused by the existence of the lower branch, which remains real to second order in $1/N$. By analogy, we would expect that *whenever* there exist two real branches of the effective potential the upper one will become complex in the next order of perturbation theory. To verify this conjecture for ϕ^6 theory in the $1/N$ expansion we require a result of Ref. 4, in which it was shown that the effective potential to second order in $1/N$ for three-dimensional ϕ^6 theory is

$$V = V_N + \frac{\hbar}{2} \int_E \frac{d^3p}{(2\pi)^3} \ln\left[1 + \rho I(p^2) + \frac{\rho\phi^2/3N}{p^2 + M^2}\right], \tag{B1}$$

where V_N is the leading-order result and the suffix E indicates a Euclidean integral. The quantities M^2, ρ, and I are given by[4]

$$M^2 = m^2 + \frac{\lambda\chi}{6} + \frac{\eta\chi^2}{5!} = \left(\frac{4\pi}{\hbar}\right)^2 \left(\chi - \frac{\phi^2}{N}\right)^2 , \tag{B2a}$$

$$\rho = \lambda + \frac{\eta\chi}{10} , \tag{B2b}$$

$$I(p^2) = \frac{\hbar}{48p}\left(\frac{2}{\pi}\right) \arcsin\left(\frac{p^2}{p^2 + 4M^2}\right)^{1/2} . \tag{B2c}$$

With Eqs. (B2) we find the arguments of the logarithm of Eq. (B1) to be (in the limit $p^2 \to 0$)

$$1 + \rho I + \frac{\rho\phi^2/3N}{p^2 + M^2} \xrightarrow[p^2\to 0]{} 1 + \left(\lambda + \frac{\eta\chi}{10}\right)\frac{\hbar}{48\pi M} + \frac{\rho\phi^2}{3NM^2} \tag{B3}$$

and (in the limit $p^2 \to \infty$)

$$1 + \rho I + \frac{\rho\phi^2/3N}{p^2 + M^2} \xrightarrow[p^2\to\infty]{} 1 . \tag{B4}$$

Therefore there will be a zero of the argument and hence a pole of the logarithm if

$$1 + \left(\lambda + \frac{\eta\chi}{10}\right)\frac{\hbar}{48\pi M} + \frac{\rho\phi^2}{3NM^2} \le 0 \tag{B5}$$

for some non-negative value of ϕ^2. Consider $\phi^2 = 0$, for which (B5) becomes

$$1 + \frac{1}{12}\left(\lambda + \frac{\eta\chi(0)}{10}\right)\left(\frac{\hbar}{4\pi}\right)^2 \frac{1}{[-\chi(0)]} \le 0 , \tag{B6}$$

or, equivalently,

$$-\chi(0) < -\frac{10\lambda}{\eta}\left(\frac{\zeta}{1-\zeta}\right) . \tag{B7}$$

From Eqs. (3.6b) and (3.7) we see that this condition is satisfied only on one of the allowed branches at $\phi^2 = 0$, and it is not difficult to show that this corresponds to the upper branch of $V(\phi)$, which therefore becomes complex to second order in $1/N$ owing to the presence of a pole in the Euclidean integral of Eq. (B1). It is obvious that this is also true for some small region near $\phi^2 = 0$, and if our interpretation is correct the entire upper branch will become complex.

The argument of the logarithm of Eq. (B1) is actually just proportional to the inverse χ propagator. This propagator has a pole for Euclidean p^2 and therefore contains a tachyon. It is the tachyon in the intermediate states that causes the upper branch to become complex to second order in $1/N$. This is similar to the behavior of the effective potential of four-dimensional $\lambda\phi^4$ theory; the crucial difference is the *additional symmetry-breaking minimum* of the effective potential of three-dimensional ϕ^6 theory, which is in general also the global minimum.

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[5]The other notation of Ref. 3, $\hbar B(\phi)$, is related to the composite field notation through $\chi(\phi) = \phi^2/N - \hbar B(\phi)$.