Scaling laws taking into account positivity properties

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Recent results concerning the possibility of scaling for the absorptive part of elastic scattering amplitudes are extended and include properties of the multiparticle generating functional. Among other results one finds a high-energy scaling law if the quantity $\langle n^2\rangle/\langle n\rangle^2$ remains bounded. Transformations which respect the existence of a dominant peak are applied both to sums of powers and to a large class of sums of orthogonal polynomials with positive coefficients, and generate the possibility of both upper and lower bounds for the peak parameters. These transformations give us the possibility of constructing from a primitive function, having a dominant peak, a set of functions with the same property. Consequently there exist different possibilities of scaling: from the weakest scaling, where only the primitive function and another one scale inside a part of the dominant peak, to intermediate scaling, where only a finite number of functions of the set scale with only a finite number of derivatives of the primitive limiting function existing, up to the strongest one where any function of the set scales and any derivative of the primitive function exists. This strongest scaling, where the primitive limiting function can be analytic with respect to the scaling variable, corresponds to scaling laws found previously, such as the case $\sigma_T \approx (\log s)^2$ or the Koba-Nielsen-Olesen scaling.

I. INTRODUCTION

The ratio of the elastic absorptive amplitude to its forward part is a function of two variables s and t (s is, as usual, the square of the c.m. energy, and t is the square of the momentum transfer in the c.m. system). We study the possibility that this ratio, when the energy goes to infinity, reduces to a function of a single scaling variable, $\tau = t\gamma(s)$, for physical $\tau<0$ values. A new approach to this problem has been briefly reported recently.[1] Taking into account the positivity properties of the elastic absorptive amplitude and also of the multiparticle generating function, we express such scaling in terms of general requirements on the relevant observables rather than to any particular model [such as $\sigma_T(\log s)^{-2} \to$ const, where σ_T is the total cross section].

As is recalled in Sec. II, this new approach must be compared with previous theoretical results,[2,3] where, from analyticity in the complex t plane when $s\to\infty$, it was shown for the reduced elastic absorptive amplitude that there exist scaling functions $f(\tau = t\gamma(s))$ analytic in the complex τ plane. In these previous cases, the scaling exists for physical $\tau<0$ as well as unphysical $\tau>0$ values. However, either it corresponds to particular cases[2] $\sigma_T(\log s)^{-2}\to$ const or it requires[3] for $\sigma_T(\log s)^{-2}\to 0$, assumptions which appear to be very difficult to test experimentally. In the second section we show, in the case $\sigma_T(\log s)^{-2}\to 0$, that in general this scaling cannot be expected to be derived from assumptions about physical observables alone. We note also the connection between these previous results[3] and a recent popular formulation of the scaling called "geometrical scaling."[4]

In the third section we recall the Arzela theorem, which appears to be very convenient for the proof of scaling properties. We establish how it can be applied to real functions having a "forward peak" due to the existence of positivity properties. Let us call $-T_\alpha(s)$ the smallest momentum transfer value such that this function is equal to a fixed positive number α. If T_α multiplied by the forward slope remains bounded when s increases, there exists a scaling function; the scaling variables are either $\tau = tT_\alpha{}^{-1}$ or $\bar\tau = t$ multiplied by the forward slope. In this way we obtain sufficient conditions for the scaling of either the generating multiparticle function or the absorptive elastic amplitude. For the latter we recall the result[1] that the scaling exists if $\sigma_{el} b/\sigma_T{}^2 \not\to \infty$, with scaling variables $\bar\tau = tb$ or $\bar\tau = t\sigma_T{}^2/\sigma_{el}$ (b is the forward slope and σ_{el} is the elastic cross section).

In the fourth section we define a class of real functions $f(z)$, keeping the existence of a dominant peak at z_0, $|f(z)| \leq f(z_0)$ for $z\in[\bar z, z_0]$ when we apply the transform

$$f \to g = [f(z) - f(z_0)]/(z-z_0)f'(z_0)$$

(we mean that g has also a dominant peak at z_0). Sums of powers, exponentials, and classical orthogonal polynomials (excluding Hermite polynomials) with positive coefficients belong to this class.

In the fifth section we consider such classes of functions (keeping the existence of a dominant peak). It is shown that inside the interval where the peak is dominant these functions have lower and upper bounds when we retain only an even or odd number of terms in the Taylor expansion around the dominant peak. For instance, the straight-line approximation with slope provided by the slope at the peak value is a lower bound, whereas the parabolic approximation where the second derivative at the peak value appears is an upper bound. As an application we get that the elastic absorptive amplitude scales if $c/b^2 \not\to \infty$ (c is the curvature) or that the generating multiparticle function scales if $\langle n^2\rangle/\langle n\rangle^2 \not\to \infty$; the limit functions are differentiable, and the scaling variables are either $\tau = tT_\alpha^{-1}$ or $\tilde{\tau} = tb$, $\tilde{\tau} = t\langle n\rangle$ ($\langle n\rangle$ is the average multiplicity).

By successive applications of the above-defined transformation we can, from a primitive function, define a set of functions having a dominant peak, and we can determine sufficient conditions in order that these functions have scaling properties. In this manner it appears that there exist different degrees of scaling: from the weakest, where the primitive function scales for $\tau < 0$, with the first derivative of the limit function existing, to intermediate scaling, where only a finite number of functions scale, with only a finite number of derivatives of the primitive limiting function existing, up to the strongest case, where any function of the set scales and any derivative of the primitive limiting function exists.

In this way it is shown that this strongest scaling corresponds for the elastic absorptive part to the one found in Refs. 2 and 3 (with scaling for $\tau > 0$) and for the generating multiparticle function to the Koba-Nielsen-Olesen scaling.[5] These properties as well as other applications of these transformations, keeping the existence of a dominant peak for the absorptive elastic amplitude or the generating multiparticle functions, are studied in Secs. VI and VII. In Sec. VIII we give the inequalities satisfied by the scaling functions and in Sec. IX we give some real-part effects.

II. STRONG SCALING OR SCALING VALID FOR COMPLEX VALUES OF THE SCALING VARIABLES IN THE ELASTIC ABSORPTIVE CASE

A. $\sigma_T(\log s)^{-2} \to$ const

In this case Auberson *et al.*[2] have shown that the elastic absorptive part $a(s,t) = A(s,t)/A(s,0) \underset{s\to\infty}{\sim} f(\tau = t(\log s)^2)$ for at least one sequence $(s_n) \to \infty$. Furthermore, $f(\tau)$ is an integer function of order $\frac{1}{2}$, and the forward slope $b \simeq (\log s)^2$.

B. $\sigma_T(\log s)^{-2} \to 0$

Let us assume[3] the following: (i) $\sigma_{el}/\sigma_T^{\,2} \simeq (\log s)^{-\gamma_0}$, $0 < \gamma_0 \leqslant 2$,

$$\gamma_0 = 2 \Rightarrow a(s,t) \underset{s\to\infty}{\sim} f(\tau = t(\log s)^2)$$

as in the previous case; (ii) $0 < \gamma_0 < 2$ and $L_{max} \simeq s^{1/2}(\log s)^{\gamma_0/2}$ (where L_{max} is the maximum number of partial waves entering effectively into the determination of the amplitude in a complex neighborhood of $t=0$). If (i) and (ii) hold, then $b \simeq (\log s)^{\gamma_0}$, $\sigma_T \simeq (\log s)^{\beta}$, $\beta \leqslant \gamma_0$, $\sigma_{el} \simeq (\log s)^{2\beta-\gamma_0}$, and $a(s,t) \underset{s\to\infty}{\sim} f(\tau = t(\log s)^{\gamma_0})$ for at least one sequence $(s_n) \to \infty$, with $f(\tau)$ being an integer function of order $\frac{1}{2}$.

In both cases we have scaling for $\tau > 0$. Moreover, in A σ_T has a very particular behavior, and in B the assumed L_{max} behavior, lower than the Froissart cutoff $L_{max} \simeq s^{1/2}(\log s)$, seems very difficult to be tested experimentally. Let us consider $\beta = \gamma_0$ and call $R^2 = (\log s)^{\gamma_0}$. It follows that $\sigma_T \simeq \sigma_{el} \simeq b \simeq R^2$. Furthermore, let us replace the condition about L_{max} in the t space by a condition in B space (impact-parameter space). Let us assume that that $a(s,B)$, the transform of $a(s,t)$, scales in the form $a(s,R) = a(B/R)$. In this way we get in the t space a scaling $f(tR^2)$. This particular type of scaling is known in the literature as "geometrical scaling."[4]

C. Assumptions about L_{max} and the yield of observables when $\sigma_T(\log s)^{-2} \to 0$

Let us ask the following question: σ_T, σ_{el}, $b, \ldots$ being given, without the assumption (ii) of Sec. IIB, can we hope to prove the scaling for $\tau > 0$ (or $t > 0$)?

The answer is probably no (taking into account only the positivity properties of the partial waves) as we shall see with the following example. Let us write the expansion near $t=0$ for a spinless absorptive elastic amplitude $a(s,t) = 1 + bt + \frac{1}{2}ct^2 +$ (positive terms for $t > 0$) and consider the following distribution of imaginary partial waves a_l: $a_l \simeq$ const for $ls^{-1/2}(\log s)^{-1/2} \in [c_1, c_2]$ and $a_l \simeq$ const $(\log s)^{-2}$ for $ls^{-1/2}(\log s)^{-1} \in [c_3, c_4]$; the other a_l are negligible. We get $\sigma_{el} \simeq \sigma_T \simeq b \simeq \log s \simeq R^2$, $c \simeq (\log s)^3$, whereas if we put $\tau = tR^2$ we get for $\tau > 0$

$$a(s,t) = 1 + \text{const} \times \tau + \text{const} \times \tau^2(\log s) + (\text{positive terms}),$$

and at fixed $\tau > 0$,

$$a(s,t) \underset{s\to\infty}{\to} \infty.$$

More generally, if the distribution of a_l near the Froissart cutoff does not decrease more than logarithmically, we can always manage a distribution of a_l such that the nth term in the $a(s,\tau)$ expansion diverges for $\tau > 0$ when $s \to \infty$.

In conclusion, if we want to find conditions where only the observables σ_{el}, σ_T, b appear as sufficient assumptions for the existence of scaling, we must give up to the $\tau > 0$ case[1,2] and seek scaling only in the $\tau < 0$ physical domain. (For further comments see the general conclusion.)

III. ARZELA THEOREM[6] AND SCALING CONDITIONS

We recall the results obtained in collaboration with Martin[1] concerning the elastic absorptive amplitude and extend these results to the generating multiparticle function.

A. Arzela theorem

Let us consider $a(s,\tau)$, a real function defined for $\tau\in[\tau_1,\tau_2]$ with s being a parameter, and we assume

$$\begin{aligned}&\text{(i)}\quad |a(s,\tau)|<C_1,\ \text{where } C_1 \text{ is independent of } s,\\ &\text{(ii)}\quad a(s,\tau) \text{ is equicontinuous.}\end{aligned} \tag{1}$$

The theorem says that from any sequence $(s_n)\to\infty$, there exists a subsequence $(s_n')\to\infty$ such that $a(s_n',\tau)\underset{n\to\infty}{\to}f(\tau)$; $f(\tau)$ is continuous on $[\tau_1,\tau_2]$. (However, for another sequence there could exist another limit.) Let us remark that the equicontinuity property (ii) is satisfied if

$$\left|\frac{\partial a}{\partial\tau}(s,\tau)\right|<C_2,\quad \text{where } C_2 \text{ is independent of } s. \tag{2}$$

B. Application to a function having a "forward peak"

Let us say that a real function $a(s,t)$ has a forward peak if for $t\in[0,-T_0]$ we have

$$\text{(i)}\quad |a(s,t)|\leqslant 1, \tag{3}$$

$$\begin{aligned}&\text{(ii)}\quad \left|\frac{\partial a}{\partial t}\right|\leqslant\left(\frac{\partial a}{\partial t}\right)_{t=0}=\dot a(0),\\ &\text{(iii)}\quad \beta=\sup_{\forall s>s_0} a(s,-T_0)<1.\end{aligned} \tag{4}$$

Let us define a fixed α value $0\leqslant|\beta|<\alpha<1$, and for each fixed s value we define $T_\alpha(s)$ $(T_\alpha<T_0)$ as the smallest momentum transfer such that

$$a(s,-T_\alpha(s))=\alpha,\quad 0\leqslant|\beta|<\alpha<1,\quad 0<T_\alpha<T_0. \tag{5}$$

1. Scaling variable

First let us define as a scaling variable

$$\tau=t/T_\alpha,\quad \tau\in[-1,0]. \tag{6a}$$

It follows that the first Arzela condition (1) is satisfied. The second condition is satisfied if

$$\left|\frac{\partial a}{\partial\tau}\right|=T_\alpha\left|\frac{\partial a}{\partial t}\right|<T_\alpha\dot a(0)\underset{s\to\infty}{\not\to}\infty. \tag{7}$$

If the condition (7) holds, $a(s,\tau)$ scales for at least one sequence $(s_n)\to\infty$. This scaling is certainly not trivial because the scaling function $f(\tau)$ is continuous and takes the values α and 1 for $\tau=-1$ and 0.

2. Another scaling variable

Secondly, let us try to choose another scaling variable

$$\tilde\tau=t\dot a(0),\quad \tilde\tau\in[0,-\dot a(0)T_0]. \tag{6b}$$

In this case, from (3) and (4) we get $|a(s,\tilde\tau)|<1$, $|\partial a/\partial\tilde\tau|<1$, and the conditions (1) and (2) of the the Arzela theorem are satisfied; however, we must verify that the scaling function is not trivially equal to 1.

In Sec. V we shall define a class of functions, keeping the existence of the peak at $t=0$ for those through particular transformations (for instance, the elastic absorptive amplitude and the generating multiparticle function belong to this class).

(i) We assume that a belongs to this class. It follows (see Sec. VI) that T_α defined by $a=\alpha\neq1$ has a lower bound $\dot a(0)T_\alpha>1-\alpha$. Let us call $\tilde\tau_\alpha=-T_\alpha\dot a(0)$ the corresponding value for which $\alpha(s,\tilde\tau_\alpha)=\alpha$.

(ii) We assume that (7) is satisfied or $T_\alpha\dot a(0)<C$, with C a fixed number. From (i) and (ii) follows that $-\tilde\tau_\alpha$ has fixed upper and lower constant bounds $1-\alpha<-\tilde\tau_\alpha<C$, and the scaling is certainly nontrivial if

$$[-(1-\alpha),-C]\subset]0,-T_0\dot a(0)[. \tag{8}$$

C. Spinless elastic absorptive amplitude

We take

$$\begin{aligned}&a(s,t)=\sum(2l+1)\left(\frac{a_l}{A(s,0)}\right)P_l(z),\\ &z=1+t/2k^2,\quad a_l>0,\quad a(s,1)=1,\end{aligned} \tag{9}$$

a has a "forward peak" for $z\in]-1,1]$ because it satisfies (3) and (4), and with the variables $\tau=tT_\alpha^{-1}$ or $\tilde\tau=tb^{\text{ABS}}$, $b^{\text{ABS}}=\dot a(0)$,

$$\text{scaling exists if}\quad b^{\text{ABS}}T_\alpha(s)\underset{s\to\infty}{\not\to}\infty. \tag{10}$$

Assuming $T_\alpha<4k^2$ and applying the unitarity relation, we get

$$\frac{\sigma_{\text{el}}^{\text{ABS}}}{\sigma_T^{\;2}}>\lambda\int_{-T_\alpha}^{0}[a(s,t)]^2dt>\lambda\alpha^2T_\alpha \tag{11}$$

(with λ a known constant).

Let us consider $\tau=tT_\alpha^{-1}$ or $\tilde\tau=tb^{\text{ABS}}$. Using (11), the condition (7) can be written with other observables:

$$\text{scaling exists if}\quad\left(\frac{\sigma_{\text{el}}^{\text{ABS}}}{\sigma_T}\right)\frac{b^{\text{ABS}}}{\sigma_T}\underset{s\to\infty}{\not\to}\infty. \tag{12}$$

Let us remark that (12) is satisfied if $\sigma_{\text{el}}^{\text{ABS}}/\sigma_T^{\;2}\simeq(\log s)^{-\gamma_0}$, $b^{\text{ABS}}\simeq(\log s)^{\gamma_0}$ or if $\sigma_{\text{el}}^{\text{ABS}}\simeq\sigma_T\simeq b^{\text{ABS}}\simeq R^2$

We recall that sufficient conditions, in order that b^{ABS} and b have the same energy behavior (if b^{ABS}

is sufficiently slowly varying at high energy and the antisymmetric amplitude is negligible), have been obtained recently.[7] From the experimental point of view, $\sigma_{el} b(\sigma_T)^{-2}$ seems bounded[8] and is less than $\sigma_T b(\sigma_{in})^{-2}$, which also seems bounded.[9]

Let us assume $\sigma_{el}^{ABS} b^{ABS}/\sigma_T^2 < C'$ and try $\tilde{\tau} = t\sigma_T^2(\sigma_{el}^{ABS})^{-1}$ as a scaling variable. We have $|a| < 1$ and $|\partial a/\partial \tilde{\tau}| < C'$. If again T_α is such that $a(s, -T_\alpha) = \alpha \neq 1$, the corresponding $\tilde{\tau}_\alpha$ value, $-\tilde{\tau}_\alpha = T_\alpha \sigma_T^2 (\sigma_{el})^{-1}$, has fixed upper and lower constant bounds $(C')^{-1}(1-\alpha) < -\tilde{\tau}_\alpha < \lambda^{-1}\alpha^{-2}$, where λ is the constant appearing in the unitarity relation (11). If bs is larger than a sufficiently large constant, then $\tilde{\tau}_\alpha$ is always inside the interval dominated by the forward peak and the scaling is nontrivial. If Eq. (12) is satisfied with σ_{el}^{ABS} replaced by σ_{el}, then we can choose $\tilde{\tau} = t\sigma_T^2 \sigma_{el}^{-1}$ as the scaling variable.

D. Multiparticle generating function

We take

$$a(s,z) = \sum_{n_0} z^n \gamma_n, \quad n_0 > 0, \quad \gamma_n = \frac{\sigma_n}{\sigma_T} > 0, \quad z \in [0,1], \qquad (13)$$
$$a(s,1) = 1, \quad z = 1+t, \quad t \in [-1, 0]$$

(σ_n is the n-particles production cross section). a has a forward peak because it satisfies (3) and (4), and for $a(s,\tau)$, taking as scaling variable either $\tau = tT_\alpha^{-1}$ or $\tilde{\tau} = t\langle n\rangle$, we get the following:

scaling exists if

$$\langle n\rangle T_\alpha(s) \not\to \infty, \quad \langle n\rangle = \frac{\partial a}{\partial t}\bigg|_{t=0} = \frac{\partial a}{\partial z}\bigg|_{z=1}.$$

Applying the "unitarity relation"

$$\sigma_T^{-2} \sum \frac{\sigma_n \sigma_m}{n+m+1} = \int_1^0 [a(s,z)]^2 dz > \int_{-T_\alpha}^0 a^2 dt > \alpha^2 T_\alpha, \qquad (14)$$

condition (14) becomes the following:

$$\text{scaling exists if } \left(\sum \frac{n\sigma_n}{\sigma_T}\right)\left(\sum \frac{\sigma_n\sigma_m}{n+m+1}\right)\sigma_T^{-2} \not\to \infty. \qquad (15)$$

If we summarize the results obtained in this section, we see that in both cases C and D the important point is the existence of a "forward peak" due to an expansion with two properties:

(i) The polynomials z^n or P_n also have a "forward peak."

(ii) The coefficients a_n or σ_n/σ_T are positive.

In the next section we try to construct more general functions having properties similar to (i) and (ii).

IV. TRANSFORMATIONS KEEPING THE EXISTENCE OF A DOMINANT PEAK

Let us define $\mathcal{S}$, the space of real functions $f(z)$, such that

(1) $f(z)$ has a "dominant peak" $|f(z)| \leq f(z_0) = 1$ for $z \in [\bar{z}, z_0]$, $|f'(z)| \leq f'(z_0)$ if $\bar{z} < z_0$, $|f'(z)| \leq -f'(z_0)$ if $\bar{z} > z_0$.

(ii) If we define the transform

$$f \xrightarrow{T} g, \quad g(z) = \frac{1-f(z)}{(z_0 - z)f'(z_0)}, \quad f'(z_0) = \frac{\partial f}{\partial z}\bigg|_{z=z_0}, \qquad (16)$$

then $g(z)$ also has a "dominant peak" $|g(z)| \leq g(1) = 1$ for $z \in [\bar{\bar{z}}, z_0]$, $|g'(z)| \leq g'(z_0)$ if $\bar{\bar{z}} < z_0$, $|g'(z)| \leq -g'(z_0)$ if $\bar{\bar{z}} > z_0$. In other words, if $f \in \mathcal{S}$, then the transform $f \xrightarrow{T} g$ keeps the existence of a "dominant peak."

Concerning the reciprocal transform, let us remark from (16) that necessarily $g(z) \geq 0$ for $z \in [\bar{z}, z_0]$. This means that from any given $f(z)$ having a "dominant peak," there does not exist, in general, a corresponding $h(z)$ such that $h \xrightarrow{T} f$.

It is outside the scope of the present paper to give a complete study of $\mathcal{S}$; we restrict ourselves to particular cases which illustrate the role of the positivity property.

Let us assume that f is a sum of real $h_n(z)$ functions with positive coefficients,

$$f(z) = \sum_{n_0} \gamma_n \frac{h_n(z)}{h_n(z_0)}, \quad f(z_0) = 1, \quad h_n(z_0) \neq 0, \quad \gamma_n > 0, \qquad (17)$$

such that h_n has a "dominant peak"

$$|h_n(z)| \leq h_n(z_0) \text{ for } z \in [\bar{z}, z_0], \qquad (18)$$
$$|h_n'(z)| \leq h_n'(z_0) \text{ if } \bar{z} < z_0, \quad |h_n'(z)| \leq -h_n'(z_0) \text{ if } \bar{z} > z_0.$$

It follows that $f(z)$ automatically has a dominant peak. If we find classes of $h_n(z)$ such that the transform defined by (16) leads to a g function, which is also a sum of $h_n(z)$ functions with positive coefficients, then $g(z)$ will also automatically have a dominant peak. In A and B below we provide such types of examples, whereas in C we give another example of $f \in \mathcal{S}$, where f and g are sums of different kinds of h_n.

A. $h_n = z^n$

Theorem I. If $f(z) = \sum_{n_0} \gamma_n z^n$, $\gamma_n > 0$, $\bar{z} = 0$, $z_0 > 0$, $n_0 \geq 0$, $f(z_0) = 1$, then g defined by (16) can be written

$$g(z) = \sum \beta_n z^n, \qquad (19)$$

with $\beta_n > 0$, $g(1) = 1$, $\beta_{n+1}/z_0 \geq \beta_n$ (the proof is given in Appendix A). So in this case $f \in \mathcal{S}$.

B. Classical orthogonal polynomials

Theorem II. Let p_n be the classical orthogonal polynomials,[10] $\langle p_n, p_n\rangle = \delta_n > 0$; k_n is the coefficient of x^n.

(i) If η, the sign of $k_n k_{n+1}$, is n independent, always $+1$ or always -1 (this property is true for Jacobi as well as Hermite or generalized Laguerre polynomials), (ii) if there exists a z_0 value such that $p_n(z_0) > 0$ for any n values, and (iii) if

$$f(z) = \sum_{n_0} \gamma_n \frac{p_n(z)}{p_n(z_0)}, \quad \gamma_n > 0, \quad f(z_0) = 1, \quad n_0 \geq 0, \tag{20}$$

then $g(z)$ defined by (16) can be written,

$$g(z) = \sum_{n_0'} \beta_n p_n(z_0) p_n(z) (\delta_n)^{-1},$$

with

$$\begin{aligned} &f'(z_0)\eta\beta_n > 0, \\ &f'(z_0)\eta(\beta_{n+1} - \beta_n) > 0, \\ &\lambda_n = \delta_n |k_{n+1}| [|k_n| p_n(z_0) p_{n+1}(z_0)]^{-1}, \\ &(f'(z_0)\eta)[\lambda_{n-1}{}^{-1}\beta_{n-1} - \beta_n(\lambda_n{}^{-1} + \lambda_{n-1}{}^{-1}) + \lambda_{n+1}{}^{-1}\beta_{n+1}] > 0. \end{aligned} \tag{21}$$

The proof is given in Appendix A; the main tool is the Christoffel-Darboux formula. For Jacobi polynomials (which include Legendre, Gegenbauer, and Tchebichef polynomials) $z_0 = 1$ and $\bar{z} = \bar{\bar{z}} < 1$ (see Appendix A for the values of $\bar{z}$), $\eta > 0$, $f'(z_0) > 0$. It follows that f as well as the Jacobi polynomials belong to $\mathcal{S}$. For generalized Laguerre polynomials, $z_0 = 0$, $\bar{z} > z_0$ (see Appendix A for $\bar{z}$), $f'(z_0) < 0$, $\eta < 0$. Here also f as well as these polynomials belong to $\mathcal{S}$. For Hermite polynomials we have not found any z_0 value such that $p_n(z)$ have for all n a dominant peak at z_0, and so this is, among the classical polynomials, the only case for which our results cannot be applied.

C. f and g are sums of different kinds of h_n functions

Clearly the above A and B cases correspond to particular classes of $f \in \mathcal{S}$. It is not necessary that f and g be sums of the same types of h_n functions. In order to illustrate this point we give now another example of $f \in \mathcal{S}$, which is an important physical case of the generating multiparticle function.

Let us consider

$$f(z) = \sum_{n_0} e^{nz}\gamma_n, \quad \gamma_n > 0, \quad \sum \gamma_n = 1. \tag{22}$$

In this case $z_0 = 0$ and $\bar{z} = -\infty$. It is easy to verify that for $z \leq 0$ we have $f(z) \leq f(0)$ and $|f'(z)| \leq f'(0)$. g defined by (16) can be written

$$g = \sum \frac{n\gamma_n}{\langle n\rangle}\left(\frac{e^{nz} - 1}{nz}\right), \quad f'(0) = \langle n\rangle = \sum n\gamma_n. \tag{23}$$

Because $(e^x - 1)x^{-1} \leq 1$ for $x \leq 0$, we see that $g(z) \leq g(0) = 1$ for $z < 0$. Similarly, $g'(z) < g'(0)$ for $z < 0$. Thus, obviously, $f \in \mathcal{S}$. Moreover, as we shall see, g itself belongs to $\mathcal{S}$. If we define $g \xrightarrow{T} h$ by the transform defined in (16), then

$$h = \sum n^2 2\gamma_n \left(\frac{1 - e^{nz} + nz}{-n^2 z^2}\right). \tag{24}$$

Because $(2/x^2)(e^x - 1 - x) \leq 1$ for $x \leq 0$, we see that, for $z < 0$, $h(z) < h(0) = 1$. Similarly, one easily finds $h'(z) < h'(0)$ for $z < 0$. More generally, let us put $f = f_0$ in Eq. (22), $f_1 = g$ in Eq. (23), $f_2 = h$ in Eq. (24), and so on; $f_0 \xrightarrow{T} f_1 \xrightarrow{T} f_2 \cdots f_{q-1} \xrightarrow{T} f_q$.
Using obvious inequalities derived from the expotential-type function it is easy to verify $f_q \in \mathcal{S}\,\forall q$.

D. Set $\{f_q\}$ of functions, keeping the existence of the forward peak

Let us define

$$f_{q-1} \xrightarrow{T} f_q = \frac{f_{q-1}(z_0) - f_{q-1}(z)}{(z_0 - z) f'_{q-1}(z_0)}, \quad q = 1, 2, \ldots,$$

where f_0 belongs to one of the three classes considered above such that not only $f \in \mathcal{S}$ but $f \xrightarrow{T} g$ with $g \in \mathcal{S}$. From f_0 we construct a set f_q of functions, keeping the existence of the forward peak. We will consider such sets $\{f_q\}$ in the following section.

V. GENERAL RESULTS FOR TRANSFORMATIONS KEEPING THE EXISTENCE OF A DOMINANT PEAK

Let us consider the set $\{f_q\}$ deduced from $f_0 \in \mathcal{S}$ by the transform

$$f_{j-1} \xrightarrow{T} f_j = \frac{1 - f_{j-1}}{(z_0 - z) f'_{j-1}(z_0)}, \quad j = 1, 2, \ldots, q-1.$$

Inside the peak $z \in [\bar{z}, z_0]$ for which $f_0 \in \mathcal{S}$, let us remark that $(z_0 - z) f'_{j-1}(z_0)$ is always negative in both cases $\bar{z} \leq z \leq z_0$ and $z_0 \leq z \leq \bar{z}$, where the peak is at the beginning or at the end of the interval.

A. Lower and upper bounds

From $f_j \leq 1$ we get

$$\left.\begin{aligned} &f_0 \geq 1 + (z - z_0) f_0'(z_0) \\ &f_0 \leq 1 + (z - z_0) f_0'(z_0) + \tfrac{1}{2}(z - z_0)^2 f_0''(z_0) \end{aligned}\right\} z \in [\bar{z}, z_0]. \tag{25}$$

Property I. More generally, if we retain an even (odd) number of terms in the Taylor expansion near z_0, we have lower (upper) bounds for $f_q(z)$ inside the peak $z \in [\bar{z}, z_0]$,

$$f_q(z) \geq 1+(z-z_0)f_q'(z_0)+\cdots+\frac{(z-z_0)^{2p+1}}{(2p+1)!}f_q^{(2p+1)}(z_0),$$

$$f_q(z) \leq 1+(z-z_0)f_q'(z_0)+\cdots+\frac{(z-z_0)^{2p+2}}{(2p+2)!}f_q^{(2p+2)}(z_0)$$

$$q=0,1,2,\ldots;\quad p=0,1,\ldots;\quad z\in[\bar{z},z_0]. \tag{26}$$

We introduce the following change of variable $(z-z_0)=t\mu$ where μ can depend on a parameter (for instance, $\mu=1/2k^2$ for the absorptive elastic amplitude). The set $\{f_q\}$ can also depend on a parameter that we do not write down at this stage. Now the peak is always at $t=0$ and is dominant on an interval $(\bar{t},0)$ corresponding to $(\bar{z},z_0)$. Here $\bar{t}<0$ or $\bar{t}>0$, depending upon whether the peak is at the end or at the beginning of the interval. In the following we always consider $\bar{t}<0$ (but it is straightforward to extend the results for $\bar{t}>0$). So $\partial^p f_q(z)/\partial z^p|_{z_0}$ as well as $\partial^p f_q(t)/\partial t^p|_{t=0}$ are positive.

Similarly, as in Eq. (5) let us define

$$f_q(s,-T_{\alpha_q})=\alpha_q,\quad 0<\alpha_q<1,\quad \bar{t}\leq -T_{\alpha_q}\leq 0, \tag{27}$$

where we assume that $\{f_q\}$ depends on some parameter s, and if we fix α_q in (27), this defines T_{α_q} as a function of this parameter.

B. Upper and lower bounds for T_{α_q} and $T_{\alpha_q}\dot{f}_q(0)$ and application to scaling

We recall that, from the Arzela theorem, the boundedness of $T_{\alpha_q}\dot{f}_q(0)$ is a sufficient condition for the existence of scaling. We put $\partial f_q/\partial t|_{t=0}=\dot{f}_q(0)$. The inequalities (25) can be written (see Fig. 1)

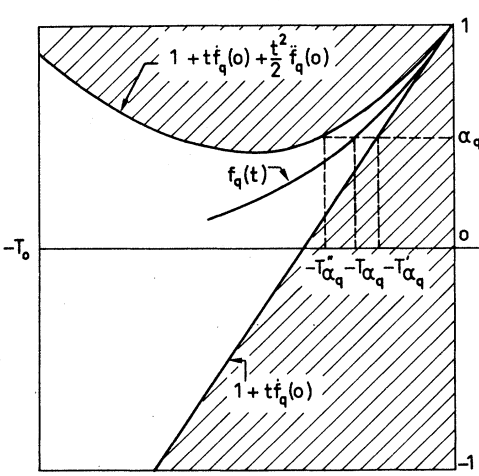

FIG. 1. First lower bound and first upper bound of $f_q(t)$.

$$1+t\dot{f}_q(0)\leq f_q(t)\leq 1+t\dot{f}_q(0)+\tfrac{1}{2}t^2\ddot{f}_q(0),\quad \bar{t}\leq t\leq 0, \tag{28}$$

$$1-T_{\alpha_q}\dot{f}_q(0)\leq \alpha_q\leq 1-T_{\alpha_q}\dot{f}_q(0)+\tfrac{1}{2}T_{\alpha_q}^{\;2}\ddot{f}_q(0). \tag{29}$$

1. Lower bound

From the inequality at the left-hand side of (29) we get

$$T_{\alpha_q}\geq\frac{1-\alpha_q}{\dot{f}_q(0)}=T'_{\alpha_q} \tag{30}$$

2. Upper bound for any $q\geq 0$ coming from the parabolic upper bound approximation

Let us assume $\ddot{f}_q(0)/[\dot{f}_q(0)]^2\not\to\infty$ when $s\to\infty$ and define

$$\sup_{\forall s}\frac{\ddot{f}_q(0)}{[\dot{f}_q(0)]^2}=\frac{1}{2(1-\bar{\alpha}_q)},\quad \bar{\alpha}_q<1\,. \tag{31}$$

In general (see property III), $\bar{\alpha}_q\neq-\infty$, and we restrict ourselves to this case. If $\bar{\alpha}_q>0$, we take $\bar{\alpha}_q<\alpha_q<1$ and otherwise $0<\alpha_q<1$. If (31) is satisfied, then the parabolic curve $y=1+t\dot{f}_q(0)+\frac{1}{2}t^2\ddot{f}_q(0)$ and the line $y=\alpha_q$ intersect at the value $-t=T''_{\alpha_q}$, which provides an upper bound for T_{α_q},

$$T_{\alpha_q}\leq T''_{\alpha_q}=\frac{\dot{f}_q(0)}{\ddot{f}_q(0)}\left[1-\left(1-2(1-\alpha_q)\frac{\ddot{f}_q(0)}{[\dot{f}_q(0)]^2}\right)^{1/2}\right]. \tag{32}$$

We maximize the right-hand side of the inequality and get

$$1-\alpha_q\leq T_{\alpha_q}\dot{f}_q(0)\leq 2(1-\bar{\alpha}_q)\left[1-\left(\frac{\alpha-\bar{\alpha}_q}{1-\bar{\alpha}_q}\right)^{1/2}\right],$$
$$T_{\alpha_q}^{\;2}\ddot{f}_q(0)\leq[2(1-\bar{\alpha}_q)]\,. \tag{33}$$

Firstly, we take $\tau=tT_{\alpha_q}^{\;-1}$. From Eq. (33) we see that $|\partial^2 f_q/\partial\tau^2|$ as well as $|\partial f_q/\partial\tau|$ are uniformly bounded, and consequently $\partial f_q(s,\tau)/\partial\tau$ is equicontinuous. By a trivial extension of the Arzela theorem one can show that from any sequence $(s_n)\to\infty$, one can extract a subsequence $(s_n')\to\infty$ such that $f_q(s_n',\tau)\to f_q(\tau)$ and $\partial f_q(\tau)/\partial\tau$ exist for $\tau\in[-1,0]$. Secondly, we take $\bar{\tau}=t\dot{f}_q(0)$ as a scaling variable. We have $|\partial f_q/\partial\bar{\tau}|<1$ and

$$\partial^2 f_q(s,\bar{\tau})/\partial\bar{\tau}^2<\ddot{f}_q(0)/[\dot{f}_q(0)]^2<[2(1-\bar{\alpha}_q)]^{-1}\,.$$

Consequently, the scaling function $f_q(\bar{\tau})$ as well as its first derivative $\partial f_q(\bar{\tau})/\partial\bar{\tau}$ exist. The scaling is not trivial because for $f_q=\alpha_q$ corresponds $t=-T_{\alpha_q}$, and $-\bar{\tau}_{\alpha}=T_{\alpha_q}\dot{f}_q(0)$ has a lower bound and an upper bound which are finite; $(1-\alpha_q)<-\bar{\tau}_{\alpha_q}<2(1-\bar{\alpha}_q)$.

Property II. If $\ddot{f}_q(0)/[\dot{f}_q(0)]^2\not\to\infty$, there exist α_q values ($0<\alpha_q<1$, $\alpha_q>\bar{\alpha}_q$) such that $T_{\alpha_q}\dot{f}_q(0)$ and $T_{\alpha_q}^{\;2}\ddot{f}_q(0)$ are bounded and f_q scales either with the scaling variables $\tau = tT_{\alpha_q}^{-1}$ or $\bar{\tau} = t\dot{f}_q(0)$; in both cases the limit functions are differentiable.

Let us define the Taylor expansion of the primitive function $f_0(s,t)$,

$$f_0 = 1 + \sum \frac{t^n}{n!} f_0^{(n)}(0)$$

and

$$D_q = \frac{f_0^{(q+1)} f_0^{(q-1)}}{(f_0^{(q)})^2}. \tag{34}$$

For $q \geq 1$ we have the relations

$$\frac{\dot{f}_q(0)}{\dot{f}_{q-1}(0)} = \frac{1}{2} \frac{\ddot{f}_{q-1}(0)}{[\dot{f}_{q-1}(0)]^2} = \frac{q}{q+1} D_q. \tag{35}$$

3. *Upper bounds in the case $q \geq 1$, taking into account the fact that* $\inf f_{q-1}$ *is bounded*

Inside the real interval of the dominant peak, we know that f_{q-1} has lower bounds: for instance, $f_0 > -1$ and $f_q \geq 0$ for $q \geq 1$. Applying the definition

$$T_{\alpha_q} \dot{f}_{q-1}(0) f_q(s, -T_{\alpha_q}) = 1 - f_{q-1}(s, -T_{\alpha_q}),$$

we get

$$T_{\alpha_q} \leq (1 - \inf f_{q-1})[\alpha_q \dot{f}_{q-1}(0)]^{-1},$$

and also

$$(1-\alpha_q) \leq T_{\alpha_q} \dot{f}_q(0) \leq \left(\frac{1-\inf f_{q-1}}{\alpha_q}\right)\left(\frac{\dot{f}_q(0)}{\dot{f}_{q-1}(0)}\right),$$

$$\frac{\dot{f}_{q-1}(0)}{\dot{f}_q(0)}(1-\alpha_q) \leq T_{\alpha_q} \dot{f}_{q-1}(0) \leq \frac{1-\inf f_{q-1}}{\alpha_q}, \tag{36}$$

$$T_{\alpha_q}^2 \ddot{f}_{q-1}(0) \leq 2\left(\frac{1-\inf f_{q-1}}{\alpha_q}\right)^2 \left(\frac{\dot{f}_q(0)}{\dot{f}_{q-1}(0)}\right).$$

Property III. If, inside the peak, T_{α_q} really exists with $\alpha_q \neq 1$ and $\alpha_q \neq 0$, then for $q \geq 1$ either $D_q \not\to 0$ or $\ddot{f}_{q-1}(0)/[\dot{f}_{q-1}(0)]^2 \not\to 0$. We recall that for the proof of Property II we have used this result, considering in Eq. (31) the case $\bar{\alpha}_q \neq -\infty$.

Property IV. If either $\dot{f}_q(0)/\dot{f}_{q-1}(0) \not\to \infty$ or $D_q \not\to \infty$ or $\ddot{f}_{q-1}(0)/[\dot{f}_{q-1}(0)]^2 \not\to \infty$, then f_{q-1} scales in either variables $\tau = tT_{\alpha_q}^{-1}$ or $\bar{\tau} = t\dot{f}_{q-1}(0)$, and in both cases the limit functions are differentiable.

First we consider $\tau = tT_{\alpha_q}^{-1}$, and by definition $f_{q-1}(s, -T_{\alpha_q}) = 1 - \alpha_q T_{\alpha_q} \dot{f}_{q-1}(0)$. If $D_q \not\to \infty$, then Eq. (36) gives $T_{\alpha_q} \dot{f}_{q-1}(0) \not\to \infty$ and $\not\to 0$ and $T_{\alpha_q}^2 \ddot{f}_{q-1}(0) \not\to \infty$. It follows that $\partial^2 f_{q-1}/\partial\tau^2$ is bounded and also that $f_{q-1}(s_1 - T_{\alpha_q}) \not\to 1$. The scaling exists and is nontrivial, and the derivative $\partial f_{q-1}(\tau)/\partial\tau$ exists.

Secondly, we consider $\bar{\tau} = t\dot{f}_{q-1}(0)$. We have $\partial f_{q-1}/\partial\bar{\tau} < 1$ and

$$\frac{\partial^2 f_{q-1}}{\partial \bar{\tau}^2} = \frac{\ddot{f}_{q-1}(0)}{[\dot{f}_{q-1}(0)]^2} \not\to \infty.$$

For $-t = T_{\alpha_q}$ we know that $f_{q-1}(s, -T_{\alpha_q}) \not\to 1$, and from (36) the corresponding $-\bar{\tau}_{\alpha_q} = T_{\alpha_q}\dot{f}_{q-1}(0)$ has constant upper and lower bounds.

Property V. If $D_q \not\to \infty$, then f_q scales in either scaling variables $\tau = tT_{\alpha_q}^{-1}$ or $t\dot{f}_q(0) = \bar{\tau}$. For $\tau = tT_{\alpha_q}^{-1}$ the results follow the inequality (36). For $\bar{\tau}$ we note that $\partial f_q(s, \bar{\tau})/\partial\tau \leq 1$, and for $f_q = \alpha_q$ corresponds $-\bar{\tau}_{\alpha_q} = T_{\alpha_q}\dot{f}_q(0)$, which has constant lower and upper bounds from Eq. (36). Summarizing Properties II, IV, and V we get the following: If

$$D_q = \frac{f_0^{(q+1)}(0) f_0^{(q-1)}(0)}{[f_0^{(q)}(0)]^2} \not\to \infty,$$

then (i) there exist α_{q-1} values such that f_{q-1} scales in the variable $\tau = t/T_{\alpha_{q-1}}$ and the limit function is differentiable, f_{q-1} scales in the variable $\tau = t/T_{\alpha_q}$ and the limit function is differentiable, and f_{q-1} scales in the variable $\bar{\tau} = t\dot{f}_{q-1}(0)$ and the limit function is differentiable. (ii) f_q scales in either the variables $\tau = t/T_{\alpha_q}$ or $\bar{\tau} = t\dot{f}_q(0)$

4. *Existence of the* qth *derivative of the scaling function*

For simplicity we consider only the derivatives of the primitive function f_0, but, of course, corresponding results could be obtained for $f_1, f_2, \ldots$. From the definition of D_q given in Eq. (34) we get the following for the $(p+1)$th derivative of f_0:

$$f_0^{(p+1)}(0) = D_p D_{p-1}^2 D_{p-2}^3 \cdots D_2^{p-1} D_1^p [\dot{f}_0(0)]^{p+1}. \tag{37}$$

Let us assume $D_q \not\to \infty$, $q = 1, 2, \ldots, p$. First we consider the scaling variable $\tau = t/T_{\alpha_1}$. From the second inequality (36), $T_{\alpha_1}\dot{f}_0(0)$ is bounded, and from (37) we see that the derivatives

$$\left|\frac{\partial^q f_0}{\partial \tau^q}(s,\tau)\right| < f_0^{(q)}(0)\, T_{\alpha_1}{}^q, \quad q = 1, 2, \ldots, p+1$$

are bounded. Secondly, we consider $\tau = tT_{\alpha_0}^{-1}$, and we know from (32) and (33) that there exist α_0 values such that $T_{\alpha_0}\dot{f}_0(0)$ is bounded. Here also

$$\left|\frac{\partial^q f_0}{\partial \tau^q}\right| < f_0^{(q)}(0)\, T_{\alpha_0}{}^q, \quad q = 1, \ldots, p+1$$

are bounded. Thirdly, we consider $\bar{\tau} = t\dot{f}_q(0)$ and

$$\left|\frac{\partial^q f_0}{\partial \bar{\tau}^q}\right| < f_0^{(q)}(0)[\dot{f}_0(0)]^{-q}$$

for $q = 1, \ldots, p+1$, are bounded if one uses Eq. (37).

Property VI. If $D_q \not\to \infty$, $q = 1, \ldots, p$ (p finite), then the derivatives of the limit functions $\partial^q f_0/\partial\tau^q$ or $\partial^q f_0/\partial\bar{\tau}^q$ ($q = 1, \ldots, p$) exist for $\tau = t/T_{\alpha_1}$ and $\tau = t\dot{f}_0(0)$. Furthermore, there exist α_0 values such that this property holds for $\tau = t/T_{\alpha_0}$.

IV. APPLICATION TO THE ELASTIC ABSORPTIVE PART

At the beginning we consider the spinless case and the expansion near the forward peak $t=0$:

$$a_0(s,t)=1+bt+\frac{ct^2}{2}+\frac{dt^3}{3!}+\cdots \qquad (38)$$

(where for simplicity in this section we omit the index ABS for $b,c,d,\ldots$).

A. Lower and upper bounds

Inside the "forward peak," $-4k^2<t\leq 0$, we have the inequalities

$$1+bt<a_0(s,t)<1+bt+\tfrac{1}{2}ct^2\,. \qquad (39)$$

From the unitarity relation we get, if $b<4k^2$,

$$\frac{\sigma_{\rm el}}{\sigma_T^{\;2}}>(16\pi)^{-1}\int_{-b^{-1}}^{0}(1+bt)^2\,dt=b^{-1}(48\pi)^{-1}\,, \qquad (40)$$

an inequality similar to the MacDowell-Martin[11] lower bound but with a worse constant $[(48\pi)^{-1}$ instead of $(36\pi)^{-1}]$.

B. Scaling[1] when $c/b^2\not\to\infty$

We consider the scaling variables $\tau=t/T_{\alpha_0}$, $\tau=t/T_{\alpha_1}$, $\tilde\tau=tb$, where $a_0\xrightarrow{T}a_1$, $a_0(s,-T_{\alpha_0})=\alpha_0$, $a_1(s,-T_{\alpha_1})=\alpha_1$, and we apply Properties II, IV, and V of the above section.

Theorem III. If $c/b^2\not\to\infty$, then

(i) there exist α_0 values such that a_0 scales in the variable $\tau=t/T_{\alpha_0}$ and the limit function is differentiable for $\tau\in[-1,0]$,

(ii) a_0 scales in the variable $\tau=t/T_{\alpha_1}$ and the limit function is differentiable,

(iii) a_1 scales in the variable $\tau=t/T_{\alpha_1}$,

(iv) a_0 scales in the variable $\tilde\tau=tb$ and the limit function is differentiable.

C. Scaling for the set $\{a_q\}$

Let us define for $a_{q-1}\xrightarrow{T}a_q$, $q=1,2,\ldots$ a set of functions keeping the existence of the forward peak:

Theorem IV. If

$$D_q=\frac{a_0^{(q+1)}(0)a_0^{(q-1)}(0)}{[a_0^{(q)}(0)]^2}\not\to\infty\,,\quad q \text{ finite}\,,$$

then

(i) $a_q, a_{q-1},\ldots,a_1$ scale in the corresponding variables $\tau=t/T_{\alpha_q}, \tau=t/T_{\alpha_{q-1}},\ldots,\tau=t/T_{\alpha_1}$,

(ii) a_0 scales in the variable $\tau=t/T_{\alpha_1}$, and the first q derivatives of the limit function exist. If $\tau=t/T_{\alpha_0}$, then α_0 values exist such that the same property holds. If $\tilde\tau=tb$, a_0 scales and the first q derivatives of the limit function exist. For the elastic spinless absorptive part, the leading term in D_q is given by

$$D_q\simeq\frac{(\sum l^{2q+3}a_l)(\sum l^{2q-1}a_l)}{(\sum l^{2q+1}a_l)^2}\,. \qquad (41)$$

The proof of the theorem (given in Appendix B) is obtained in successive steps:

(i) $D_q<C_q=\dfrac{(\sum l^{2q+3}a_l)(\sum la_l)^q}{(\sum l^3a_l)^{q+1}}$.

(ii) $C_q<(C_{q+1})^{q/q+1}$.

(iii) If $C_q\not\to\infty$, then $C_{q-1}\not\to\infty,\ldots,C_1\not\to\infty$ and $D_q\not\to\infty, D_{q-1}\not\to\infty,\ldots,D_1\not\to\infty$.

(iv) Applying Property V of the above section, we see that $a_0,a_1,\ldots,a_q$ scale.

(v) We apply Property VI.

D. Application to one case of the strong scaling studied in Ref. 3 (see Appendix B)

Theorem V. If

$$\sigma_T\simeq(\log s)^\gamma$$

and

$$L_{\max}\leq \text{const}\times s^{1/2}(\log s)^{\gamma/2},\quad 0<\gamma\leq 2,$$

then for any integer q we have $C_q<(\text{const})^q$. It follows that any functions of the set $\{a_q\}$ scale, and any derivative of the limiting function of a_0 exists. Furthermore, $b\simeq(\log s)^\gamma$, and if we define $\tilde\tau=\text{const}\times tb$ we have $|a_0(s,\tilde\tau)|<\exp(\text{const}\times|\tilde\tau|^{1/2})$ in the complex $\tilde\tau$ plane around $\tilde\tau=0$, and the limiting function of a_0 is analytic in the $\tilde\tau$ variable. This establishes for this particular example the link between the results of Ref. 3 and those of the present paper.

E. Derivative of the absorptive spinless amplitude

Let us define

$$\frac{\dfrac{\partial}{\partial z}a_0(s,z)}{\left(\dfrac{\partial}{\partial z}a_0(s,z)\right)_{z=1}}=\hat a_0(s,z) \qquad (42)$$

or

$$\hat a_0(s,t)=1+\frac{ct}{b}+\left(\frac{d}{b}\right)\frac{t^2}{2}+\cdots\,. \qquad (43)$$

Because $\hat a_0(s,z)$ is a sum of $P_l'(z)$ with positive coefficients and P_l' are particular Gegenbauer polynomials, all the properties obtained above in Secs. IV and V can be applied. There exist lower and upper bounds, and sufficient conditions for the scaling could be obtained. As an example we have the following lower and upper bounds for $-4k^2<t\leq 0$:

$$1+\frac{ct}{b}<b^{-1}\frac{\partial a_0}{\partial t}<1+\frac{ct}{b}+\left(\frac{d}{b}\right)\frac{t^2}{2}. \qquad (44)$$

F. Spin case[1]

We have shown in collaboration with Martin[1] that the ratio of the two-body absorptive part of the elastic cross section for particles with spin, divided by its forward part, can be written

$$\langle A\rangle^2=\sum C_n\cos(n\theta),\quad C_n\geqslant 0,\quad \sum C_n=1.$$

Since the sum of Tchebichef polynomials $T_n=\cos(n\theta)$ with positive coefficients is in the class of functions, all the properties established above can be applied if we keep the existence of the forward peak. We have the set of lower and upper bounds by retaining even or odd numbers of terms in the Taylor expansion. For instance, for $-4k^2<t\leqslant 0$, we have

$$\langle A\rangle^2>1+2bt,\quad b=\frac{\partial\langle A\rangle}{\partial t}\bigg|_{t=0}.$$

If we use the unitarity relation and put $D/2k^2=\int_{-1}^{+1}\langle A\rangle^2 d\cos\theta$, then $D>\int_{-(2b)^{-1}}^{0}(1+2bt)\,dt$ or $b>(4D)^{-1}$, which is a generalization in the spin case of the MacDowell-Martin lower bound.[11] All the properties of scaling can be extended to the spin case, leading to results similar to those of the spinless case.

VII. APPLICATION TO THE MULTIPARTICLE GENERATING FUNCTION

From the initial multiparticle generating function $a_0(s,z)=\sum z^n\gamma_n$, $\gamma_n=\sigma_n/\sigma_T>0$, $a_0(s,1)=1$ we can associate other functions. For instance, we consider

$$a_0^{(1)}(s,t)=\sum(1+t)^n\gamma_n=\sum\frac{\langle n(n-1)\cdots(n-q+1)\rangle}{q!}t^q,\qquad t\in[-1,0],\quad (45)$$

$$a_0^{(2)}(s,t)=\sum e^{nt}\gamma_n=\sum\frac{\langle n^q\rangle}{q!}t^q,\qquad t\leqslant 0. \qquad (46)$$

We want to apply the results of Sec. V to $a_0^{(i)}$, $i=1,2$. In the first case, $t=0$ is a dominant peak for $t\in[-1,0]$, and the repeated application of the transform $a_q^{(1)}\xrightarrow{T}a_{q+1}^{(1)}$ gives a set $\{a_q^{(1)}\}$, $q=0,1\ldots$, keeping this existence of a dominant peak at $t=0$ in the same interval. Similarly, in the second case $a_0^{(2)}(s,t)$, the corresponding set $\{a_q^{(2)}\}$, $q=0,1,\ldots$ has a dominant peak at $t=0$ for $t\leqslant 0$.

A. Lower and upper bounds

We consider only the first two inequalities for the primitive function $a_0^{(i)}$:

$$1+\langle n\rangle t<a_0^{(i)}(s,t),\qquad i=1,\ t\in[-1,0],\quad i=2,\ t\leqslant 0,$$
$$1+\langle n(n-1)\rangle\tfrac{1}{2}t^2+\langle n\rangle t>a_0^{(1)}(s,t), \qquad (47)$$
$$1+\langle n\rangle t+\langle n^2\rangle\tfrac{1}{2}t^2>a_0^{(2)}(s,t).$$

More generally, taking an even (odd) number of terms in the last term at the right-hand side of (46) for $i=1$ or at the right-hand side of (47) for $i=2$, we get a lower bound (upper bound) for, respectively, $t\in[-1,0]$, $i=1$, or $t\leqslant 0$, $i=2$. From the lower bound in Eq. (47) we can obtain (if $\langle n\rangle\geqslant 1$) a lower bound for $\langle n\rangle$, which in the multiparticle case is similar to the MacDowell-Martin lower bound for the slope b^{ABS};

$$\int_0^1[a_0^{(1)}(s,z)]^2dz\geqslant\int_{-\langle n\rangle^{-1}}^{0}(1+t\langle n\rangle)^2dt=\tfrac{1}{3}\langle n\rangle^{-1}$$

If

$$\langle n\rangle>1,$$

then

$$\langle n\rangle>\tfrac{1}{3}\sigma_T{}^2\left(\sum_{n,m}\frac{\sigma_n\sigma_m}{n+m+1}\right)^{-1}. \qquad (48)$$

If $\langle n\rangle$ is the average multiplicity corresponding to $\gamma_n=\sigma_n/\sigma_{\text{in}}$, $a_0(s,1)=1$, then we have only to replace σ_T by σ_{in} in Eq. (48).

B. Scaling for $a_0^{(i)}(s,t)$ and $a_1^{(i)}(s,t)$, $a_0^{(i)}\xrightarrow{T}a_1^{(i)}$

Theorem III′. If $\langle n^2\rangle/\langle n\rangle^2\not\to\infty$, we consider the scaling variables

$$\tau^{(i)}=t/T_{\alpha_0}^{(i)},\quad \tau^{(i)}=t/T_{\alpha_1}^{(i)},\quad \tilde{\tau}=t\langle n\rangle$$

and get the following:

(i) There exists in both cases $i=1,2$, α_0 values such that $a_0^{(i)}(s,t)$ scale in the variables $\tau^{(i)}=t/T_{\alpha_0}^{(i)}$ and the limit functions are differentiable,

(ii) $a_0^{(i)}(s,t)$ scale in the variables $\tau^{(i)}=t/T_{\alpha_1}^{(i)}$ and the limit functions are differentiable,

(iii) $a_1^{(i)}$ scale in the variables $\tau^{(i)}=t/T_{\alpha_1}$,

(iv) $a_0^{(i)}$ scale in the variable $\tilde{\tau}=t\langle n\rangle$ and the limit function is differentiable.

C. Scaling for the set $\{a_q^{(i)}\}$, $i=1,2;q=0,1,2,\ldots$, $(a_{q-1}^{(i)}\xrightarrow{T}a_q^{(i)})$

In order to simplify the conditions, let us assume in the case $i=1$ that $\langle n\rangle\underset{s\to\infty}{\to}\infty$. In this case

$$D_q^{(i)} = \frac{\left(\frac{\partial^{q+1}}{\partial t^{q+1}} a_0^{(i)}(s,t)\right)_{t=0} \left(\frac{\partial^{q-1}}{\partial t^{q-1}} a_0^{(i)}(s,t)\right)_{t=0}}{\left(\frac{\partial^q}{\partial t^q} a_0^{(i)}(s,t)\right)^2}$$

$$\simeq \frac{\langle n^{q-1}\rangle\langle n^{q+1}\rangle}{\langle n^q\rangle^2}.$$

Theorem IV′. If either $D_q \not\to \infty$ or

$$C_q \simeq \frac{\langle n^{q+1}\rangle}{\langle n\rangle^{q+1}} \not\to \infty, \quad q \text{ finite},$$

then $a_q^{(i)}, a_{q-1}^{(i)}, \ldots, a_1^{(i)}$ scale in the corresponding variables

$$\tau^{(i)} = \frac{t}{T_{\alpha_q}^{(i)}}, \quad \tau^{(i)} = \frac{t}{T_{\alpha_{q-1}}^{(i)}}, \quad \ldots, \quad \tau^{(i)} = \frac{t}{T_{\alpha_1}^{(i)}}, \quad \tau^{(i)} = \frac{t}{T_{\alpha_0}^{(i)}}.$$

$a_0^{(i)}$ scale in the variable $\tau^{(i)} = t/T_{\alpha_1}^{(i)}$, and the qth derivatives of the limit function exist. If $\tau^{(i)} = t/T_{\alpha_0}^{(i)}$, there exist α_0 values such that the same property holds. If $\bar\tau = t\langle n\rangle$, $a_0^{(i)}$ scale and the first q derivatives of the limit function exist.

The proof of the theorem (given in Appendix B) is obtained in successive steps:

(i) $D_q < C_q < (C_{q+1})^{q/(q+1)}$;

(ii) if $C_q \not\to \infty$, then $C_{q-1} \not\to \infty, \ldots, C_1 \not\to \infty$ and $D_q \not\to \infty, \ldots, D_1 \not\to \infty$;

(iii) applying Properties V and VI of Sec. X, we see that $a_0, a_1, \ldots, a_q$ scale.

D. Application to the strong-scaling case

If for any integer q value, $\langle n^q\rangle/\langle n\rangle^q \not\to \infty$, then any function of the set $\{a_q\}$ scales. It is worthwhile to recall that $C_q \not\to \infty\ \forall_q$ is a property satisfied in the Koba-Nielsen-Olesen scaling.

Let us show now that if C_q is uniformly bounded in s and satisfies

$$C_q = \frac{\langle n^{q+1}\rangle}{\langle n\rangle^{q+1}} < (\text{const})^q, \tag{49}$$

then we have a strong scaling in the sense that there exists a scaling function analytic in the scaling variables near $\tau = 0$.

Let us consider $a_0^{(2)}(s,t)$ in a *complex neighborhood of* $t=0$. We get with the help of (49),

$$|a_0^{(2)}(s,t)| < \lambda_1 \exp(\lambda_2 |t| \langle n\rangle),$$

where λ_1 and λ_2 are s-independent constants or if we write $\bar\tau = t\langle n\rangle$,

$$|a_0^{(2)}(s,\bar\tau)| < \lambda_1 \exp(\lambda_2 |\bar\tau|).$$

We have a uniform upper bound, which is s independent, and then for at least one sequence $(s_n) \to \infty$, $a_0^{(2)}(s_n, \bar\tau) \underset{s_n\to\infty}{\to} f(\bar\tau)$, and $f(\bar\tau)$ is analytic near $\bar\tau = 0$. However, we could have a trivial limiting function: for instance, a constant equal to 1. However, from (36)–(49) we have const < const $\times C_1^{-1} \leq T_{\alpha_1}\langle n\rangle <$ const. This means that for T_{α_1} such that $a_1^{(2)}(s, -T_{\alpha_1}) = \alpha_1$, the corresponding $\bar\tau_{\alpha_1}$ value $-\bar\tau_{\alpha_1} = T_{\alpha_1}\langle n\rangle$ satisfies const $\leq -\bar\tau_{\alpha_1} \leq$ const. On the other hand, for this same $\bar\tau_{\alpha_1}$ value, $a_0(s, \bar\tau_{\alpha_1} = -T_{\alpha_1}\langle n\rangle) = a_0(s, -T_{\alpha_1}) \not\to 1$ following Property IV. It follows that the scaling is nontrivial.

VIII. INEQUALITIES SATISFIED BY THE LIMIT SCALING FUNCTIONS

From the set of lower and upper bounds (26) satisfied by functions belonging to $\mathcal{E}$, we can deduce in the τ scaling variables corresponding inequalities for the scaling function and its existing derivatives. We consider $a_0 \in \mathcal{E}$ such that $a_0 \mathcal{I} a_1$ keeps the existence of a dominant peak:

(i) We assume $D_1 = \ddot a_0(0)/[\dot a_0(0)]^2 \not\to \infty$ and consider a fixed $\tau < 0$ value. We know that $a_0(\tau)$ and $\partial a_0(\tau)/\partial\tau$ exist with scaling variables $t\dot a_0(0)$ or $tT_{\alpha_0}^{-1}$ or $tT_{\alpha_1}^{-1}$. For any given $\epsilon_\tau > 0$ arbitrarily small there exists a set (s_n), $n > N_{\epsilon_\tau}$ such that

$$a_0(\tau) + \epsilon_\tau > a_0(s_n, \tau) > a_0(\tau) - \epsilon_\tau,$$

$$\left(\frac{\partial a_0}{\partial\tau}(s_n,\tau)\right)_{\tau=0} > \left(\frac{\partial a_0(\tau)}{\partial\tau}\right)_{\tau=0} - \epsilon_\tau.$$

Moreover, the first lower bound in the t variable can be written in the scaling variable;

$$a_0(s_n,\tau) \geq 1 + \tau\left(\frac{\partial a_0}{\partial\tau}(s_n,\tau)\right)_{\tau=0}.$$

It follows that

$$a_0(\tau) + \epsilon' \geq 1 + \tau\left(\frac{\partial a_0(\tau)}{\partial\tau}\right)_{\tau=0},$$

where $\epsilon' = \epsilon_\tau(1+\tau)$ is also arbitrarily small for τ *finite*. Moreover, from the Arzela theorem, $a_0(s_n,\tau)$ converges uniformly to $a_0(\tau)$, and we can choose ϵ_τ independently of τ.

In conclusion, if $\ddot a_0(0)/[\dot a_0(0)]^2 \not\to \infty$ ($c/b^2 \not\to \infty$ for the elastic absorptive amplitude or $\langle n^2\rangle/\langle n\rangle^2 \not\to \infty$ for the multiparticle generating function), then the scaling function also has the lower bound $a_0(\tau) \geq 1 + \tau\partial a_0(\tau)/\partial\tau$ for $\tau < 0$, $|\tau|$ finite.

(ii) More generally, if D_q defined in Eq. (34) is such that $D_q \not\to \infty$ for $q = 1, 2, \ldots, p$, then for the $\tau \leq 0$ values such that the scaling exists we have the inequalitites

$$1 + \tau\left(\frac{\partial a_0(\tau)}{\partial\tau}\right)_{\tau=0} \leq a_0(\tau) \leq 1 + \tau\left(\frac{\partial a_0(\tau)}{\partial\tau}\right)_{\tau=0} + \frac{\tau^2}{2}\left(\frac{\partial^2 a_0(\tau)}{\partial\tau^2}\right)_{\tau=0}, \ldots,$$

$$1+\cdots+\frac{\tau^{2n+1}}{(2n+1)!}\left(\frac{\partial^{2n+1}a_0(\tau)}{\partial\tau^{2n+1}}\right)_{\tau=0}\leq a_0(\tau)\leq 1+\cdots+\frac{\tau^{2n+2}}{(2n+2)!}\left(\frac{\partial^{2n+2}a_0(\tau)}{\partial\tau^{2n+2}}\right)_{\tau=0};$$

the last inequality is a lower bound if $p=2n+1$ or an upper bound if $p=2n+2$.

(iii) In the elastic absorptive case or in the multiparticle generating function case we recall that if either $D_p\not\to\infty$ or $C_p\not\to\infty$, then $D_q\not\to\infty$, $q\leq p$. It follows that in these cases the p inequalities are satisfied. This is, for instance, the case if $\langle n^p\rangle/\langle n\rangle^p\not\to\infty$.

(iv) Finally, in the strong-scaling case where the scaling functions exist also for $\tau>0$ and where any derivative at $\tau=0$ exists, we see that for $\tau\leq 0$, if we retain in the Taylor expansion of the scaling function around $\tau=0$ an even or odd number of terms, they provide lower or upper bounds. In conclusion, we see that for $a_0(s,t)\in\mathcal{E}$ then the scaling function, if it exists, cannot be entirely arbitrary because it has to satisfy well-defined inequalitites.

IX. INCLUSION OF THE REAL PART

Because of the lack of positivity of the real part of partial-wave amplitudes our method cannot be directly applied, and the theoretical results obtained are poor compared with those of the absorptive part. A summary of the present situation is presented in Ref. 1, and here we want to add some remarks concerning the strong scaling case.

A. The odd amplitude contribution

First we want to recall that if the odd amplitude is not negligible compared to the even amplitude, then many changes can occur. Let us assume[12]

(i) $\lambda(s)=\dfrac{|F(s,0)|^2}{s^2\sigma_{el}}\simeq(\log s)^{\gamma_0},\quad 0<\gamma_0\leq 2$,

(ii) $L_{max}\simeq s^{1/2}(\log s)^{\gamma_0/2}$ in a neighborhood of $t=0$, where $F(s,t)$ is the scattering amplitude. We recall[3] that $b<(\log s)^{\gamma_0}$, and that at least for one sequence $(s_n)\to\infty$, $F(s_n,t)/F(s_n,0)\to f(\tau=t(\log s)^{\gamma_0})$; the scaling is valid for complex τ values. So the scaling variable is $t\lambda(s)$, and if the odd amplitude is dominant in such a way that $|\mathrm{Re}F(s,0)|/\mathrm{Im}F(s,0)\to\infty$, then the scaling variable is $t[\mathrm{Re}F(s,0)]^2/s^2\sigma_{el}$ instead of $t\sigma_t^2/\sigma_{el}$.

B. Symmetric scattering process

We must first assume that the limit scaling function is reached $\forall s\geq s_0$. The even crossing properties or the dispersion relation give us in principle the possibility to get the real part from the absorptive part. At least the existence of the first derivative of the scaling function is necessary; however, many difficulties still remain, and so we do not consider what we have called weak scaling. We consider the "strong-scaling case" of Ref. 3, knowing that all the derivatives of the scaling function at $\tau=0$ exist. We extend a recipe which has been made previously in different cases[13]: We do not even try to justify it; the only interest is that it gives an indication of what kind of relation we could hope to obtain at best. From an absorptive symmetric amplitude assumed to be of the kind

$$A(s,t)\underset{s\ \text{large}}{\simeq}s\sigma_t(s)f(\tau),$$

$$\tau=tb(s),$$

$$\forall n,\ \left(\frac{\partial^n f(\tau)}{\partial\tau^n}\right)_{\tau=0}\ \text{exists},$$

in order to get the symmetric amplitude, we associate

$$F(s,t)\simeq-(se^{-i\pi/2})\sigma_T(se^{-i\pi/2})f(tb(se^{-i\pi/2})).$$

We introduce the rapidity variable $y=\log s$ and define $b(s)=\tilde b(y)$, $\sigma_T(s)=\tilde\sigma_T(y)$. Expanding and taking into account first-order corrective terms, one finds

$$a(s,t)=\frac{A(s,t)}{A(s,0)}$$

$$\simeq f(\tau)-\frac{\pi^2}{4}\tau\frac{\partial f}{\partial\tau}\left(\frac{\partial\log\tilde\sigma_T(y)}{\partial y}\right)\left(\frac{\partial\log\tilde b(y)}{\partial y}\right),$$

$$r(s,t)=\frac{R(s,t)}{R(s,0)}$$

$$\simeq f(\tau)+\tau\frac{\partial f}{\partial\tau}\ \frac{\partial\log\tilde b(y)/\partial y}{\partial\log\tilde\sigma_T(y)/\partial y}.$$

If, for instance, $\tilde\sigma_T=y^\beta$, $b=y^\gamma$, $0<\beta\leq\gamma\leq 2$, then we get

$$r(s,t)\simeq f(t)+\frac{\gamma}{\beta}\tau\frac{\partial f}{\partial\tau},$$

$$a(s,t)\simeq f(\tau)-\frac{\pi^2}{4}\frac{\beta\gamma}{(\log s)^2}\tau\frac{\partial f}{\partial\tau},$$

and if $\beta=\gamma$, we recover the previous results of Ref. 14. One can also determine the contribution of the absorptive part and the real part to the forward slope;

$$\frac{\partial a}{\partial t}(s,t)\bigg|_{t=0}\simeq b\frac{\partial f}{\partial\tau}\bigg|_{\tau=0},$$

$$\frac{\partial r}{\partial t}(s,t)\bigg|_{t=0}\simeq b\frac{\partial f}{\partial\tau}\bigg|_{\tau=0}\left[1+\frac{\partial\log\tilde b(y)/\partial y}{\partial\log\tilde\sigma_T(y)/\partial y}\right].$$

If we put $(\partial f/\partial\tau)_{\tau=0}=1$, then b is the absorptive slope. If, further, $\tilde{b}=y^{\gamma}$, $\tilde{\sigma}_T=y^{\beta}$, $0<\beta\leqslant\gamma\leqslant 2$, then

$$\frac{\partial r}{\partial t}(s,t)\Big|_{t=0}=b\left(1+\frac{\gamma}{\beta}\right)$$

We must, of course, assume that b can be prolongated analytically from s to $se^{i\pi}$.

X. CONCLUSION

In this paper, through the mathematical details which were necessary in order to distinguish clearly what can be proved from what can be guessed (see, for instance,[4] the confusion between the existence of an upper bound and the existence of a limit scaling function) there emerge two or three main ideas.

First we consider what we have called "weak scaling," where the scaling is insured only for physical scaling values. In order to obtain it, the knowledge of a few observables is sufficient: $b, c, \sigma_e, \sigma_T, \langle n\rangle, \langle n^2\rangle, \ldots$. Experimentally, this scaling seems well verified, and it is satisfied by current phenomenological models. In this way, the existence of scaling in hadron physics appears as a sufficiently natural and weak property so that perhaps we shall not learn too much from it.

Secondly, we consider the "strong scaling" case where the scaling is also valid for unphysical scaling values. Although it has very nice properties (for instance, the scaling functions must satisfy a well-defined set of lower and upper bounds, the real part can be obtained in the symmetric elastic case, etc.), it is, in general, difficult to prove its validity from the knowledge of observables. We have given counterexamples considering explicit positive imaginary partial-wave distributions leading to well-defined behavior for some observables. The underlying difficulty is mainly that, in general, from a finite number of observables we can control only a finite number of derivatives of the elastic amplitude at $t=0$, whereas the scaling for unphysical scaling values requires the specification for them all. Note that the analyticity in a neighborhood of $t=0$ cannot help, because the proof of strong scaling requires the knowledge of a uniform upper bound for complex scaling values, and, again, the control of all the derivatives at $t=0$. If we want to introduce the contraints up to $t=4\mu^2$, then the recent work by Haan and Mütter[14] shows that only partial waves of the order of $s^{3/2}$ or more are constraints, and so, again, the introduction cannot help for this problem.

Why does this difficulty not occur if $\sigma_T\simeq(\log s)^2$, or more generally, when the unitary constaints are saturated? Owing to the existence of the Froissart cutoff, in this case all the moments $\sum l^{2p+1}a_l$ are known, with the further specification that they have lower and upper bounds with similar behavior. It follows that we control all the moments, or, equivalently, all the derivatives at $t=0$ of the elastic absorptive amplitude. Let us consider now $\sigma_T\simeq(\log s)^{\gamma_0}$, $0<\gamma_0<2$, and a trivial calculation shows that only if $L_{\max}$, the maximum number of partial waves, is $s^{1/2}(\log s)^{\gamma_0/2}$, are we in the same favorable situation as above. (This was the reason why this assumption was introduced in Ref. 3.) Concerning the proof of "strong scaling" we have, in principle, the same type of difficulty in the multiparticle case, and the theoretical situation could be even worse because we do not know the equivalent of the Froissart cutoff.

Thirdly, we consider the phenomenological situation which is quite different in both cases. In the multiparticle case, the coefficient σ_n/σ_T of the sum is directly observable, many moments $\langle n^q\rangle$ are experimentally known, and so we have a good experimental confirmation of the boundedness of $\langle n^q\rangle/\langle n\rangle^q$ for many q values. Personally, I have the feeling that in the multiparticle case the experimental results seem to indicate that, really, the strong-scaling case exists in nature, and in this respect the success of Koba, Nielsen, and Olesen is perhaps not merely an accident. For the elastic case there are many drawbacks. On the one hand, our present results apply mainly to the absorptive part. On the other hand, the partial waves are not directly accessible by experiments (perhaps a fine analysis in the impact-parameter space could help) and we know *only a few moments* $\sum la_l$, $\sum l^3a_l$, $\sum l^5a_l$, without the expectation of obtaining more in the near future. In the elastic case, even if one has no doubts that the actual experimental results are compatible with weak scaling, one cannot be as sure concerning the existence of strong scaling.

Finally, it may be useful to add three remarks. First, the scaling, if it exists, is not described by entirely arbitrary functions because they have to satisfy well-defined inequalities for physical scaling values. Secondly, in this paper there exists only a *mathematical analogy* between the possibility of scaling in the elastic case and scaling in the multiparticle case, but perhaps there exist deeper physical links between the concept of scaling in both situations. Thirdly, the method and the results presented here can be easily extended to the inelastic overlap function.[15]

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APPENDIX A

We study the transformation

$$f \xrightarrow{T} g, \quad g(z) = \frac{f(z_0) - f(z)}{(z_0 - z) f'(z_0)}, \tag{A1}$$

where $f(z_0) = 1$ and $f(z)$ is either a polynomial or a sum of polynomials. First we want to get g as a sum of *polynomials* of the *same type* at those of f. Secondly, we choose z_0 such that the *polynomials* are *maximum* at z_0 for z in some interval. Thirdly, we consider the case when the *coefficients* in the sum defining f are *positive*.

1. Power case

Let us consider

$$f(z) = \sum_{n_0} \gamma_n z^n, \quad f(z_0) = 1, \quad z_0 > 0, \quad n_0 \geq 0. \tag{A2}$$

We get for $g(z)$,

$$g(z) = [f'(z_0)]^{-1} \sum_{n_0'} \gamma_n (z_0^{n-1} + z z_0^{n-2} + \cdots + z^{n-1})$$

$$= \sum_0 \beta_n z^n, \tag{A3}$$

with $n_0' = 1$ if $n_0 = 0$, otherwise $n_0' = n_0$. Identifying the two expansions, we get

$$f'(z_0)\beta_n = z_0^{-(n+1)} \sum_{p = n_0' + n} \gamma_p z_0^p,$$

$$\beta_{n-1} - z_0 \beta_n = \gamma_{n_0' + n} z_0^{n_0'} [f'(z_0)]^{-1}.$$

Furthermore, let us assume $\gamma_n > 0$; then $f'(z_0) > 0$, $\beta_n > 0$, $\beta_{n-1} - z_0\beta_n > 0$.

2. Classical orthogonal polynomials

a. f(z) is a polynomial

Let $p_n(z)$ be the classical orthogonal polynomials,[10] with $\langle p_n, p_n \rangle = \delta_n > 0$, and k_n the coefficient of x^n in p_n. Let us assume $p_n(z_0) \neq 0$ and define

$$\lambda_n = \frac{\delta_n |k_{n+1}|}{p_{n+1}(z_0) p_n(z_0) |k_n|}, \quad \eta_n = \text{sign}\left(\frac{k_{n+1}}{k_n}\right),$$

$$q_n = \frac{1}{(z_0 - z)} \left(1 - \frac{p_n(z)}{p_n(z_0)}\right), \quad \hat{p}_n(z) = \frac{p_n(z) p_n(z_0)}{\delta_n}. \tag{A4}$$

The Christoffel-Darboux formula can be written

$$\lambda_n \eta_n \sum_0^n \hat{p}_\nu(z) = \frac{1}{z_0 - z} \left(\frac{p_n(z)}{p_n(z_0)} - \frac{p_{n+1}(z)}{p_{n+1}(z_0)}\right) = c_n,$$

$$q_n(z) = \sum_0^{n-1} \eta_\nu c_\nu = \sum_0^{n-1} \hat{p}_m(z) \sum_{r=m}^{r=n-1} \lambda_r \eta_r. \tag{A5}$$

It follows that if $f(z) = p_n(z)/p_n(z_0)$, then $f \xrightarrow{T} g = p_n(z_0) q_n^{(z)}/p_n'(z_0)$, where q_n is given in (A4) and (A5). In conclusion we have the following:

(i) If $\forall n\, p_n(z_0) > 0$ and η_n *is independent of* n (always $+1$ or always -1), then $q_n(z)[p_n'(z_0)]^{-1}$ is a linear combination of $\hat{p}_n$, with coefficients having always the same sign.

(ii) If, further, $\eta_n p_n'(z_0) \geq 0$ and $p_n(z)$ in some interval $[\bar{z}, z_0]$ has a maximum at z_0, then the same property holds for $q_n(z)/p_n'(z_0)$. If (i) and (ii) hold, then the general results of Sec. V can be applied; in particular, there exists the set of lower[16] and upper bounds for $p_n(z)$ by retaining an even or odd number of terms in the Taylor expansion at z_0;

$$\left.\begin{array}{l} p_n(z_0) + (z - z_0)p_n'(z_0) < p_n(z) < p_n(z_0) + (z - z_0)p_n'(z_0) + \tfrac{1}{2}(z - z_0)^2 p_n''(z_0) \\ p_n(z_0) + \cdots + \dfrac{(z - z_0)^{2q-1}}{(2q-1)!} p_n^{(2q-1)}(z_0) < p_n(z) < p_n(z_0) + \cdots + \dfrac{(z - z_0)^{2q}}{2q!} p_n^{(2q)}(z_0) \end{array}\right\} z = [\bar{z}, z_0]. \tag{A6}$$

Let us discuss the assumptions (i) and (ii).

(i) For Jacobi (including Legendre, Gegenbauer, and Tchebicheff) polynomials we find $p_n(z_0) > 0$ for $z_0 = 1$. [Let us remark, for instance, that for Legendre polynomials $p_n(z_0) > 0\ \forall z_0 > 1$.] For generalized Laguerre polynomials we find $p_n(z_0) > 0$ for $z_0 = 0$. For the Hermite polynomials we have not found such values, and thus we exclude this case in the following. (ii) For Jacobi (including, Legendre, Gegenbauer, and Tchebicheff) polynomials, $\eta_n = 1$, $f'(z_0) \geq 0$, and $f'(z_0)\eta_n \geq 0$. For generalized Laguerre polynomials $\eta_n = -1$, but $f'(z_0) \leq 0$, and so $\eta_n f'(z_0) \geq 0$.

For the following polynomials we want to make precise the intervals in which they have a dominant peak with the meaning

$$\left.\begin{array}{l} |p_n(z)| \leq p_n(z_0) \\ |p_n'(z)| \leq |p_n'(z_0)| \end{array}\right. \quad \text{for } z \in [\bar{z}, z_0]. \tag{A7}$$

Generalized Laguerre polynomials $L_n^{\alpha}(z)$. The inequalities (A7) are satisfied for $0 \leq z \leq \alpha + \frac{1}{2}$.

Jacobi polynomials $p_n^{(\alpha,\beta)}(z)$. If $\alpha > -\frac{1}{2}$, $\beta > -\frac{1}{2}$, then

$$| p_n^{(\alpha,\beta)}(z)| \leq p_n^{(\alpha,\beta)}(1)$$

for

$$\frac{\beta-\alpha}{\beta+\alpha+1} \leq z \leq 1,$$

$$\left|\frac{d}{dz}\, p_n^{(\alpha,\beta)}(z)\right| \leq \frac{d}{dz}\, p_n^{(\alpha,\beta)}(z)\bigg|_{z=1}$$

for

$$\frac{\beta-\alpha}{\beta+\alpha+3} \leq z \leq 1.$$

Gegenbauer polynomials $C_n^{\lambda}(z)$. For $\lambda > 0$, (A7) is satisfied for $-1 \leq z \leq 1$.

Tchebicheff polynomials. (They correspond to particular cases of Jacobi or Gegenbauer polynomials.) We have two cases. $U_n(z) = C_n^1(z)$ and (A7) are satisfied for $-1 \leq z \leq 1$; $T_n(z) = \cos n\theta$, $z = \cos\theta$, $(d/dz\, T_n(z) = nU_{n-1}(z)$, and (A7) are satisfied for $-1 \leq z \leq 1$.

b. f(z) is a sum of orthogonal polynomials studied in a

Let us put

$$f(z) = \sum_{n_0} \gamma_n \frac{p_n(z)}{p_n(z_0)}\,, \qquad g(z) = \sum \hat{p}_q(z)\beta_q\,. \tag{A8}$$

Substituting q_n given by (A5) in (A1), we get

$$g(z)f'(z_0) = \sum \gamma_n \sum_0^{n-1} \hat{p}_m(z) \sum_{r=m}^{r=n-1} \lambda_r \eta_r\,. \tag{A9}$$

If we identify (A8) and (A9), we get

$$\begin{aligned}
\beta_0 f'(z_0) &= \gamma_1\eta_0\lambda_0 + \cdots + \gamma_n(\eta_0\lambda_0 + \cdots + \eta_{n-1}\lambda_{n-1}) + \cdots,\\
\beta_1 f'(z_0) &= \gamma_2\eta_1\lambda_1 + \cdots + \gamma_n(\eta_1\lambda_1 + \cdots + \eta_{n-1}\lambda_{n-1}) + \cdots,\\
\beta_n f'(z_0) &= \gamma_n(\eta_{n-1}\lambda_{n-1}) + \gamma_{n+1}(\eta_{n-1}\lambda_{n-1} + \eta_n\lambda_n) + \cdots,
\end{aligned} \tag{A10}$$

and

$$\frac{f'(z_0)}{\eta_{n-1}}\left(\frac{\beta_{n-1} - \beta_n}{\lambda_{n-1}}\right) = \sum_{i=n} \gamma_i\,, \tag{A11}$$

$$\frac{\gamma_n}{f'(z_0)} = \frac{\beta_{n-1}}{\lambda_{n-1}\eta_{n-1}} - \beta_n\left(\frac{1}{\eta_n\lambda_n} + \frac{1}{\eta_{n-1}\lambda_{n-1}}\right) + \frac{\beta_{n+1}}{\lambda_n\eta_n}\,.$$

Let us assume now that $\gamma_n > 0\ \forall n$. In all considered cases, Jacobi, Legendre, Gegenbauer, Tchebicheff, and generalized Laguerre polynomials, we have $f'(z_0)\eta_n > 0$. It follows that $\beta_n \geq 0$, $\beta_{n-1} \geq \beta_n$, and another inequality given by (A11)

$$\frac{\gamma_n}{|f'(z_0)|} = \frac{\beta_{n-1}}{\lambda_{n-1}} - \beta_n\left(\frac{1}{\lambda_n} + \frac{1}{\lambda_{n-1}}\right) + \frac{\beta_{n+1}}{\lambda_n} > 0.$$

In some cases the expressions (A8) and (A10), are simple and we write them down.

Legendre Polynomials: In this case (A8) and (A10) can be written for $z_0 = 1$

$$f(z) = \sum_{n_0} \gamma_n p_n(z),$$

$$g(z) = \sum (2n+1)p_n(z)\beta_n\,,$$

$$\beta_{n-1}f'(1) = \gamma_n\left(\frac{1}{n}\right) + \gamma_{n+1}\left(\frac{1}{n} + \frac{1}{n+1}\right) + \cdots + \gamma_{n+p}\left(\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{n+p}\right).$$

Laguerre Polynomials $L_n^{(0)}$ *and* $z_0 = 0$:

$$f(z) = \sum \gamma_n L_n^{(0)}(z)\,,$$

$$g(z) = \sum \beta_n L_n^{(0)}(z),$$

$$-\beta_{n-1}f'(0) = \gamma_n\left(\frac{1}{n}\right) + \cdots + \gamma_{n+p}\left(\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{n+p}\right) + \cdots.$$

In all other cases, β_n can be calculated easily from the expression δ_n, k_n, $p_n(z_0)$, given in textbooks,[10] if we take for z_0 the value $z_0 = 1$ for Jacobi polynomials or $z_0 = 0$ for generalized Laguerre polynomials. But our results can be applied for other z_0 values such that the p_n satisfy assumptions (i) and (ii) of 1, for instance, for the Legendre polynomials with $z_0 > 1$.

Let us recall that the Bessel function $J_0(z)$ can be obtained as a limit of Legendre polynomials, and, consequently,[17] there exists also for $J_0(z)$, $z > 0$, the set of lower and upper bounds when we retain an even or odd number of terms in the Taylor expansion around $z = 0$. More generally, from limit of Jacobi polynomials it is well known that we can obtain functions linked to other Bessel functions, and so our results can be also applied in these cases;

$$\lim_{n\to\infty} n^{-\alpha} p_n^{(\alpha,\beta)}\left(\cos\frac{x}{n}\right) = \left(\frac{2}{x}\right)^{\alpha} J_{\alpha}(x).$$

APPENDIX B

1. Absorptive part of spinless elastic amplitude

In order to simplify the calculation we replace $(2l+1)a_l$ by la_l in the partial-wave expansion; similarly, for the derivative we retain only the leading l-power behavior. We define

$$D_q=\frac{(\sum l^{2q+3}a_l)(\sum l^{2q-1}a_l)}{(\sum l^{2q+1}a_l)^2}, \quad a_l>0$$
$$C_q=\frac{(\sum l^{2q+3}a_l)(\sum la_l)^q}{(\sum l^3a_l)^{q+1}}, \tag{B1}$$

We want to show $D_q \leq C_q$ by many uses of the Schwarz inequality. Let us define

$$\frac{D_q}{C_q}=\eta_q=\frac{(\sum l^{2q-1}a_l)(\sum l^3a_l)^{q+1}}{(\sum l^{2q+1}a_l)^2(\sum la_l)^q}$$

$\eta_1=1, \quad D_1=C_1,$

and

$$\epsilon_q=\frac{(\sum l^{2q-1}a_l)(\sum l^3a_l)}{(\sum l^{2q+1}a_l)(\sum la_l)}.$$

Firstly, we want to show that $\epsilon_q<1$,

$$\epsilon_q=\left(\frac{(\sum l^{2q-1}a_l)^2}{(\sum l^{2q+1}a_l)(\sum l^{2q-3}a_l)}\right)\left(\frac{(\sum l^{2q-3}a_l)(\sum l^3a_l)}{(\sum l^{2q-1}a_l)(\sum la_l)}\right). \tag{B2}$$

The first term in the product at the right-hand side of (B2) is less than 1. It follows that

$$\epsilon_q<\epsilon_{q-1}<\cdots<\epsilon_2=\frac{(\sum l^3a_l)^2}{(\sum l^5a_l)(\sum la_l)}<1.$$

η_q can be written

$$\eta_q=\left(\frac{(\sum l^{2q-1}a_l)^4}{(\sum l^{2q+1}a_l)^2(\sum l^{2q-3}a_l)^2}\right)\eta_{q-1}\epsilon_{q-1}. \tag{B3}$$

Using the Schwarz inequality, we get

$$\eta_q<\eta_{q-1}\epsilon_{q-1}<\eta_{q-1}<\eta_{q-2}\epsilon_{q-2}<\cdots<\eta_2=\frac{(\sum l^3a_l)^4}{(\sum l^5a_l)^2(\sum la_l)^2}<1.$$

It follows that $\eta_q \leq 1$ and $D_q \leq C_q$.

We want to show that $C_q<C_{q+1}{}^{q/(q+1)}$ for $q\geq 1$, where C_q is given in (B1). First, for $q=1$ it is easy to get $C_1{}^2<C_2$. Secondly, we get

$$C_q{}^2=\frac{(\sum la_l)^{2q}}{(\sum l^3a_l)^{2q+2}}\left(\sum l^{2q+3}a_l\right)^2 < \frac{(\sum la_l)^{2q}}{(\sum l^3a_l)^{2q+2}}\left(\sum l^{2q+5}a_l\right)\left(\sum l^{2q+1}a_l\right),$$

or $C_q{}^2<C_{q-1}C_{q+1}$. Assuming that the inequality is true for $q-1$, we get $C_q{}^2<C_q{}^{(q-1)/q}C_{q+1}$ and $C_q<C_{q+1}{}^{q/(q+1)}$.

We study one case of strong scaling. Let us assume

(i) $1\geq a_l\geq 0, \quad \sum l^p a_l \simeq \sum^{L_{\max}} l^p a_l,$

$L_{\max}\leq\sqrt{e_2}\,s^{1/2}(\log s)^{\gamma/2}, \quad p$ integer, $\gamma>0,$

(ii) $e_1(\log s)^\gamma<s^{-1}\sum la_l<e_2(\log s)^\gamma, \qquad$ (B4)

where

$e_1>0$ and $e_2>0$ are constants,

(iii) $$|a(s,t)|\simeq\sum\frac{|t|^n}{s^n}\frac{(\text{const})^n}{(n!)^2}\frac{\sum l^{2n+1}a_l}{\sum la_l}.$$

First we want to show that

$$b\simeq s^{-1}\frac{\sum l^3a_l}{\sum la_l}\simeq \text{const}\times(\log s)^\gamma.$$

$\sum l^3a_l<e_2{}^2s^2(\log s)^{2\gamma}$ has an upper bound; it has also a lower bound, as we shall see,

$$s^{-1}\sum_0^{[e_1s(\log s)^\gamma/2]^{1/2}} la_l<\tfrac{1}{2}e_1(\log s)^\gamma \Rightarrow s^{-1}\sum_{[e_1s(\log s)^\gamma/2]^{1/2}} la_l>\tfrac{1}{2}e_1(\log s)^\gamma, \tag{B5}$$

$$\sum l^3a_l>\sum_{[e_1s(\log s)^\gamma/2]^{1/2}}^\infty l^3a_l>s\tfrac{1}{2}e_1(\log s)^\gamma\sum_{[e_1s(\log s)^\gamma/2]^{1/2}}^\infty la_l>(\tfrac{1}{2}e_1)^2s^2(\log s)^{2\gamma}\Rightarrow\frac{e_1{}^2}{2e_2}(\log s)^\gamma<b<\frac{e_2{}^2}{e_1}(\log s)^\gamma.$$

Secondly, with (B4) and (B5) we show that C_q is bounded for any q finite

$$C_q<\left(\frac{2e_2}{e_1}\right)^{2q+2}<(\text{const})^q.$$

Thirdly, we get an upper bound (t complex around $t=0$) for $|a(s,t)|$,

$$\frac{\sum l^{2n+1}a_l}{\sum la_l}=C_{n-1}\left(\frac{\sum l^3a_l}{\sum la_l}\right)^n,$$

and it follows that

$$|a(s,t)|<\sum\frac{(|t|\,\text{const}\,b)^n}{(n!)^2}=\exp(\text{const}|\tau|^{1/2}),$$

with $\tau=tb$.

2. Multiparticle generating function

Let us define for $q=1,2,..$

$$D_q=\frac{\langle n^{q+1}\rangle\langle n^{q-1}\rangle}{\langle n^q\rangle^2},\quad C_q=\frac{\langle n^{q+1}\rangle}{\langle n\rangle^{q+1}}.$$

We want to show $D_q\leq C_q$ using the Schwarz inequality. Let us define

$$\frac{D_q}{C_q}=\eta_q=\frac{\langle n\rangle^{q+1}\langle n^{q-1}\rangle}{\langle n^q\rangle^2},$$

$$\epsilon_q=\frac{\langle n^{q-1}\rangle\langle n\rangle}{\langle n^q\rangle}.$$

Firstly, we want to show that $\epsilon_q<1$,

$$\epsilon_q=\left(\frac{\langle n^{q-1}\rangle^2}{\langle n^q\rangle\langle n^{q-2}\rangle}\right)\frac{\langle n^{q-2}\rangle\langle n\rangle}{\langle n^{q-1}\rangle}<\epsilon_{q-1}<\cdots<\epsilon_1=1.$$

Secondly, η_q can be written

$$\eta_q=\left(\frac{\langle n^{q-1}\rangle^4}{\langle n^q\rangle^2\langle n^{q-2}\rangle^2}\right)\left(\frac{\langle n^{q-2}\rangle^2\langle n\rangle^{q+1}}{(\langle n^{q-1}\rangle)^3}\right). \tag{B6}$$

The first term in the product at the right-hand side of (B7) is bounded by 1. It follows that

$$\eta_q\leq\eta_{q-1}\epsilon_{q-1}\leq\eta_{q-1}\leq\cdots\leq\eta_1=1\Rightarrow D_q\leq C_q.$$

The property $C_q<(C_{q+1})^{q/(q+1)}$ for $q\geq 1$ is very well known. Firstly, for $q=1$ we get $C_1{}^2<C_2$, and secondly,

$$C_q{}^2=\frac{\langle n^{q+1}\rangle^2}{\langle n\rangle^{2q+2}}<\frac{\langle n^{q+2}\rangle\langle n^q\rangle}{\langle n\rangle^{q+2}\langle n\rangle^q}=C_{q-1}C_{q+1},$$

and assuming that the relation is true for $q-1$, we get the same for q.

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[17] I thank André Martin, who kindly pointed out to me this result.