Models with quark confinement and linear trajectories without parity doubling*

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We consider simple dynamic models where the quarks satisfy the Dirac equation with a confining potential. We show that if the confining potential is a Lorentz scalar, the solutions to the Dirac equation satisfy the MacDowell symmetry. We discuss the slope of Regge trajectories in these models and show that in order for the trajectory to be linear (in $s = E^2$) the potential must have a specific j dependence, which produces a cut in j of the type first proposed by Carlitz and Kislinger. In other words, the same mechanism which produces linear trajectories in the first place also produces the cut which removes the parity partners. The specific j dependence is expected in "bag" models, as well as relativistic soliton solutions.

I. INTRODUCTION

The quark model has been very successful in predictions for hadron spectroscopy, high-energy and large-transverse-momentum behavior of cross sections, and current algebra. In any consistent theory of hadrons based on the quark model the question of quark confinement is of crucial importance. That there may be a sufficient infrared catastrophe to provide confinement in an asymptotically free gauge theory of quarks and gluons with an exact SU(3) color symmetry, remains a most exciting possibility.[1] On a more direct level the dynamic models proposed by MIT[2] and SLAC[3] groups, in which the quarks are confined to a region of space called a "bag," provide new insight into the question of confinement as well as into the properties of the hadronic states.

In these "bag" models the quarks satisfy Dirac's equation with either covariant boundary conditions at the surface of the bag or a self-consistent confining "potential" through their interaction with a scalar field.[2,3] So far, exact solutions have been obtained only for the lowest hadronic states. In order to study the Regge trajectories in these models, however, we shall need solutions for the higher excited states, which are much more difficult because of the lack of spherical symmetry in general, as demanded by the self-consistency or the boundary conditions. As a first step in studying this problem, we therefore limit ourselves to a much simpler problem here, namely, what *general* properties the confining "potential" should have, without worrying about the self-consistency or the boundary conditions, if the resulting Regge trajectory is to be linear in $s = E^2$, especially near $E = 0$.[4]

It has been pointed out[3] that if the confining potential is a Lorentz scalar, there is no Klein paradox as occurs for sharp localization of a Dirac particle in a strong vector (Coulomb) potential.[5] We find that the scalar potential, surprisingly, also ensures that solutions of the Dirac equation would satisfy the MacDowell symmetry,[6] which demands the partial-wave amplitudes of opposite parity are related through $f_j^+(E) = f_j^-(-E)$, where the plus (minus) sign denote $j = l + \frac{1}{2}$ ($j = l - \frac{1}{2}$). It is well known that MacDowell symmetry together with linear (in E^2) baryon trajectories would seem to imply parity doublets, which are experimentally absent.[7] The Dirac equation with a scalar confining potential therefore provides a simple dynamical laboratory to study this question.

We find that in order for the trajectory to be linear near $E = 0$ the potential must have a specific j dependence, which produces a cut in j of the type first proposed by Carlitz and Kislinger[8] and the trajectory moves to another sheet. In other words, the *same* mechanism which produces linear trajectories in the first place in the Dirac equation, also produces the cut which removes MacDowell partners. Furthermore, the specific j dependence is not unexpected in the more realistic "bag" models that have been proposed.

In Sec. II we show that the solutions of the Dirac equations with a scalar potential satisfy the MacDowell symmetry. In Sec. III we discuss the slope of a Dirac trajectory and demonstrate that a linear trajectory is inconsistent with a simple j-independent potential and show the j dependence demanded by linearity. In Sec. IV we give some simple examples with this specific j dependence and discuss the possibility that such j dependence might emerge in more realistic models.

II. MacDOWELL SYMMETRY AND SCALAR POTENTIAL

Consider the simple dynamic model in which the quarks satisfy the Dirac equation with a static confining potential. In order to have infinitely rising trajectories[4] and to avoid Klein's paradox,[3] one finds that the confining potential should be a Lorentz scalar rather than the fourth component of a four-vector. A perhaps unexpected advantage of a scalar potential is that the scattering matrix in such a model automatically satisfies the MacDowell symmetry. Thus we have a model to study the question of linearly rising trajectories and how the experimentally unobserved parity partners are removed.

First, let us establish the MacDowell symmetry of the scattering amplitude. Consider the Dirac equations for both parity states with a spherically symmetric scalar potential:

$$\frac{d}{dr}(rf_+) - \frac{j+\frac{1}{2}}{r}(rf_+) - (m+E-V)rg_+ = 0, \qquad j = l + \tfrac{1}{2} \quad (2.1)$$
$$\frac{d}{dr}(rg_+) + \frac{j+\frac{1}{2}}{r}(rg_+) - (m-E-V)rf_+ = 0,$$

$$\frac{d}{dr}(rf_-) + \frac{j+\frac{1}{2}}{r}(rf_-) - (m+E-V)rg_- = 0, \qquad j = l - \tfrac{1}{2} \quad (2.2)$$
$$\frac{d}{dr}(rg_-) - \frac{j+\frac{1}{2}}{r}(rg_-) - (m-E-V)rf_- = 0.$$

Comparing Eqs. (2.1) and (2.2) we immediately get

$$f_+(j,E,r) = g_-(j,-E,r), \qquad g_+(j,E,r) = f_-(j,-E,r) \quad (2.3)$$

to within a normalization constant.

The partial-wave scattering amplitude is obtained from the asymptotic behavior of the f's and g's (see Ref. 9):

$$f \sim N(E+M)^{1/2}\sin[kr - \tfrac{1}{2}l\pi + \delta(j,E)], \qquad g \sim N(E-M)^{1/2}\sin[kr - \tfrac{1}{2}l'\pi + \delta(j,E)], \quad l' = 2j - l. \quad (2.4)$$

Using the relation (2.3), we have

$$e^{2i\delta_+(j,E)} = e^{2i\delta_-(j,E)}$$

or (2.5)

$$S_+(j,E) = S_-(j,-E),$$

which is the MacDowell symmetry.

As an explicit example, the scattering amplitudes for the *scalar* Coulomb potential $V = \beta/r$ is given by

$$S_+(j,E) = \frac{j+\frac{1}{2} - i\eta}{\gamma + i\eta'}\frac{\Gamma(\gamma - i\eta')}{\Gamma(\gamma + i\eta')}\exp[i\pi(j+\tfrac{1}{2}-\gamma)],$$
$$S_-(j,E) = \frac{j+\frac{1}{2} + i\eta}{\gamma + i\eta'}\frac{\Gamma(\gamma - i\eta')}{\Gamma(\gamma + i\eta')}\exp[i\pi(j+\tfrac{1}{2}-\gamma)], \quad (2.6)$$

where $\gamma = [(j+\frac{1}{2})^2 + \beta^2]^{1/2}$, $\eta = \beta E/(E^2 - m^2)^{1/2}$, and $\eta' = \beta m/(E^2-m^2)^{1/2}$. For the ordinary Coulomb potential $V = \alpha/r$, which is the fourth component of a four-vector, the scattering amplitude is obtained if, in (2.6), we replace γ by $[(j+\frac{1}{2})^2 + \alpha^2]^{1/2}$, and interchange η and η'.[10] We see that only the scalar Coulomb potential gives MacDowell-symmetric scattering amplitudes.

III. THE SLOPE OF THE REGGE TRAJECTORIES FROM THE DIRAC EQUATION

It is instructive to examine the scattering amplitudes for the scalar Coulomb potential in a little more detail. The leading Regge trajectory

$$\alpha_+(E) = \alpha_-(E) = -\tfrac{1}{2} + i\eta = \begin{cases} -\frac{1}{2} + i\frac{\beta E}{(E^2-m^2)^{1/2}}, & E > m \\ -\frac{1}{2} + \frac{\beta E}{(m^2-E^2)^{1/2}}, & E < m \end{cases} \quad (3.1)$$

does *not* have MacDowell partners because of the factors $(j + \frac{1}{2} \pm i\eta)$ in the numerator of the partial-wave amplitudes. In fact the "trajectory" $\alpha_+(E)$ does not exist at all. Unfortunately, this particular mechanism for removing the parity partners, although it depends only on the existence of a spin-orbit coupling, obviously does not apply to the case of an (idealized) trajectory that is linear in E^2:

$$\alpha(E) = \alpha_0 + \alpha' E^2. \quad (3.2)$$

In fact we can prove a pedagogical theorem which shows the impossibility of having an idealized linear trajectory at least near $E = 0$ in models based on the Dirac equation with an ordinary (namely, j-independent, see below) potential but at the same time suggest how, in the more general bag models, parity partners of linear trajectories disappear through cuts in the angular momentum plane.[8]

Theorem. The slope of a given Regge trajectory is given by

$$\frac{\Delta j}{\Delta E} = \mp \frac{1}{\int_0^\infty r^2 dr (f^* g + f g^*)/r} \quad \text{for } j = l \pm \tfrac{1}{2}. \quad (3.3)$$

Proof. Consider the case $j = l + \frac{1}{2}$. Letting

$$\psi = \begin{pmatrix} f \\ g \end{pmatrix}$$

be the solution of the Dirac equation of a given E and j, we have after a little rearrangement

$$\begin{pmatrix} m - V & -\frac{d}{dr} - \frac{j+\frac{1}{2}}{r} \\ \frac{d}{dr} - \frac{j+\frac{1}{2}}{r} & -(m-V) \end{pmatrix}\begin{pmatrix} rf \\ rg \end{pmatrix} = E\begin{pmatrix} rf \\ rg \end{pmatrix}. \quad (3.4)$$

Treat the change in j, Δj, when we go to a neighboring state on the trajectory, as a perturbation and calculate the corresponding change in E. To first order, one can use the unperturbed solution

ψ. Hence we have $\langle\psi, (\Delta H)\psi\rangle = \langle\psi, (\Delta E)\psi\rangle$ or

$$\int r^2 dr(f^*g + g^*f)\left(-\frac{\Delta j}{r}\right) = \Delta E \int r^2 dr(f^*f + g^*g). \tag{3.5}$$

Using the normalization condition

$$\int_0^\infty r^2 dr(f^*f + g^*g) = 1, \tag{3.6}$$

we have

$$\frac{\Delta j}{\Delta E} = -\frac{1}{\int_0^\infty r^2 dr(f^*g + fg^*)/r}.$$

To calculate the slope we only need infinitesimal Δj; thus the first-order perturbation result is exact. The same calculation goes through for the $j = l - \frac{1}{2}$ case.

Results similar to the above in the case of the Schrödinger equation are well known.[11]

Now if the trajectory has the idealized linear form (3.2)

$$\frac{\Delta j}{\Delta E} \xrightarrow[E\to 0]{} 0. \tag{3.7}$$

Hence, the integral in the denominator must diverge as $E \to 0$. This is impossible because ψ is normalizable.

Thus the problem of removing MacDowell partners for a linear (in E^2) trajectory is not so much the removal of the partners, which are generally absent for top trajectories in the Dirac equation (because of the spin-orbit coupling), but the problem is the creation of linearly rising trajectories, which near $E = 0$ impossible with a conventional static scalar potential.

To see how we can realize the idealized linear trajectory in more general models, we notice that we might consider (3.3) as defining a length of the system, especially if we write (3.3) as

$$\frac{\Delta j}{\Delta E} = \pm\frac{\int_0^\infty r^2 dr(f^*f + g^*g)}{\int_0^\infty r^2 dr(f^*g + g^*f)/r} \equiv L\,. \tag{3.8}$$

The condition (2.7) seems to suggest that we consider models with a length L proportional to E:

$$L \to \text{constant} \times |E|. \tag{3.9}$$

The absolute-value sign is necessary because of the MacDowell-symmetry condition (2.3). This violates the analyticity requirement. An alternative, however, suggests itself, by considering the inverse of the trajectory function, $j = \alpha(E)$, given by $E = \alpha^{-1}(j)$. For the ideal linear trajectory (3.2), the requirement (3.9) becomes

$$L \to \text{constant} \times (j - \alpha_0)^{1/2}. \tag{3.10}$$

We may consider, for instance, a j-dependent potential. In the scalar Coulomb case we identify L with the "Bohr radius," which then should have the form (3.10)

$$a_0 = \frac{\beta}{m} = \text{constant} \times (j - \alpha_0)^{1/2}, \tag{3.11}$$

which implies a j-dependent β:

$$\beta = g^2(j - \alpha_0)^{1/2}. \tag{3.12}$$

The top trajectory will now be even in E, but the absence of parity partners along that trajectory does not violate the MacDowell symmetry, because the residue of α_- is now proportional to

$$(\alpha + \tfrac{1}{2})^{1/2} + g^2E/(m^2 - E^2)^{1/2}$$

and vanishes for negative E. In other words the trajectory moves to another sheet for negative E, as in the models of Ref. 8.

An alternative way is to consider a bag model with L identified as the bag size that is j-dependent. Both these possibilities will be considered in the next section.

IV. MODELS WITH QUARK CONFINEMENT AND LINEAR TRAJECTORIES WITHOUT PARITY DOUBLING

The simplest model turns out to be exactly soluble and is again the scalar Coulomb potential with a j-dependent coupling constant. We find from (2.6) the leading trajectory, with $\beta = g^2(j - \alpha_0)^{1/2}$, $\alpha_0 = -\frac{1}{2}$, is given by

$$\alpha(E^2) = \alpha_0 + \frac{(g^2)^2E^2}{(m^2 - E^2)^{1/2}}, \quad E < m. \tag{4.1}$$

We can obtain a linear trajectory and quark confinement if $m \gg 1$ GeV and $g^2 \gg 1$, with $\alpha(E^2) = -\frac{1}{2} + \alpha' E^2$ and

$$\alpha' = \left(\frac{g^2}{m}\right)^2 \approx 1\ \text{GeV}^{-2}. \tag{4.2}$$

It is interesting to note that if $g^2 \approx 137$, then $m \approx 137$ GeV. Together with the $1/r$ behavior of the potential they are very suggestive, although it is not clear at all how a magnetic charge could lead to a scalar potential.

In any case, while one must not take the particular form of the potential that leads to the partial-wave scattering amplitudes (2.6) too literally, the explicit form with $\beta = g^2(j - \alpha_0)^{1/2}$, $g^2/m \to \alpha'$, and $m \to \infty$ could have useful phenomenological applications.

The second way of introducing j dependence is by making the bag size a function of j. For simplicity we shall assume absolute confinement by having $V(r) = \infty$ for $r > R$. To discuss the MacDowell symmetry in this case we need to use a limiting process. Let $V(r) = M$ for $r > R$, and join the interior ($r < R$) solution rf_+ and rg_+ to the exterior solution. For the case $j = l + \frac{1}{2}$, we have

$$f_+ = [A^+h_j^{(1)} + \tfrac{1}{2}(KR) + B^+h_j^{(2)} + \tfrac{1}{2}(KR)],$$
$$g_+ = \left(\frac{E-M}{E+M}\right)^{1/2}[A^+h_j^{(1)} - \tfrac{1}{2}(KR) + B^+h_j^{(2)} - \tfrac{1}{2}(KR)], \tag{4.3}$$

where $h^{(1)}, h^{(2)}$ are the spherical Hankel functions and $K^2 = E^2 - M^2$.

For large M the S matrix is given by

$$S^+ = \frac{B^+}{A^+} \to e^{-2MR}e^{-i\pi(j+1/2)}\frac{f_+(j,E,R) + [(E-m)/(E+m)]^{1/2}g_+(j,E,R)}{f_+(j,E,R) - [(E-m)/(E+m)]^{1/2}g_+(j,E,R)} \tag{4.4}$$

when $M \to \infty$, $S^+ \to 0$, reflecting the fact that the particle is confined. The function

$$\bar{S}^+ \equiv \lim_{M\to\infty} e^{2MR}S^+$$

remains finite and correctly gives the bound states of the problem. The case of $j = l - \frac{1}{2}$ can be handled the same way. The MacDowell symmetry is satisfied by the functions $\bar{S}^+$ and $\bar{S}^-$.

The requirement of a bag size,

$$L \equiv R = \text{constant} \times (j - \alpha_0)^{1/2}, \tag{4.5}$$

actually is not unexpected. Using semiclassical arguments, one can show[12] that in the MIT bag model, for large j,

$$L = (6/\pi^2 g^2 B)^{1/4}\sqrt{j},$$

with $g^2/4\pi$ the color gluon coupling constant and B the magnitude of the volume tension of the bag. Although in this case, the hadrons are of tubular shape and rotating in space with a uniform angular velocity, one may consider the spherical static bag with a j-dependent size as a (time-averaged) approximation to the more realistic situation.

For the simplest case, when

$$V(r) = \begin{cases} \infty, & r > R \\ -m, & r < R \end{cases} \tag{4.6}$$

we have

$$\bar{S}^+ = e^{-i\pi(j+1/2)}\frac{J_{j+1}(ER) + J_j(ER)}{J_{j+1}(ER) - J_j(ER)}, \tag{4.7}$$

where J_ν is the Bessel function of order ν. With

$$R = R_0(j+1)^{1/2}$$

we have for $ER \ll 1$

$$\bar{S}^+ \approx \frac{ER/2 + j + 1}{ER/2 - j - 1} = \frac{E + (2/R_0)(j+1)^{1/2}}{E - (2/R_0)(j+1)^{1/2}} \tag{4.8}$$

or

$$\alpha(E^2) = 1 + \left(\frac{R_0}{2}\right)^2 E^2. \tag{4.9}$$

Again the degeneracy between opposite parity states is removed.

The $\sqrt{j}$ dependence of the potential must be the reflection of a fully relativistic theory. In this connection it is informative to consider the "potential" $V(r)$ in the discussion so far to be the functional form arising from a fully relativistic self-binding nonlinear spinor field[13,14]; i.e., a soliton field. In this case Ψ^2 or some other bilinear combination plays the role of the potential. Thus, the length L in (3.8), which is closely related to the range of the potential, is in turn closely related to the "radius" of the solution.

Using a variational approach the classical field equations can be solved for large j in the limit of large bare spinor field mass and strong self-coupling. The solutions peak sharply at the Compton wavelength of the soliton providing a narrow spherical shell of potential of radius $L \approx \sqrt{j}$. The trajectory is indeed linear and given by $E = \text{constant} \times \sqrt{j}$.

Soliton solutions are outside of the scope of this paper and will be discussed in the future.

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