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Gauge-symmetry hierarchies*

Eldad Gildener
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138
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It is shown that one cannot artifically establish a gauge hierarchy of any desired magnitude by arbitrarily adjusting the scalar-field parameters in the Lagrangian and using the tree approximation to the potential; radiative corrections will set an upper bound on such a hierarchy. If the gauge coupling constant is approximately equal to the electromagnetic coupling constant, the upper bound on the ratio of vector-meson masses is of the order of $\alpha^{-1/2}$, independent of the sclar-field masses and their self-couplings. In particular, the usual assumption that large scalar-field mass ratios in the Lagrangian can induce large vector-meson mass ratios is false. A thus far unsuccessful search for natural gauge hierarchies is briefly discussed. It is shown that if such a hierarchy occurred, it would have an upper bound of the order of $\alpha^{-1/2}$.

I. INTRODUCTION

A gauge-symmetry hierarchy is said to occur if some of the gauge symmetries of a theory are much more strongly broken than others. The importance of such hierarchies is that they are essential to making sense out of attempts to embed the weak and electromagnetic interactions[1], or, more ambitiously, the weak, electromagnetic, and strong interactions into spontaneously broken gauge theories based on a simple gauge group.[2] In addition to their aesthetic virtue, such theories offer objective advantages. For example, since such theories possess only one free gauge coupling constant, the mixing angle of the gauge coupling constants of an $SU(2)\otimes U(1)$ weak-electromagnetic subgroup is determined. However, such theories demand the presence of superheavy vector bosons in order to suppress unobserved interactions. Thus symmetry breakdown occurs at two mass scales, one associated with the W and Z bosons, the other with the superheavy vector bosons.

In a theory where the gauge symmetry is spontaneously broken by the vacuum expectation values of a set of weakly coupled elementary scalar fields, it has traditionally been assumed that one could artificially arrange for any desired gauge hierarchy by arbitrarily adjusting the scalar-field parameters in the Lagrangian and using the tree approximation to the effective potential.[3] However, the precision with which one can specify the scalar self-couplings is limited by the effects of the one-loop contributions to the effective potential. Our main result, contained in Sec. II, is to show (in the context of a simple model) that the radiative corrections set a bound on the tree-approximation gauge hierarchy, independent of the scalar-field masses or their self-couplings. If the gauge coupling constant is approximately equal to the electromagnetic coupling constant, then the upper bound on the ratio of vector-meson masses is shown to be of the order of $\alpha^{-1/2}$. In particular, contrary to what is usually assumed, a large ratio of scalar-field masses in the Lagrangian cannot be used to induce a large ratio in the vector-meson masses. The problem of gauge hierarchies is steeped in the radiative corrections: The larger the gauge hierarchy, the greater the number of loop diagrams one must include in the approximation. The question of whether there is an inherent bound on the gauge hierarchy for any particular model is very interesting, but completely open.

In Sec. III, we discuss a thus far unsuccessful search for naturally induced gauge hierarchies. It is shown that even if such hierarchies were found, they *would* have an upper bound of the order of $\alpha^{-1/2}$. We explain why pseudosymmetries do not lead to such natural hierarchies. Section IV contains our conclusions.

II. CONTRIVED GAUGE HIERARCHIES

A. The bound on tree-approximation hierarchies

Consider the following simple model: The gauge group is $O(n)$, and the scalar fields transform as two n-vectors, $\vec{\chi}$ and $\vec{\eta}$. If the values of $\vec{\chi}$ and $\vec{\eta}$ at the minimum of the potential (designated $\vec{\chi}_m$ and $\vec{\eta}_m$) are such that one of them has a component which is orthogonal and small compared to the other, then there is a gauge-symmetry hierarchy: $O(n)$ is strongly broken down to $O(n-1)$ and weakly down to $O(n-2)$.

For simplicity let us assume that the theory possesses the discrete symmetry $\vec{\chi}\to-\vec{\chi}$. The tree approximation to the effective potential is

$$V_0=-\tfrac{1}{2}m_1^2\vec{\chi}^2-\tfrac{1}{2}m_2^2\vec{\eta}^2+\tfrac{1}{4}f_1(\vec{\chi}^2)^2+\tfrac{1}{4}f_2(\vec{\eta}^2)^2+\tfrac{1}{2}f_3\vec{\chi}^2\vec{\eta}^2+\tfrac{1}{4}f_4(\vec{\chi}\cdot\vec{\eta})^2. \quad (2.1)$$

Since only the last term depends on the relative orientation of $\vec{\chi}$ and $\vec{\eta}$, it is obvious that a minimum of V_0 for $f_4 < 0$ will occur at $\vec{\chi}_m \parallel \vec{\eta}_m$, whereas if $f_4 > 0$, it will occur at $\vec{\chi}_m \perp \vec{\eta}_m$. But only the latter can result in a gauge hierarchy. In order that V_0 be bounded from below, one must have $f_1 > 0$ and $f_2 > 0$. (Of course the effective potential could be bounded from below even if its zero-loop approximation is not, but for now we are interested in minimizing V_0.) Thus we shall assume that f_1, f_2, and f_4 are positive. The minimization of V_0 is somewhat detailed and thus outlined in an appendix. We will freely refer to the relevant results of that appendix.

The symmetry-breaking stationary points of V_0 are

$$\vec{\chi}_a = 0, \quad \vec{\eta}_a^{\,2} = m_2^2/f_2, \tag{2.2a}$$

$$\vec{\eta}_b = 0, \quad \vec{\chi}_b^{\,2} = m_1^2/f_1, \tag{2.2b}$$

and

$$\vec{\chi}_c^{\,2} = \frac{f_2 m_1^2 - f_3 m_2^2}{f_1 f_2 - f_3^2}, \quad \vec{\eta}_c^{\,2} = \frac{f_1 m_2^2 - f_3 m_1^2}{f_1 f_2 - f_3^2}, \tag{2.2c}$$

with $\vec{\chi}_i \cdot \vec{\eta}_i = 0$ $(i = a, b, c)$. Of course only the third solution can produce a gauge hierarchy.

To quantify the gauge hierarchy, we consider the vector-meson masses. We have calculated the zero-loop approximation to the gauge-field mass-squared matrix, and the result for $\vec{\chi}_m \cdot \vec{\eta}_m = 0$ is that the nonzero eigenvalues are

$$\begin{aligned} &1 \text{ eigenvalue} = g^2(\vec{\chi}_m^{\,2} + \vec{\eta}_m^{\,2}), \\ &n-2 \text{ eigenvalues} = g^2\vec{\chi}_m^{\,2}, \\ &n-2 \text{ eigenvalues} = g^2\vec{\eta}_m^{\,2}, \end{aligned} \tag{2.3}$$

where g is the gauge coupling constant. If $\vec{\chi}_m^{\,2} \gg \vec{\eta}_m^{\,2}$ (or vice versa), then there is a gauge hierarchy. If we define the magnitude of the hierarchy to be the ratio of the heavy-vector-meson masses to the light ones, then

$$\frac{M_H}{M_L} \simeq \left(\frac{\vec{\chi}_m^{\,2}}{\vec{\eta}_m^{\,2}}\right)^{1/2}, \tag{2.4}$$

where the equality is exact for all but one of the mesons.

Let us first focus on the case $f_3 = 0$. With this constraint $\vec{\chi}_m^{\,2} = \vec{\chi}_c^{\,2}$, $\vec{\eta}_m^{\,2} = \vec{\eta}_c^{\,2}$, so that

$$\frac{\vec{\chi}_m^{\,2}}{\vec{\eta}_m^{\,2}} = \frac{f_2 m_1^2}{f_1 m_2^2}. \tag{2.5}$$

We assume that $m_1^2 \geq m_2^2$ and that f_1 and f_2 are always chosen such as to enhance the hierarchy, i.e., $f_2 \geq f_1$. Thus we have the conventional wisdom that a gauge hierarchy of any magnitude can simply be induced in tree approximation by an appropriate choice of the scalar masses and self-couplings; in particular, a large scalar mass ratio can be used to induce a large gauge hierarchy. However, this conclusion is false. The reason is that the hierarchy in tree approximation is critically dependent upon the value of f_3, but since the one-loop gauge-field contribution includes a term of the order of $g^4\vec{\chi}^2\vec{\eta}^2$, it sets a limit (of order g^4) to the precision with which the term proportional to f_3 can be specified. Thus it is the radiative corrections that determine the gauge hierarchy.

To understand how this occurs, one needs to consider the minimum of V_0 as a function of f_3. The result is graphically presented in Fig. 1, where $V_0(\vec{\chi}_c^{\,2}, \vec{\eta}_c^{\,2})$ is plotted against f_3. The values of $V_0(\vec{\chi}_a^{\,2}, \vec{\eta}_a^{\,2})$ and $V_0(\vec{\chi}_b^{\,2}, \vec{\eta}_b^{\,2})$ are indicated by dashed horizontal lines. The only values of f_3 for which the minimum of V_0 occurs at $\vec{\chi}_c^{\,2}, \vec{\eta}_c^{\,2}$, with both $\vec{\chi}_c^{\,2}$ and $\vec{\eta}_c^{\,2}$ positive, are $-(f_1 f_2)^{1/2} < f_3 < f_1 m_2^2/m_1^2$. Thus only in this domain are gauge hierarchies possible.

The magnitude of the gauge hierarchy was defined in terms of $\vec{\chi}_m^{\,2}/\vec{\eta}_m^{\,2}$ $(= \vec{\chi}_c^{\,2}/\vec{\eta}_c^{\,2}$ in the domain of interest). Note that

$$\frac{\partial(\vec{\chi}_m^{\,2}/\vec{\eta}_m^{\,2})}{\partial f_3} = \frac{f_2 m_1^4 - f_1 m_2^4}{(f_1 m_2^2 - f_3 m_1^2)^2} > 0. \tag{2.6}$$

Thus the gauge hierarchy increases monotonically as f_3 increases. Also,

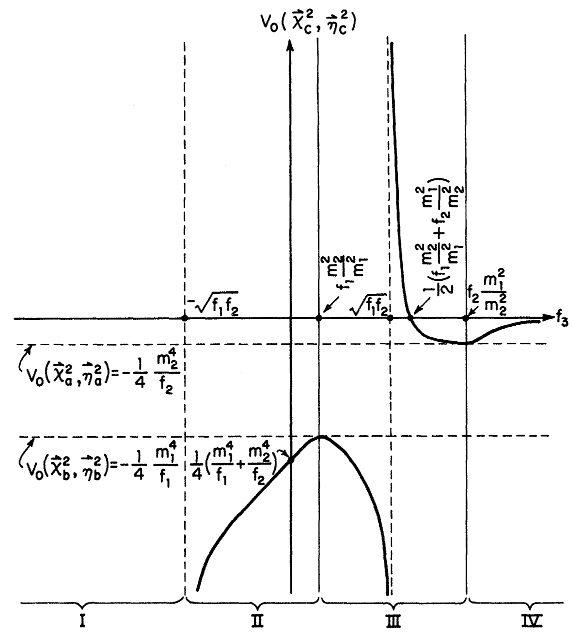

FIG. 1. In region I, $\vec{\chi}_c^{\,2}$ and $\vec{\eta}_c^{\,2}$ are negative. In region III, $\vec{\chi}_c^{\,2}$ or $\vec{\eta}_c^{\,2}$ is negative. In both regions II and IV, $\vec{\chi}_c^{\,2}$ and $\vec{\eta}_c^{\,2}$ are positive, but in region IV, the maximum of V_0 occurs at $\vec{\chi}_b^{\,2}, \vec{\eta}_b^{\,2}$, whereas in region II it occurs at $\vec{\chi}_c^{\,2}, \vec{\eta}_c^{\,2}$. Thus only in region II are gauge hierarchies possible.

$$\frac{\vec{\chi}_m{}^2}{\vec{\eta}_m{}^2}=\frac{f_2 m_1{}^2 - f_3 m_2{}^2}{f_1 m_2{}^2 - f_3 m_1{}^2} \xrightarrow[f_3 \to f_1 m_2{}^2/m_1{}^2]{} \infty . \tag{2.7}$$

Hence (ignoring for the moment the validity of the approximation) the zero-loop potential is unbounded. Let us investigate this situation more carefully. It is convenient to express f_3 as $f_3 \equiv f_1 m_2{}^2/m_1{}^2 - \Delta f_3$ (where $\Delta f_3 > 0$). Then

$$\vec{\eta}_m{}^2 = \frac{\Delta f_3 m_1{}^2}{f_1 f_2 - f_3^2} \tag{2.8a}$$

and

$$\vec{\chi}_m{}^2 = \frac{f_2 m_1{}^2[1 - f_1 m_2{}^4/f_2 m_1{}^4 + (\Delta f_3/f_2) m_2{}^2/m_1{}^2]}{f_1 f_2 - f_3{}^2}. \tag{2.8b}$$

As $\Delta f_3 \to 0$, $\vec{\eta}_m{}^2 \to 0$ and the symmetry-breaking pattern goes from $O(n) \to O(n-2)$ to $O(n) \to O(n-1)$. This might seem to be inconsistent with the Georgi-Glashow theorem,[4] which can be restated as follows: Suppose that $V(\varphi) = V_a(\varphi) + \delta V(\varphi)$ and that $V_a(\varphi)$ has a minimum at $\varphi = \lambda$ and $V(\varphi)$ has one at $\varphi = \lambda + \delta\lambda$. Next assume that for $\langle\varphi\rangle_0 = \lambda$, the symmetry of the theory is spontaneously broken, but an unbroken subgroup persists. Then if $\delta V(\varphi)$ and the induced $\delta\lambda$ are small, and if all the zero eigenvalues of the lowest-order scalar-field mass matrix of the asymmetric theory, $\partial^2 V_a(\lambda)/\partial\varphi_i \partial\varphi_j$, are associated with Goldstone bosons, the exact vacuum expectation values, $\lambda + \delta\lambda$, will leave unbroken the same subgroup as λ. It is this theorem that justifies using lowest-order perturbation theory to establish the symmetry-breaking pattern of a theory. The exceptions to the theorem are determined by the lowest-order scalar-field mass matrix. We have evaluated this matrix in our model for $\Delta f_3 = 0$ and have found that it has $2n-2$ zero eigenvalues. But in spontaneously breaking the theory from $O(n) \to O(n-1)$, there are $n-1$ Goldstone bosons. Thus V_0 is flat at its minimum in $n-1$ non-Goldstone directions, so that one should not be surprised that an infinitesimal change in f_3 can alter the symmetry-breaking pattern.

Now we focus on the effects of the radiative corrections. The one-loop gauge-field and scalar-field interaction contributions are such that (in the Landau gauge) $V_g^{(1)} = O(g^4)$ and $V_s^{(1)} = O(f^2)$,[5] respectively. If g approximately equals the electromagnetic coupling constant, then the tree approximation to the potential is only valid for $\alpha^2 \ll f \ll 1$. But more importantly, the radiative corrections set a bound on the precision with which the term proportional to f_3 can be specified in a valid tree approximation. In particular, there is an effective lower bound on Δf_3, which we shall denote as $\Delta f_3^{(e)}$.

To see how this bounds the gauge hierarchy in the tree approximation, note that

$$\frac{\vec{\chi}_m{}^2}{\vec{\eta}_m{}^2} = \frac{f_2}{\Delta f_3}\left(1 - \frac{f_3 m_2{}^2}{f_2 m_1{}^2}\right) < 2\frac{f_2}{\Delta f_3} < 2\frac{f_2}{\Delta f_3^{(e)}}. \tag{2.9}$$

In order to maximize the hierarchy—subject to this constraint—consider the following three cases:

(1) $1 \gg f_2 \gg \alpha \Rightarrow V^{(1)} = O(f_2{}^2) \Rightarrow \Delta f_3^{(e)} = O(f_2{}^2)$

$$\Rightarrow \frac{\vec{\chi}_m{}^2}{\vec{\eta}_m{}^2} \ll O(\alpha^{-1}),$$

(2) $f_2 \simeq \alpha \Rightarrow V^{(1)} = O(\alpha^2) \Rightarrow \Delta f_3^{(e)} = O(\alpha^2)$

$$\Rightarrow \frac{\vec{\chi}_m{}^2}{\vec{\eta}_m{}^2} < O(\alpha^{-1}),$$

(3) $\alpha \gg f_2 \gg \alpha^2 \Rightarrow V^{(1)} = O(\alpha^2) \Rightarrow \Delta f_3^{(e)} = O(\alpha^2)$

$$\Rightarrow \frac{\vec{\chi}_m{}^2}{\vec{\eta}_m{}^2} \ll O(\alpha^{-1}).$$

This implies that

$$M_H/M_L < O(\alpha^{-1/2}). \tag{2.10}$$

So within its range of validity, the tree-approximation gauge hierarchy has an upper bound of the order of $\alpha^{-1/2}$, independent of the scalar masses or their self-couplings; an arbitrarily large scalar mass ratio cannot be used to induce a correspondingly large gauge hierarchy.

Though our results were derived for a specific model, they can be generalized. In general, if one tries to establish a gauge hierarchy by adjusting the parameters of a Lagrangian with scalar fields transforming as more than one irreducible representation of the symmetry group, there will always be terms proportional to the product of two quadratic invariants of different representations (e.g., the f_3 term of our example). These terms connect the representations in the potential. The radiative corrections place an effective lower bound on the precision with which one can specify the scalar self-couplings. Thus, as in our example, the notion that one can arbitrarily specify the scalar parameters of the Lagrangian so as to construct any desired gauge hierarchy in tree approximation is false. One would, quite generally, expect a valid tree-approximation gauge hierarchy to have an upper bound of the order of $\alpha^{-1/2}$.

B. The inclusion of the radiative corrections

Let us now approach the problem of the radiative corrections. It is convenient to refer to the scalar fields collectively as φ. The one-loop approximation to the potential is $V(\varphi) \simeq V_0(\varphi) + V_1(\varphi)$; assume that the minimum of V_0 occurs at φ_0, while that of V occurs at $\varphi_0+\Delta\varphi$, where $\Delta\varphi$ is small. Now consider the following schematic (all indices suppressed) Taylor's series expansion:

$$\frac{\partial V(\varphi_0+\Delta\varphi)}{\partial\varphi}=\frac{\partial V_0(\varphi_0)}{\partial\varphi}+\Delta\varphi\,\frac{\partial^2 V_0(\varphi_0)}{\partial\varphi^2}+\frac{(\Delta\varphi)^2}{2}\frac{\partial^3 V_0(\varphi_0)}{\partial\varphi^3}+\cdots+\frac{\partial V_1(\varphi_0)}{\partial\varphi}+\cdots=0. \quad (2.11)$$

The first term of this expansion is zero. Recall that V_1 need be included only when $\Delta f_3 \lesssim \Delta f_3^{(e)}$, but then by an appropriate choice of the renormalization point one can always absorb the term proportional to Δf_3 into V_1. If one makes this choice, then (as stated in part A of this section) $\partial^2 V_0(\varphi_0)/\partial\varphi^2$ will have $(n-1)$ zero eigenvalues in non-Goldstone directions. Thus, solving our equations for these directions (in our schematic notation),

$$\frac{M_H}{M_L}\simeq\frac{\varphi_0}{\Delta\varphi}\simeq\left(-\varphi_0{}^2\frac{\partial^3 V_0}{\partial\varphi_0{}^3}\Big/\frac{\partial V_1(\varphi_0)}{\partial\varphi}\right)^{1/2}. \quad (2.12)$$

In general, if one were to apply arguments similar to those following Eq. (2.9) to this result, one would expect a maximum hierarchy of the order of $\alpha^{-1/2}$. However, it might be possible to vary the scalar-field parameters such that the denominator of Eq. (2.12) goes continuously to zero, in which case there is no bound on the hierarchy in the one-loop approximation. (To actually determine whether this is possible, even in this simple model, would be quite messy.) If one cannot vary the scalar-field parameters in such a manner, then the theory has an intrinsic upper bound on the gauge hierarchy of the order of $\alpha^{-1/2}$. On the other hand, even if one could so vary the parameters, the two-loop contribution to the potential would bound the one-loop hierarchy. That is, we could recycle our arguments to show that within its range of validity, the one-loop approximation gauge hierarchy has an upper bound of the order of α^{-1}. One would then have to study the two-loop potential to see whether the theory has an intrinsic upper bound on the hierarchy of order α^{-1}. The argument can be continued *ad infinitum*.

Thus any attempt to perturbatively establish a gauge hierarchy whose magnitude is greater than the order of $\alpha^{-1/2}$ by adjusting the scalar-field parameters in the Lagrangian is rendered fruitless by computational complexity. The question of whether such a hierarchy is possible is completely open.

III. NATURALLY INDUCED GAUGE HIERARCHIES

A. A possibility

As disturbing as it is to realize that one cannot simply set up an arbitrary gauge hierarchy by appropriately adjusting parameters in the Lagrangian, a gauge hierarchy would be physically compelling only if it occurred naturally. We are familiar with mechanisms that lead to a natural hierarchy of global symmetry breakdown; approximate isotopic-spin conservation can be explained by such mechanisms.[6] In such models, a zeroth-order symmetry is a symmetry of only part of the Lagrangian, and thus broken in higher orders. Such a scheme cannot be adapted to produce gauge hierarchies, since a gauge symmetry must be a symmetry of the entire Lagrangian.

The possibility that radiative corrections could alter the symmetry-breaking pattern of the tree approximation, producing a gauge hierarchy, is severely constrained by the Georgi-Glashow theorem (discussed in Sec. II). We conclude from that theorem that the only hope for a radiatively induced gauge hierarchy is that the lowest-order potential be so flat in non-Goldstone directions as to imply zero-mass Higgs bosons in lowest order. To just impose such a condition would be very artificial. However, there is a symmetry-breaking mechanism which suggests that such a possibility might occur naturally. If one considers a region of field-strength space where the classical scalar fields are much larger than any masses or dimensional coupling constants, then the one-loop approximation to the effective potential is the same as for the massless theory. It has been shown that for this case symmetry breakdown can be radiatively induced,[7,8] and it was suggested that such a mechanism provides a natural explanation of superstrong symmetry breaking.[7] Since the symmetric theory is effectively massless, it inspires the hope that the desired zero-mass Higgs bosons occur naturally, and that a gauge hierarchy results. In fact, a zero-mass Higgs boson associated with the radial direction does always naturally occur,[7,8] but the symmetry-breaking pattern is independent of the radial coordinate. For the models considered in the original paper, no natural gauge hierarchies occurred.[7]

Now let us very briefly consider radiatively induced symmetry breaking for the model of Sec. II.[9] For large field strengths, we have

$$V_0=\tfrac{1}{4}f_1(\vec{\chi}^2)^2+\tfrac{1}{4}f_2(\vec{\eta}^2)^2+\tfrac{1}{2}f_3\vec{\chi}^2\vec{\eta}^2+\tfrac{1}{4}f_4(\vec{\chi}\cdot\vec{\eta})^2. \quad (3.1)$$

It has been shown that the symmetry can be broken only if V_0 has minima along a ray in field-strength space, and that this condition imposes one constraint on the coupling constants.[8,9] However, such constraints are not unnatural; the evolution of the coupling constants (in the sense of the renormalization group) make it probable that such constraints are satisfied in some region of field-strength space for a wide class of theories.[7] For our particular model, V_0 has four types of stationary points, each with its corresponding constraint on the couplings:

Stationary point	Constraint
(1) $\vec{\chi}\perp\vec{\eta}$, $\vec{\chi}^2/\vec{\eta}^2=(f_2/f_1)^{1/2}$	$f_3=-(f_1f_2)^{1/2}$
(2) $\vec{\chi}\parallel\vec{\eta}$, $\vec{\chi}^2/\vec{\eta}^2=(f_2/f_1)^{1/2}$	$2f_3+f_4=-2(f_1f_2)^{1/2}$
(3) $\vec{\chi}=0$, $\vec{\eta}\neq 0$	$f_2=0$
(4) $\vec{\eta}=0$, $\vec{\chi}\neq 0$	$f_1=0$

For case (1), the symmetry is broken down to $O(n-2)$ in zeroth order, with no particular reason for the $O(n-1)$ subgroup being broken less strongly than $O(n)$. The other three cases reduce $O(n)$ to $O(n-1)$ in zeroth order. However, if one evaluates the eigenvalues of the second-derivative matrix of V_0, one finds no eigenvalues that are zero in a transverse, non-Goldstone direction. Thus this model cannot produce natural gauge hierarchies. We have investigated several other models, but our results have been negative.

If such a natural hierarchy were found, it would suffer from one severe constraint: It would have an upper bound of the order of $\alpha^{-1/2}$. This bound can be determined by arguments that parallel those leading to and following Eq. (2.12).

B. Pseudosymmetries

There is a well-established mechanism that very naturally introduces zero-mass Higgs bosons in the tree approximation: that is, for the scalar-field polynomial in the Lagrangian to be forced by gauge invariance and renormalizability to have a larger group of symmetries, $\bar{G}$, than the gauge group G. Such a theory will contain pseudo-Goldstone bosons associated with the broken generators of $\bar{G}$ which are not contained in G.[10] Though the zeroth-order mass of the pseudo-Goldstone bosons vanishes, it picks up finite contributions from higher-order effects. An obvious question is whether radiative corrections can induce a shift of the minimum of the potential in the pseudo-Goldstone direction such as to produce a gauge hierarchy; the unfortunate answer is no. The reason is that although the second derivatives of the potential at the minima are zero in pseudo-Goldstone direction, what one actually has is a continuous set of degenerate physically inequivalent minima, generated by applying the elements of $\bar{G}$ that are not in G to one of the minima.[5] Thus, the effect of the one-loop contribution to the potential (which is symmetric under the gauge group only, not $\bar{G}$) is not to induce a slight shift of the zeroth-order minimum in some pseudo-Goldstone direction, but rather to pick the correct vacuum from this over-rich set.

IV. CONCLUSION

Our analysis undermines the traditional assumption that one could always establish a gauge-symmetry hierarchy of any magnitude by arbitrarily adjusting the scalar-field parameters of the Lagrangian and then considering the tree approximation to the potential; the radiative corrections will set an upper bound of the order of $\alpha^{-1/2}$ on the tree-approximation gauge hierarchy. Of course, such an artifical approach to gauge hierarchies is not physically compelling. Unfortunately, the hope of radiatively inducing a natural gauge hierarchy has been frustrated by our attempts. But even if a model with a natural hierarchy were found, we have shown that it would have an upper bound of the order of $\alpha^{-1/2}$.

This bound presents a barrier to understanding the attempts to unify the weak, electromagnetic, and strong interactions into a theory based on a simple gauge group, since for such theories superstrong symmetry breakdown typically occurs for masses in the range of 10^{11}–10^{15} GeV,[11] and the W and Z bosons have masses of about 50 GeV, so that the magnitude of the expected gauge hierarchy greatly exceeds our bound. There are natural symmetry-breaking mechanisms which have been speculated about which are not limited by our bound. For example, the superstrong symmetry breakdown might occur via some as yet unknown mechanism (perhaps gravitational[11]) and result in an effective field theory[12] at "ordinary" energies with only *massless* scalar fields[8]; the second-stage symmetry breaking can then be radiatively induced.[7,8] Or perhaps one must abandon perturbation theory and assume that the weak symmetry breakdown is dynamical, despite the problems implied by this possibility.[13] The problem of gauge hierarchies is perplexing but central, for once one has chosen a model based on a simple gauge group to unify the interactions, the core of the physics lies in the spontaneous symmetry breakdown.

ACKNOWLEDGMENT

I am grateful for useful conversations with Steven Weinberg.

APPENDIX: THE MINIMIZATION OF V_0

In this Appendix we outline the minimization of the potential (2.1). (Recall that we assumed that

f_1, f_2, and f_4 are positive, that $m_1^2 \geq m_2^2$, and that $f_2 \geq f_1$.) If f_3 is also positive, then V_0 is bounded from below, and one can conclude that the stationary point of V_0 for which V_0 is most negative is the minimum of V_0. For f_3 negative, V_0 will be bounded from below if

$$\tfrac{1}{4}f_1(\vec{\chi}^2)^2 + \tfrac{1}{4}f_2(\vec{\eta}^2)^2 + \tfrac{1}{2}f_3\vec{\chi}^2\vec{\eta}^2 > 0, \tag{A1}$$

for all values of $\vec{\chi}$ and $\vec{\eta}$. This holds if and only if

$$(\sqrt{f_1}\,\vec{\chi}^2 - \sqrt{f_2}\,\vec{\eta}^2)^2 + 2\vec{\chi}^2\vec{\eta}^2(\sqrt{f_1 f_2} + f_3) > 0, \tag{A2}$$

which in turn maintains if

$$f_3^2 < f_1 f_2. \tag{A3}$$

Our argument has, of course, been such that this condition is sufficient, but not necessary. We have purposely singled out this constraint because it will be required shortly for other reasons.

For $\vec{\chi}_i \perp \vec{\eta}_i$, the stationary points satisfy

$$\frac{\partial V_0}{\partial \chi_i} = \chi_i(-m_1^2 + f_1\vec{\chi}^2 + f_3\vec{\eta}^2) = 0 \tag{A4a}$$

and

$$\frac{\partial V_0}{\partial \eta_i} = \eta_i(-m_2^2 + f_2\vec{\eta}^2 + f_3\vec{\chi}^2) = 0. \tag{A4b}$$

The three symmetry-breaking solutions of these equations are those of (2.2).

Only for f_3's such that $\vec{\chi}_c^2, \vec{\eta}_c^2$ is the minimum of V_0 is there a hierarchy. By manipulating inequalities, one can convince oneself that the constraint that $\vec{\chi}_c^2$ and $\vec{\eta}_c^2$ be positive implies that

$$-\sqrt{f_1 f_2} < f_3 < f_1 m_2^2/m_1^2 \tag{A5}$$

or

$$f_3 > f_2 m_1^2/m_2^2. \tag{A6}$$

The requirement that V_0 evaluated at $\vec{\chi}_c^2, \vec{\eta}_c^2$ be less than that evaluated at $\vec{\chi}_a^2$, $\vec{\eta}_a^2$ and $\vec{\chi}_b^2$, $\vec{\eta}_b^2$ further restricts the domain of interest. It turns out that one can conveniently express V_0 evaluated at each of these three points as

$$V_0(\vec{\chi}_i^2, \vec{\eta}_i^2) = -\tfrac{1}{4}(m_1^2\vec{\chi}_i^2 + m_2^2\vec{\eta}_i^2)\,. \tag{A7}$$

If $f_3 = f_1 m_2^2/m_1^2$, then $\vec{\chi}_c^2 = \vec{\chi}_b^2$ and $\vec{\eta}_c^2 = \vec{\eta}_b^2$. One can compute the following results:

$$\left.\frac{\partial V_0(\vec{\chi}_c^2, \vec{\eta}_c^2)}{\partial f_3}\right|_{f_3 = f_1 m_2^2/m_1^2} = 0 \tag{A8}$$

and

$$\left.\frac{\partial^2 V_0(\vec{\chi}_c^2, \vec{\eta}_c^2)}{\partial f_3^2}\right|_{f_3 = f_1 m_2^2/m_1^2} = -\frac{(f_1 f_2 + f_1^2 m_2^4/m_1^4)^2}{2f_1(f_1 f_2 - f_1^2 m_2^4/m_1^4)^3} < 0. \tag{A9}$$

Combining these facts one can then argue that for $-\sqrt{f_1 f_2} < f_3 < f_1 m_2^2/m_1^2$, the minimum of V_0 does in fact occur at $\vec{\chi}_c^2, \vec{\eta}_c^2$. (The results of this paragraph are graphically represented in Fig. 1, which in turn should make the omitted arguments manifest.) Similarly, if $f_3 = f_2 m_1^2/m_2^2$, then $\vec{\chi}_c^2 = \vec{\chi}_a^2$ and $\vec{\eta}_c^2 = \vec{\eta}_a^2$. Because of the symmetries of the problem, the following results can be obtained by interchanging $f_1 \leftrightarrow f_2$ and $m_1^2 \leftrightarrow m_2^2$ in Eqs. (A8) and (A9):

$$\left.\frac{\partial V_0(\vec{\chi}_c^2, \vec{\eta}_c^2)}{\partial f_3}\right|_{f_3 = f_2 m_1^2/m_2^2} = 0 \tag{A10}$$

and

$$\left.\frac{\partial^2 V_0(\vec{\chi}_c^2, \vec{\eta}_c^2)}{\partial f_3^2}\right|_{f_3 = f_2 m_1^2/m_2^2} = -\frac{(f_1 f_2 + f_2^2 m_1^4/m_2^4)^2}{2f_2(f_1 f_2 - f_2^2 m_1^4/m_2^4)^3} > 0. \tag{A11}$$

These results imply that for $f_3 > f_2 m_1^2/m_2^2$, the minimum of V_0 occurs at $\vec{\chi}_b^2, \vec{\eta}_b^2$. Thus our conclusion is that only for $-\sqrt{f_1 f_2} < f_3 < f_1 m_2^2/m_1^2$ are gauge hierarchies possible.

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