Nonexistence of spherically symmetric monopoles with multiple magnetic charge*

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We show that in a spontaneously broken SU(2) gauge theory there are no field configurations with magnetic charge >1 which are both spherically symmetric and of finite energy.

Some time ago, 't Hooft[1] and Polyakov[2] showed the existence of nonsingular solutions with unit magnetic charge (in units of $1/e$) in a spontaneously broken SU(2) gauge theory containing a triplet of scalar mesons.[3] By choosing to look for spherically symmetric solutions, they were able to effect an enormous simplification of the field equations; it seems quite natural to employ the same strategy in looking for solutions with higher magnetic charge. The purpose of this note is to show that any such attempt is futile. More specifically, we shall show that in a spontaneously broken SU(2) gauge theory there are no field configurations with magnetic charge >1 which are both spherically symmetric and of finite energy.

We begin by considering the case where the scalar mesons belong to the triplet representation; the extension to other representations will be discussed later. We first define spherical symmetry more precisely. In a gauge theory, the transformations of the fields under rotations will in general be a combination of a "naive" rotation and a gauge transformation. Thus, the scalar field $\vec{\chi}$ and the gauge field $\vec{W}_i$ should transform under rotations about the l axis according to[4] (vector notation refers to internal indices)

$$\delta_l\vec{\chi}(\vec{r})=\mathfrak{D}_l\,\chi(\vec{r})+\vec{\Lambda}_l(\vec{r})\times\vec{\chi}(\vec{r})\,, \tag{1}$$

$$\delta_l\vec{W}_i(\vec{r})=\mathfrak{D}_l\vec{W}_i(\vec{r})+\epsilon_{lij}\vec{W}_j(\vec{r})+\vec{\Lambda}_l(\vec{r})\times\vec{W}_i(\vec{r})-(1/e)\partial_i\vec{\Lambda}_l(\vec{r})\,, \tag{2}$$

where we have defined

$$\mathfrak{D}_l=\epsilon_{lmn}r_m\partial_n\,. \tag{3}$$

We shall say that a field configuration is spherically symmetric if there is a choice of $\vec{\Lambda}_l$ such that $\delta_l\vec{\chi}$ and $\delta_l\vec{W}_i$ vanish.

It is worth noting the inhomogeneous term in (2). If this term is absent, a spherically symmetric configuration can be obtained by taking the asymptotic forms of the fields (presumably symmetric) and multiplying them by a function of r; this procedure can be used to obtain the 't Hooft-Polyakov ansatz. However, if the $\vec{\Lambda}_l$ are not constants, the transformations are nonlinear and this simple procedure is not valid.

It will be convenient to work in a particular gauge; we do not lose any generality by doing so, since our result is a gauge-invariant statement. We begin by specifying, as a gauge condition, the isotopic-spin direction of the scalar field at each point in space, i.e.,

$$\vec{\chi}(\vec{r})=k(\vec{r})\vec{v}(\theta,\phi)\,, \tag{4}$$

where the unit isovector $\vec{v}$ is a definite function of angle having the topological properties appropriate to magnetic charge n. A convenient choice is

$$\vec{v}=\begin{pmatrix}\sin\theta\cos n\phi\\ \sin\theta\sin n\phi\\ \cos\theta\end{pmatrix}. \tag{5}$$

For $|n|\geq 2$, this choice leads to singularities in the spatial derivatives of $\vec{v}$ along the z axis; for this reason, we impose (5) only for $\delta<\theta<\pi-\delta$, and assume a smooth continuation along the z axis. In our arguments, we will avoid the region near the z axis, so the explicit form of the continuation will not be needed. We note for later reference that this choice of $\vec{v}$ has the property that

$$\partial_i\vec{v}\times\partial_j\vec{v}\cdot\vec{v}=n\epsilon_{ijk}\frac{r_k}{r^3}\,. \tag{6}$$

We are still free to make gauge transformations with $\vec{\Lambda}$ parallel to $\vec{v}$; we use this freedom to require

$$r_i\vec{W}_i\cdot\vec{v}=0\,,\quad r>\epsilon\,. \tag{7}$$

(We exclude a sphere about the origin in order to avoid possible singularities.) There is still the freedom to make r-independent gauge transformations along the direction of $\vec{v}$; we shall exploit this later.

Now let us consider the asymptotic form of the fields. Finiteness of the energy requires that as $r\to\infty$ the magnitude of the scalar field approach a constant, and that $D_i\vec{\chi}$ and thus $D_i\vec{v}$ tend to zero ($D_i\equiv\partial_i+e\vec{W}_i\times$). The last requirement leads to the asymptotic form of the gauge field

$$\vec{W}_i^\infty=\frac{1}{e}\partial_i\vec{v}\times\vec{v}+c_i\vec{v}\,, \tag{8}$$

where the gauge condition (7) requires $c_i r_i = 0$. We now recall the gauge-invariant definition of the electromagnetic field[1,5]

$$F_{\mu\nu} = \hat{\chi}\cdot\vec{G}_{\mu\nu} - (1/e)\hat{\chi}\cdot D_\mu\hat{\chi}\times D_\nu\hat{\chi}$$
$$= \partial_\mu(\hat{\chi}\cdot\vec{W}_\nu) - \partial_\nu(\hat{\chi}\cdot W_\mu) - (1/e)\hat{\chi}\cdot\partial_\mu\hat{\chi}\times\partial_\nu\hat{\chi}\,, \tag{9}$$

where

$$\vec{G}_{\mu\nu} = \partial_\mu\vec{W}_\nu - \partial_\nu\vec{W}_\mu + e\vec{W}_\mu\times\vec{W}_\nu \tag{10}$$

and

$$\hat{\chi} = \frac{\vec{\chi}}{|\vec{\chi}|}. \tag{11}$$

[Note that the identity of the two forms of (9) does not depend on the form of χ.] Since we want our configuration to be of magnetic charge n, (6) and (9) show that $\epsilon_{ijk}\partial_j c_k = 0$, at least outside the excluded solid angle near the z axis. But c_i must then be the gradient of a function of angle, which we can eliminate by using our remaining gauge freedom.

Now let us determine the $\vec{\Lambda}_l$ which appear in (1) and (2). By requiring that the gauge condition (4) be preserved, we find

$$\vec{\Lambda}_l = \mathfrak{D}_l\hat{v}\times\hat{v} + f_l(\vec{r})\hat{v}. \tag{12}$$

Further, the second gauge condition (7) leads to requirement

$$\frac{\partial f_l}{\partial r} = 0. \tag{13}$$

Since the f_l are independent of r, they may be fixed by requiring that the asymptotic form $\vec{W}_i^\infty$ be left invariant under rotations. Substitution of $\vec{W}_i^\infty$ into (2) leads to the requirement

$$\partial_i f_l = \partial_i\hat{v}\times\mathfrak{D}_l\hat{v}\cdot\hat{v}$$
$$= -\frac{n}{r}\left(\delta_{il} - \frac{r_i r_l}{r^2}\right), \tag{14}$$

where the second line is obtained by using (3) and (6). Since the right side of (14) is symmetric in i and l, f_l must be curl-free; we can therefore write $f_l = \partial_l g$, where g must satisfy

$$\nabla^2 g = -\frac{2n}{r}. \tag{15}$$

The general solution of this equation consistent with (13) is

$$g = -nr - h_j r_j\,, \tag{16}$$

where h_j is an arbitrary constant vector. Hence,

$$f_l = -\frac{n r_l}{r} - h_l. \tag{17}$$

Having determined the f_l, we are now ready to determine the requirements which spherical symmetry places on the form of the gauge field. If we write

$$\vec{W}_i = \vec{W}_i^\infty + \vec{B}_i\,, \tag{18}$$

then, since $\delta_l\vec{W}_i^\infty$ vanishes, we must require

$$0 = \mathfrak{D}_l\vec{B}_i + \epsilon_{lij}\vec{B}_j + \vec{\Lambda}_l\times\vec{B}_i. \tag{19}$$

If we now make us of the identity $r_i\mathfrak{D}_i = 0$, we find

$$\epsilon_{lij}\frac{r_l}{r}\vec{B}_j = -\frac{r_l}{r}\vec{\Lambda}_l\times\vec{B}_i$$
$$= \left(n + \frac{h_i r_i}{r}\right)\hat{v}\times\vec{B}_i. \tag{20}$$

It immediately follows that

$$0 = \epsilon_{lij}\frac{r_l}{r}\vec{B}_j\cdot\hat{v}. \tag{21}$$

This, plus the gauge condition (7), implies that $\vec{B}_j\cdot\hat{v} = 0$.

If we now square (20) and sum over both spin and isospin indices, we find

$$\vec{B}_j\cdot\vec{B}_j - \left(\frac{r_j}{r}\vec{B}_j\right)^2 = \left(n + \frac{h_i r_i}{r}\right)^2\vec{B}_j\cdot\vec{B}_j. \tag{22}$$

and hence

$$-\left(\frac{r_j}{r}\vec{B}_j\right)^2 = \left[\left(n + \frac{h_i r_i}{r}\right)^2 - 1\right]\vec{B}_j\cdot\vec{B}_j. \tag{23}$$

Clearly, (23) can be satisfied with nonzero $\vec{B}_j$ only if the quantity in brackets is not positive. For $n = 0$, ± 1, this is easily arranged by choosing $h = 0$. For $|n| \geq 2$, we see that there must be a hemisphere in which $h_i r_i$ is positive; in this hemisphere the entire quantity in brackets must be positive, and so $\vec{B}_j$ must vanish (except possibly in the excluded region about the z axis).

Thus, we have a region of finite solid angle in which the gauge field must have its asymptotic form for all $r > \epsilon$. Since $\vec{W}_i^\infty$ is proportional to $1/r$, this leads to an energy density which increases near the origin as $1/r^4$. By choosing ϵ sufficiently small, the energy can be made arbitrarily large; no finite-energy configurations exist.

These arguments can be easily extended to include the case where the Higgs mesons belong to other representations [provided, of course, that the unbroken symmetry group is U(1)]. The unit isovector $\hat{v}$ must now be chosen to be the eigenvector of zero eigenvalue of the vector-meson mass matrix. With $\hat{v}$ defined in this manner, the expression for the electromagnetic field given above is still valid, while the arguments of Shankar[6] show that the asymptotic form of the gauge field is unchanged. The remainder of the argument given above is independent of the representation of the scalar fields.

To make these results more concrete, let us consider the consequences of continuing the asymptotic fields for finite r, but with the fields modified by functions of r; i.e.,

$$\vec{\chi}(\vec{r}) = k(r)\vec{v}\,,$$
$$\vec{W}_i(\vec{r}) = (1/e)f(r)\partial_i\vec{v}\times\vec{v}\,. \tag{24}$$

(For $n=1$, this is just the 't Hooft-Polyakov ansatz.) Clearly, the magnetic field, given by (9), is spherically symmetric. Similarly, the energy density can easily be seen to be a function only of r. However, let us consider the tensor $T_{ij} = D_i\vec{v}\cdot D_j\vec{v}$. This quantity is gauge-invariant and must therefore be manifestly rotationally invariant (i.e., without recourse to gauge transformations). We calculate

$$T_{ij} = [1-f(r)]^2\left[\frac{1}{r^4}(\delta_{ij}r^2 - r_{ij}) + \frac{n^2-1}{r^2}u_i u_j\right]\,, \tag{25}$$

where

$$u_i = (-\sin\phi, \cos\phi, 0)\,. \tag{26}$$

Thus, T_{ij} is an invariant tensor only if $n=\pm 1$. Similarly, one can show by substitution in the equations of motion that this ansatz can yield a static solution only for $n=\pm 1$.

Now that we have eliminated the possibility of spherically symmetric solutions, what are the prospects for finding solutions with multiple magnetic charge? Any simplifying assumptions must be consistent with the symmetry of the theory. Thus, we may look for an axially symmetric solution, but in doing so we must consider the most general axially symmetric form; if we also impose the requirement of definite parity, we are left with a set of equations involving five functions of r and θ. Even at this point, there is no guarantee that a solution of the form we want exists; for example, the only axially symmetric solution for $n=2$ might be a pair of infinitely separated unit monopoles. On the other hand, a variational calculation by Bais,[7] using the ansatz (24), leads to a value of the mass of the double monopole which is 2.13 times that of the single monopole; a better choice of ansatz might bring this value below 2 and thus show the existence of a localized double-monopole solution.

Finally, we note that if there are classically stable solutions for monopoles with multiple magnetic charge the spectrum of the quantum theory will be richer than that implied by the unit monopole. Just as in the case of the unit monopole, there will be a series of states labeled by different values of the electric charge.[8-10] In addition, since the classical solution will necessarily be noninvariant under rotations, there will be a further multiplicity of states labeled by different values of angular momentum.

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