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Strong-coupling field theories. II. Fermions and gauge fields on a lattice*

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This paper (the second in a series) reports our recent progress in the study of strong-coupling quantum field theories on a lattice. In particular we study theories involving fermions and gauge fields and pay special attention to the peculiar problems encountered when one formulates theories of fermions on a lattice. It is unique to our approach that we preserve local chiral symmetry and at the same time correctly count the number of fermionic states. We demonstrate how our formalism works with the lattice Thirring and Schwinger models, whose continuum limits are solvable in one space and one time dimension. We show in the strong-coupling limit that these theories are equivalent to a Heisenberg antiferromagnetic chain. We also discuss briefly some general features of non-Abelian gauge theories of quarks and gluons in three space and one time dimension. The most interesting results we have to report at this stage are as follows: (i) The only "gauge-invariant states" which remain at low mass in the limit of very strong gauge coupling have the quantum numbers of physical hadrons. (ii) The resulting "effective strong-coupling" theory preserves the full chiral symmetry of the exact theory [SU(3) × SU(3) if we introduce three flavors of quarks each with three colors] and describes a theory of "massless bare hadrons" interacting with one another through a quark interchange mechanism of finite strength.

I. INTRODUCTION

The goal of explaining the observed properties of hadrons starting from a field theory of elementary quark constituents has motivated the search for reliable ways to study strong-coupling field theories. This paper is number two in a series reporting our efforts along these lines.

In our earlier paper[1] we applied variational methods to study low-lying states and possible phase transitions of boson field theories that are rendered finite by formulating them on a spatial lattice. In particular in order to learn about the reliability of our methods we analyzed in some detail the conditions for the occurence of spontaneous symmetry breakdown in scalar ϕ^4 theory in one space and one time dimension. This model has little physical content and so, building on this experience, we now turn our attention to theories involving fermions. Since current opinion holds that non-Abelian theories of quarks coupled to color-gauge gluons comprise the class from which "the theory" will emerge, these models are of particular interest. The most interesting general results we have to report at this stage in our studies of such gauge models are as follows:

(i) The only "gauge-invariant states" which remain at low mass in the limit of very strong gauge coupling have the quantum numbers of physical hadrons.

(ii) The resulting "effective strong-coupling" theory preserves the full chiral symmetry of the exact theory [SU(3) × SU(3) if we introduce three flavors of quarks each with three colors] and describes a theory of "massless bare hadrons" interacting with one another through a quark-interchange mechanism of finite strength.

In formulating gauge theories of fermions on a lattice two kinds of problems are faced. First there is the general question of how one introduces gauge fields on a lattice so that the theory has full "gauge invariance." Our approach to this question is to adopt the prescription of Wilson[2] and Kogut and Susskind,[3] according to which the gauge field is defined not at the individual lattice sites but on links joining lattice points.

The second problem concerns the prescription for describing the fermion field on a lattice. In this we differ from previous approaches in a way which is crucial for obtaining the two general results given above.

This difference is a consequence of our way of resolving a problem peculiar to theories of fermions on a lattice; namely the usual transcription of the gradient in the Dirac Hamiltonian as a difference operator (let $\Lambda = 1/a$ be the reciprocal lattice spacing)

$$(\nabla\psi)_j = \Lambda(\psi(j+1) - \psi(j)) \tag{1.1}$$

leads directly to a doubling of the fermionic degrees of freedom. In particular for a free Dirac particle the energy-momentum dispersion relation based on (1.1) in one space and one time dimension is

$$E(k) = [m^2 + \Lambda^2 \sin^2(k/\Lambda)]^{1/2}, \tag{1.2}$$

where $-\pi \leq k\Lambda \leq \pi$. As illustrated in Fig. 1, this formula shows that to each eigenvalue E there correspond *two distinct states of* $k > 0$ *and two of* $k < 0$; hence the spectrum of states possesses a

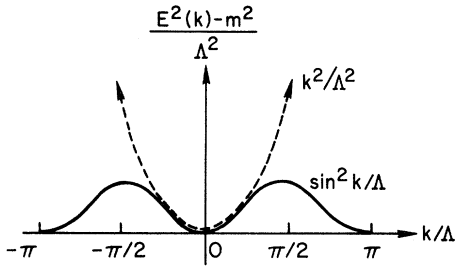

FIG. 1. Energy-momentum dispersion relation for a free Dirac particle with the gradient replaced on a lattice by the difference operator.

doubling of levels not encountered in the continuum theory.

Kogut and Susskind[4] have proposed one technique for avoiding this problem in two dimensions. They simply put the upper (lower) components of Dirac spinors on even (odd) lattice sites. The advantage of their procedure is calculational, in that only nearest-neighbor sites are coupled together by the gradient operator (1.1). The disadvantage of this procedure is that it makes it impossible to write down locally chiral-invariant interactions since one does not have both particles and antiparticles at the same point. In higher dimensions their procedure is very contrived since it becomes also necessary to split spin components in two space and one time dimension and to double the number of fermions (say proton and neutron) in three space and one time dimension. An alternate projection operator technique recently introduced by Wilson[5] in his action formulation also destroys local γ_5 invariance.

We avoid this difficulty by defining the gradient operator via the prescription of Ref. 1; i.e., for

$$f(j)=\sum_k f(k)e^{ikj/\Lambda} \tag{1.3}$$

we define

$$\begin{aligned}(\nabla f)_j &= \sum_k ikf(k)e^{ikj/\Lambda} \\ &= \sum_{j'} f(j')\left[\sum_k \frac{ik}{(2N+1)}\, e^{ik(j-j')/\Lambda}\right] \\ &\equiv \sum_{j'} f(j')[-\delta'(j-j')],\end{aligned} \tag{1.4}$$

where $(2N+1)$ is the number of sites in the lattice. In three space dimensions (1.4) becomes

$$(\nabla_x f)_{j_x j_y j_z}=\sum_{j'_x} f(j'_x, j_y, j_z)[-\delta'(j_x - j'_x)]. \tag{1.5}$$

As in the case of the free boson field discussed in Ref. 1 this prescription yields the exact relativistic Einstein energy-momentum relation for a free fermion of mass m,

$$E(k)=(k^2+m^2)^{1/2}. \tag{1.6}$$

Thus, the only difference between the lattice "free fermion" theory and the continuum theory is that in the lattice version we have a maximum allowable momentum, $|k_{\max}|=\pi\Lambda$. On the basis of (1.5) and (1.6) there is no doubling of energy levels and no need to split field components onto different lattice sites. We can therefore easily incorporate exact γ_5 (chiral) invariance into theories with this formalism. The only cumbersome feature of (1.5) is that it couples all lattice sites along the direction of each component of the gradient instead of coupling only nearest-neighbor sites as in (1.1).

Since our formulation and conclusions depend crucially on the way in which we define the gradient operator, we turn first to the study of simple soluble models with fermions. These are the Thirring model,[6] both in one space and one time dimension and three space and one time dimension, and the Schwinger model.[7] We isolate the important features and compare the results of our formalism with the known properties of these models in their continuum version. In addition we show that the fact that our formulation of the lattice theory preserves full chiral invariance leads to very different results in the strong-coupling limit from those based upon (1.1) and the method of Kogut and Susskind.

In Sec. II and III we develop the general formalism and discuss the spectrum of low-lying states for the Thirring models. In particular, in Sec. II we show that for the strong-coupling theory the effective-potential method introduced by Nambu and Jona-Lasinio,[8] in their pioneering work on developing a dynamical model of nucleons, fails to describe correctly the symmetry properties of the theory. This is in agreement with similar conclusions made much earlier by Ichimura, Kikkawa, and Yazaki[9] using different techniques.

In Sec. III we relate the strong-coupling two-dimensional Thirring model to the linear Heisenberg antiferromagnetic chain with more than nearest-neighbor interactions. We discuss some general properties of the ground state building upon the wealth of knowledge developed about this spin system starting with Bethe in 1931. In particular the crucial role of being able to locate fermions and antifermions at the same lattice site will be apparent in this model. Bound pairs of fermions and antifermions on individual lattice sites are present in the ground state and their "spin waves" form a massless excitation spectrum.

In Sec. IV we turn to the simplest Abelian gauge theory—i.e., the Schwinger model, or QED in

one space and one time dimension. In the strong-coupling limit we find for the low-lying states essentially the spectrum of the Thirring model. In addition there are the high-lying excitations when flux links, corresponding to "massive photons," are present. This spectrum again depends crucially on (1.4) and the fact that our formulation permits locating fermions and antifermions on the same lattice site. The relation of these results to the continuum Schwinger model for weak and strong coupling is also described.

Finally, Sec. V is devoted to some preliminary discussion of non-Abelian color-gauge theories for which we have not yet carried out a detailed analysis.

II. FERMION MODELS ON A LATTICE

We introduce a spatial lattice as in Ref. 1 by replacing the continuum variable $\vec{x}$ by points on a discrete lattice of linear dimension L and minimum spacing $a = 1/\Lambda$, chosen so that there are $2N+1$ points in each direction, i.e.,

$$L = (2N+1)/\Lambda, \qquad V = L^p. \tag{2.1}$$

Time remains a continuous variable and $p = 1$, 2, or 3 is the dimension of the lattice in the model being analyzed. The lattice points are labeled by

$$\vec{j} = (j_1, \ldots, j_p), \quad -N \le j_i \le +N \tag{2.2}$$

and as in Ref. 1, the momentum variables are labeled by

$$\vec{k} = (k_1, \ldots, k_p), \quad k_i = \frac{2\pi}{L} n_i, \quad -N \le n_i \le +N. \tag{2.3}$$

The gradient defined in (1.4) sums over lattice points only along the direction of its vector components.

The Hamiltonian for a free massive Dirac field is in this notation

$$H_0 = \frac{1}{\Lambda^p}\left\{\sum_{\vec{j}_1,\vec{j}_2} \psi^\dagger(\vec{j}_1)\left[-\Lambda\vec{\delta}'(\vec{j}_1 - \vec{j}_2)\cdot\frac{\vec{\alpha}}{i}\right]\psi(\vec{j}_2) + m\sum_{\vec{j}} \psi^\dagger(\vec{j})\beta\psi(\vec{j})\right\}, \tag{2.4}$$

where $\vec{\alpha}$, β are anticommuting Dirac matrices.

As discussed in Sec. I, H_0 is diagonalized in a basis

$$\psi(\vec{j}) = \frac{1}{\sqrt{V}}\sum_{\vec{k}} e^{i\vec{k}\cdot\vec{j}/\Lambda}\psi(\vec{k}), \qquad \psi(\vec{k}) = \sum_\alpha [u_\alpha(\vec{k})b_\alpha(\vec{k}) + v_\alpha(-\vec{k})d^\dagger_\alpha(-\vec{k})], \tag{2.5}$$

where $u_\alpha(\vec{k})$ [$v_\alpha(\vec{k})$] are the linearly independent positive- (negative-) energy solutions to the Dirac equation

$$(\vec{\alpha}\cdot\vec{k} + \beta m)u_\alpha(k) = E(k)u_\alpha(k), \qquad (\vec{\alpha}\cdot\vec{k} + \beta m)v_\alpha(-k) = -E(k)v_\alpha(-k), \tag{2.6}$$

with

$$E(k) \equiv (k^2 + m^2)^{1/2}. \tag{2.7}$$

Equation (2.7) gives the correct energy-momentum relation up to the lattice cutoff

$$k_{\max} = \frac{2\pi}{L}N = \frac{2\pi N}{2N+1}\Lambda.$$

A. Lattice Thirring models

We first study lattice versions of the continuum models based upon the chirally invariant Hamiltonian

$$H = \int d^p x\left\{\psi^\dagger\frac{\vec{\alpha}\cdot\vec{\nabla}}{i}\psi - \frac{g_0}{2}[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2]\right\}. \tag{2.8}$$

In one space and one time dimension (2.8) describes the massless Thirring model.[6] H can also be Fierz-transformed into a current-current interaction

$$H = \int d^p x\left\{\psi^\dagger\frac{\vec{\alpha}\cdot\vec{\nabla}}{i}\psi + \tfrac{1}{4}g_0[(\bar{\psi}\gamma_\mu\psi)^2 - (\bar{\psi}\gamma_\mu\gamma_5\psi)^2]\right\}. \tag{2.9}$$

In either version H is invariant under the transformations

$$\psi \to e^{i\theta}\psi, \quad \bar{\psi} \to \bar{\psi}e^{-i\theta}, \qquad \psi \to e^{i\gamma_5\theta}\psi, \quad \bar{\psi} \to \bar{\psi}e^{-i\gamma_5\theta}, \tag{2.10}$$

where θ is an arbitrary constant. Hence their generators

$$N \equiv \int \psi^\dagger\psi\, d^p x, \qquad Q_5 \equiv \int \psi^\dagger\gamma_5\psi\, d^p x \tag{2.11}$$

commute with H. It is also the usual conclusion that the local currents

$$j_\mu = \bar{\psi}\gamma_\mu\psi, \qquad j_{5\mu} = \bar{\psi}\gamma_\mu\gamma_5\psi \tag{2.12}$$

are conserved. The Hamiltonian (2.8) has dimensions of energy, and the canonical dimension for the field ψ is $(\text{mass})^{p/2}$, where p is the lattice dimension. The coupling constant g_0 has dimension $(\text{mass})^{-p+1}$. In transcribing to a lattice version of (2.8) we introduce dimensionless variables $\chi(\vec{j})$, $\bar{\chi}(\vec{j})$, and g via

$$\psi(\vec{j}) \equiv \Lambda^{p/2}\chi(\vec{j}),$$
$$\bar{\psi}(\vec{j}) \equiv \Lambda^{p/2}\bar{\chi}(\vec{j}), \tag{2.13}$$
$$g_0 \equiv g\Lambda^{1-p},$$

with the canonical anticommutation relations

$$\{\chi^\dagger(\vec{j}_1), \chi(\vec{j}_2)\} = \underline{1}\,\delta_{\vec{j}_1,\vec{j}_2}. \tag{2.14}$$

In terms of these variables we write the lattice Hamiltonian

$$H = \Lambda\Big(\sum_{\vec{j}_1,\vec{j}_2} \chi^\dagger(\vec{j}_1) i\vec{\alpha}\cdot\vec{\delta}'(\vec{j}_1 - \vec{j}_2)\chi(\vec{j}_2) - \frac{g}{2}\sum_{\vec{j}}\{[\bar{\chi}(\vec{j})\chi(\vec{j})]^2 - [\bar{\chi}(\vec{j})\gamma_5\chi(\vec{j})]^2\}\Big). \tag{2.15}$$

The lattice "conserved charges" are

$$N = \sum_{\vec{j}} \chi^\dagger(\vec{j})\chi(\vec{j}),$$
$$Q_5 = \sum_{\vec{j}} \chi^\dagger(\vec{j})\gamma_5\chi(\vec{j}), \tag{2.16}$$

and the field equation for the lattice field $\chi(\vec{j})$ as computed directly from the Heisenberg equation $i\dot{\chi}(\vec{j}) = [H, \chi(\vec{j})]$ is

$$i(\gamma_0\partial_0 + \Lambda\vec{\gamma}\cdot\vec{\nabla})\chi(\vec{j}) = -2\Lambda g[\bar{\chi}(\vec{j})\chi(\vec{j}) - \bar{\chi}(\vec{j})\gamma_5\chi(\vec{j})\gamma_5]\chi(\vec{j}), \tag{2.17}$$

with $\vec{\nabla}\chi(j)$ defined by (1.4). We can also construct conserved but nonlocal currents on the lattice

$$j_\mu(\vec{j}) \text{ and } j_{5\mu}(\vec{j})$$

which are given by

$$j_0(\vec{j}) = \chi^\dagger(\vec{j})\chi(\vec{j}), \quad j_{50}(\vec{j}) = \chi^\dagger(\vec{j})\gamma_5\chi(\vec{j}),$$
$$\vec{j}(\vec{j}) = \chi^\dagger(\vec{j})\vec{\alpha}\chi(\vec{j}) + \sum_{\vec{j}_1,\vec{j}_2} S(\vec{j};\vec{j}_1,\vec{j}_2)\chi^\dagger(\vec{j}_1)\vec{\alpha}\chi(\vec{j}_2), \tag{2.18}$$
$$\vec{j}_5(\vec{j}) = \chi^\dagger(\vec{j})\vec{\alpha}\gamma_5\chi(\vec{j}) + \sum_{\vec{j}_1,\vec{j}_2} S(\vec{j};\vec{j}_1,\vec{j}_2)\chi^\dagger(\vec{j}_1)\vec{\alpha}\gamma_5\chi(\vec{j}_2).$$

$S(\vec{j};\vec{j}_1,\vec{j}_2)$ is uniquely defined in terms of the gradient operator (1.4) by the condition that for any two functions

$$\nabla\Big[f(\vec{j})g(\vec{j}) + \sum_{\vec{j}_1,\vec{j}_2} S(\vec{j};\vec{j}_1,\vec{j}_2)f(\vec{j}_1)g(\vec{j}_2)\Big] = [\nabla f(\vec{j})]g(\vec{j}) + f(\vec{j})[\nabla g(\vec{j})]. \tag{2.19}$$

An explicit formula for $S(\vec{j};\vec{j}_1,\vec{j}_2)$ is derived in Appendix A. As defined $j_\mu(\vec{j})$ and $j_{5\mu}(\vec{j})$ satisfy

$$\partial_0 j_0 + \vec{\nabla}\cdot\vec{j}(j) = \partial_0 j_{5,0} + \vec{\nabla}\cdot\vec{j}_5 = 0, \tag{2.20}$$

and the charges (2.16) are conserved. The nonlocal terms in the space components of (2.18) arise from the free-field gradient terms in H. If we evaluate commutators involving time and space components of these currents we obtain a nonlocal term which, in the continuum limit, becomes the familiar Schwinger term.

B. Variational solution in a momentum-space basis

Our goal is to develop reliable methods for diagonalizing (2.15). In particular, we wish to calculate the ground state and low-lying excitation spectrum of H. We have already seen in (2.4) and (2.7) that the $g \to 0$ free-field limit of this theory is readily solved by diagonalizing H in a momentum basis. Recalling the analysis of Ref. 1, we anticipate that it might also be a valid approximation to perform a variational calculation for the upper bound of the ground-state energy in a momentum basis with the mass as variational parameter if we are in the weak-coupling limit of $g \ll 1$ in (2.15). However, in the light of Ref. 1, we also may expect this approach to fail for intermediate or strong coupling $g \gtrsim 1$. Just as the spontaneous symmetry breaking was found to be incorrectly represented in the scalar ϕ^4 theory by such an approach, it would come as no surprise to find here that momentum-space variational methods fail to describe the γ_5-symmetry properties of the ground state for (2.15). In fact the principal conclusions of Ref. 1 are that we should work in a configuration-space basis to construct reliable approximate solutions to (2.15) for intermediate and strong couplings.

Nambu and Jona-Lasinio[8] constructed an approximate solution of this problem in their pioneering attempt to develop a dynamical model of elementary particles based on a cutoff version of (2.8) or (2.9). In their calculation the one-loop contribution to the mass operator was computed, and a finite nonvanishing fermion mass was shown to exist when certain inequalities on the coupling parameter g were satisfied. Their approach is identical to performing a variational calculation for a bound on the ground-state eigenvalue of (2.8) in a momentum basis. To obtain their equation for the mass gap we take the expectation value of (2.8) in the trial ground state $|\Psi_{\text{trial}}(m)\rangle$ defined by

$$b_m(k)|\Psi_{\text{trial}}(m)\rangle = 0,$$
$$d_m(k)|\Psi_{\text{trial}}(m)\rangle = 0, \tag{2.21}$$

where the $b_m(k)$ and $d_m(k)$ are the fermion and antifermion annihilation operators defined by the plane-wave expansion (2.5), (2.6), and (2.7) in terms of a mass parameter that is chosen arbitrarily. Either the continuum theory with a $k_{\max}=\pi\Lambda$ cutoff or the lattice theory can be used since, as formulated, they lead to the same dispersion relation (1.6). Minimizing this expectation value with respect to the variation parameter, m, leads directly to

$$m\left[1-4g\frac{\Lambda^{1-p}}{L^p}\sum_k\frac{1}{(k^2+m^2)^{1/2}}\right]=0. \tag{2.22}$$

Following from (2.22) either $m=0$ or the "gap equation" of Nambu and Jona-Lasinio must be satisfied. Whenever g has values such that there exists a solution of the gap equation, the $m\neq 0$ solution corresponds to the energy minimum and $m=0$ to a local maximum. In particular (2.22) has an $m\neq 0$ solution for all finite values of the volume V. For $p>1$ (i.e., for a two- or three-space-dimensional lattice model) there is a critical value of g such that as $V\to\infty$, for $g<g_{\text{crit}}\approx 1$, $m=0$ is the only solution, whereas for $g\geq g_{\text{crit}}$, $m\neq 0$ is the true minimum. In general for $g\approx 1$, this value of m satisfies $0<m\ll\Lambda$; on the other hand, $m\sim g\Lambda$ when $g\gg 1$. When $p=1$ (i.e., for the one-space-dimensional Thirring model) there is a finite-mass solution of the gap equation (2.22) for all values of g. In particular for $p=1$

$$\begin{aligned} &g\sim 1/\ln(\Lambda/m) \text{ for } \Lambda/m\gg 1,\\ &g\sim m/\Lambda \text{ for } m/\Lambda\gg 1. \end{aligned} \tag{2.23}$$

As argued in Ref. 8 when $m\neq 0$, corresponding to the existence of massive fermion states, the ground state of (2.8) is (continuously) infinitely degenerate, implying the existence of massless Goldstone bosons. However, this result is in conflict with Coleman's theorem[10] for $p=1$. Hence this technique, which is identical to the one-loop effective-potential method, must be misleading for determining chiral properties of the ground state. It is evident that the application of such techniques, which have been proposed for studying spontaneous breakdown of γ_5 invariance and the formation of dynamical Goldstone bosons, is open to challenge. To see why this is so recall from Ref. 1 that this technique for calculating the ground-state energy in k space diagonalizes the gradient term, but makes a Hartree-Fock approximation to the quartic self-interaction potential, viz.,

$$\langle(\bar\psi\psi)^2\rangle\to 2\langle\bar\psi\psi\rangle^2. \tag{2.24}$$

In our earlier study of the scalar ϕ^4 theory we found that such a variational analysis was reliable for weak coupling, where the gradient term is the more important one, but can be very misleading when applied in the strong-coupling regime, which requires a more accurate treatment of the nonlinear potential effects.

We turn therefore to a configuration-space approach to construct reliable approximate solutions to (2.15) when $g\gtrsim 1$. It will turn out that a site basis gives lower (i.e., better) ground-state energies than the above method, and, furthermore, we realize the γ_5 symmetry of the theory in the "normal way"; i.e., we find a spectrum of massive, parity-doubled bosons and fermions rather than massless Goldstone bosons and infinitely degenerate ground states. Similar conclusions were stated by Ichimura, Kikkawa, and Yazaki[9] in 1966 using a truncated Hamiltonian.

C. Configuration-space analysis of the "Thirring model" in three space and one time dimension

We begin by studying (2.15) in the strong-coupling limit in three space and one time dimension because it is formally simpler, though less interesting, than the Thirring model in one space and one time dimension. The subtleties of the latter and a comparison of its continuum and lattice versions are explored in Sec. III. A convenient representation for finding the ground state and excitation spectrum in the strong-coupling limit, $g\gg 1$, is

$$\chi_{\vec j}=\begin{pmatrix}B_{\vec j}\\ D^\dagger_{\vec j}\end{pmatrix},\quad B_{\vec j}=\begin{pmatrix}b_{j+}\\ b_{j-}\end{pmatrix},\quad D_{\vec j}=\begin{pmatrix}d_{j+}\\ d_{j-}\end{pmatrix},\quad n_{b_{j\pm}}=b^\dagger_{j\pm}b_{j\pm},\quad n_{d_{j\pm}}=d^\dagger_{j\pm}d_{j\pm} \tag{2.25}$$

$$\gamma_0=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \gamma^l=\begin{pmatrix}0&\sigma^l\\-\sigma^l&0\end{pmatrix},\quad \gamma_5=\begin{pmatrix}1&0\\0&-1\end{pmatrix},$$

in terms of which

$$H=-2g\Lambda\sum_{\vec{j}}\left[(n_{b_{j+}}+n_{d_{j+}}-1)^2+(n_{b_{j-}}+n_{d_{j-}}-1)^2+2(b_{j+}^{\dagger}d_{j+}^{\dagger}d_{j-}b_{j-}+d_{j-}^{\dagger}b_{j-}^{\dagger}b_{j+}d_{j+})\right]$$
$$+\Lambda\sum_{\vec{j},\vec{j}'}\left[iB_{\vec{j}}^{\dagger}\vec{\sigma}\cdot\vec{\delta}'(\vec{j}-\vec{j}')B_{\vec{j}'}-iD_{\vec{j}}^{\dagger}\vec{\sigma}\cdot\vec{\delta}'(\vec{j}-\vec{j}')D_{\vec{j}'}^{\dagger}\right]$$
$$\equiv H_0+K \tag{2.26}$$

and the conserved charges Q and Q_5 take the form

$$Q\equiv N-\sum_{\vec{j}}2=\sum_{\vec{j}}(n_{b_{j+}}+n_{b_{j-}}-n_{d_{j+}}-n_{d_{j-}}),$$
$$Q_5\equiv\sum_{\vec{j}}(n_{b_{j+}}+n_{b_{j-}}+n_{d_{j+}}+n_{d_{j-}}-2). \tag{2.27}$$

In the strong-coupling limit we diagonalize the first or "potential" term in (2.26), denoted H_0, and treat the kinetic term iteratively as a perturbation.[11] H_0 is diagonal in a site basis and the eigenstates, energy eigenvalues, and corresponding charges at a single site are listed in Table I. Our first observation is that with the conventional choice, $g>0$, there is a unique ground state with $Q=Q_5=0$. The gradient term commutes with Q and Q_5 and mixes states only within the same (Q,Q_5) sector of the Fock space. Hence it splits the degenerate excited states but induces only negligible corrections $\sim 1/g$ to the nondegenerate ground-state structure. Table I also shows that the lowest excited states are a chiral pair of spin-zero "bosons" separated from the ground state by a gap of $4g\Lambda$. There are also parity-doubled fermion states separated from the ground states by $6g\Lambda$. Evidently there is little content to this model of physical interest since, for $g\gg 1$, all particle states are very massive, since $m\sim g\Lambda$ which is larger than the cutoff. The importance of our conclusion that there exist no Goldstone bosons, but a *unique ground state* plus massive degenerate chiral multiplets of bosons and fermions, lies in its difference from the widely applied effective-potential technique. As noted, the loopwise effective-potential expansion leads to (2.22) in the one-loop approximation and to the prediction of Goldstone bosons when $g\gtrsim 1$ and $m\neq 0$ for the energy minimum. Further evidence in support of our analysis comes from the following two observations:

1. The site basis gives a lower value for the ground-state energy than the upper bound obtained from (2.21) and (2.22). According to Table I the ground-state energy for a cubic lattice of $(2N+1)^3=V\Lambda^3$ sites is, for $g\gg 1$,

$$E_0^{\text{site}}=-8g\Lambda(2N+1)^3=-8g\Lambda^4V. \tag{2.28}$$

The corresponding result using (2.21) and filling the negative-energy sea with fermions of mass m gives for the ground state

$$E_0^{\text{mom}}=-2V\int^{(p_M=\pi\Lambda)}\frac{d^3p}{(2\pi)^3}(p^2+m^2)^{1/2}. \tag{2.29}$$

In the strong-coupling limit of the gap equation (2.22)

$$m\simeq\frac{2\pi}{3}g\Lambda\gg\Lambda, \tag{2.30}$$

and (2.29) becomes

$$E_0^{\text{mom}}=-\frac{\pi}{3}\Lambda^3Vm\simeq-\frac{2\pi^2}{9}g\Lambda^4V, \tag{2.31}$$

which is higher than the site basis result (2.28) by a factor of greater than 3.5.

2. In the weak-coupling limit of $g\ll 1$, (2.15) becomes the Hamiltonian of free massless fermions, and as a result of our treatment of the gradient operator we obtain the correct relativistic energy-momentum relation

$$E(k)=|k|.$$

No finite iterative treatment of the interaction terms in powers of g can lead to the prediction of Goldstone bosons. Although we have not attempted a systematic study of (2.15) for $g\sim 1$, it is difficult to understand how the symmetry structure of the theory can change so radically in this region, leading to Goldstone bosons that must cleverly hide themselves in both the strong- and weak-coupling regions.

III. THE THIRRING MODEL IN TWO DIMENSIONS

In this section we analyze the two-dimensional lattice Thirring model defined by the Hamiltonian (2.15) for the case $p=1$. This theory has important features in common with gauge theories, and so this analysis will prove useful to our subsequent discussions. In particular we find that the strong-coupling limit of this model in one space and one time dimension, in contrast to the preceding discussion for $p=3$, describes a system of massless fermion-antifermion bound states in addition to supermassive charged fermions of mass $\sim g\Lambda\gg\Lambda$. Hence this model provides a concrete example of a theory for which, as $g\to\infty$, the original fermionic degrees of freedom become "frozen out," but new massless degrees of free-

TABLE I. A list of the eigenstates of H_0 defined in (2.26) and their corresponding eigenvalues as well as their Q and Q_5 eigenvalues.

State	(per site) Q	Q_5	Energy $\langle H_0\rangle$
$\frac{1}{\sqrt{2}}(b_+^\dagger d_+^\dagger + b_-^\dagger d_-^\dagger)\|0\rangle$	0	0	$-8g_0\Lambda$
$\frac{1}{\sqrt{2}}(b_+^\dagger d_+^\dagger - b_-^\dagger d_-^\dagger)\|0\rangle$	0	0	0
$\frac{1}{\sqrt{2}}(b_+^\dagger d_-^\dagger + b_-^\dagger d_+^\dagger)\|0\rangle$	0	0	0
$\frac{1}{\sqrt{2}}(b_+^\dagger d_-^\dagger - b_-^\dagger d_+^\dagger)\|0\rangle$	0	0	0
$\|0\rangle$	0	−2	$-4g_0\Lambda$
$b_-^\dagger b_+^\dagger d_-^\dagger d_+^\dagger\|0\rangle$	0	+2	$-4g_0\Lambda$
$b_-^\dagger b_+^\dagger\|0\rangle$	2	0	0
$d_-^\dagger d_+^\dagger\|0\rangle$	−2	0	0
$b_+^\dagger\|0\rangle$	+1	−1	$-2g_0\Lambda$
$b_-^\dagger\|0\rangle$	+1	−1	$-2g_0\Lambda$
$d_+^\dagger\|0\rangle$	−1	−1	$-2g_0\Lambda$
$d_-^\dagger\|0\rangle$	−1	−1	$-2g_0\Lambda$
$d_+^\dagger b_+^\dagger b_-^\dagger\|0\rangle$	+1	+1	$-2g_0\Lambda$
$d_-^\dagger b_+^\dagger b_-^\dagger\|0\rangle$	+1	+1	$-2g_0\Lambda$
$b_+^\dagger d_+^\dagger d_-^\dagger\|0\rangle$	−1	+1	$-2g_0\Lambda$
$b_-^\dagger d_+^\dagger d_-^\dagger\|0\rangle$	−1	+1	$-2g_0\Lambda$

dom are left behind. In addition, we can "track" these massless states into the weak-coupling region of the theory formulated on a few lattice sites.

A. Strong-coupling calculation on a lattice

For $g \gg 1$ we follow our strategy of first diagonalizing the quartic part of the Hamiltonian (2.15) exactly. For this purpose a convenient two-component representation is

$$\gamma_5 = \alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.1}$$

and

$$\chi(j) = \begin{pmatrix} b(j) \\ d^\dagger(j) \end{pmatrix}, \tag{3.2}$$

with $b(j)$ and $d(j)$ satisfying the standard anticommutation relations, viz.,

$$\{b(j), b^\dagger(j')\} = \{d(j), d^\dagger(j')\} = \delta_{jj'}, \quad \{b(j), d(j')\} = 0, \text{ etc.} \tag{3.3}$$

Substituting these formulas into (2.15) gives [using $\delta'(j_1 - j_2) = -\delta'(j_2 - j_1)$]

$$H = \Lambda\left\{\sum_{j_1, j_2} i\delta'(j_1 - j_2)(b_{j_1}^\dagger b_{j_2} - d_{j_1}^\dagger d_{j_2}) - g\sum_j [n_b(j) + n_d(j) - 1]^2\right\} \equiv K + V, \tag{3.4}$$

where

$$n_b(j) \equiv b^\dagger(j)b(j), \quad n_d(j) \equiv d^\dagger(j)d(j) \tag{3.5}$$

are particle and antiparticle number operators, respectively. As before, since the potential V is a sum of commuting single-site terms we can diagonalize each term separately and form a product basis over all sites.

There are only four states for each j corresponding to the different choices $n_b(j) = 1, 0$ and $n_d(j) = 1, 0$. If we define $|0(j)\rangle$ by

$$b(j)|0(j)\rangle = d(j)|0(j)\rangle = 0, \tag{3.6}$$

we find the four eigenstates of the interaction term listed in Table II. Even at the one-site level the ground state is two-fold degenerate with

$$E_0(j) = E_\pm(j) = -g\Lambda \tag{3.7}$$

corresponding to having nothing, with $Q(j) = 0$, $Q_5(j) = -1$, or a particle-antiparticle pair, $Q(j) = 0$, $Q_5(j) = +1$, at a site. One also sees that the single-site charged states, with $Q(j) = \pm 1$, $Q_5(j) = 0$, lie high above the ground state from which they are separated by a gap $\sim g\Lambda$.

This two-fold degeneracy of the single-site eigenstates, which did not occur in the three-dimensional lattice, means that in the absence of the gradient term, the ground state of the strong-interaction part of H is 2^{2N+1}-fold degenerate, since there can be either nothing or a "bound" fermion-antifermion pair at each lattice site. The total electric charge of these degenerate states

$$Q = \sum_j [n_b(j) - n_d(j)] \tag{3.8}$$

is zero, and their γ_5 charge

$$Q_5 = \sum_j [n_b(j) + n_d(j) - 1] \tag{3.9}$$

can take any odd-integer value from $-(2N+1)$ to $+(2N+1)$ depending on the number of sites occupied by pairs. Note that any neutral state which contains an unbound pair with a fermion and an antifermion split to different lattice sites will lie higher in energy by an amount equal to $2g\Lambda$ for

TABLE II. The eigenstates of V defined in (3.4), their corresponding eigenenergies, and eigenvalues of Q and Q_5.

State	$Q(j) \equiv n_b(j) - n_d(j)$	$Q_5(j) \equiv n_b(j) + n_d(j) - 1$	Eigenvalue of $-g\Lambda[n_b(j) + n_d(j) - 1]^2$
$\|0\rangle$	0	−1	$-g\Lambda$
$\|+\rangle \equiv b^\dagger \|0\rangle$	1	0	0
$\|-\rangle \equiv d^\dagger \|0\rangle$	−1	0	0
$\|\pm\rangle \equiv b^\dagger d^\dagger \|0\rangle$	0	+1	$-g\Lambda$

each such split pair. The states in this sector and their eigenvalues are listed in Table III for a three-site lattice, where we have introduced the notation

TABLE III. The eigenstates of V defined in (3.4) for the special case of a three-site lattice. The states are labeled according to the notation of (3.10) and list the eigenenergies $E_{\text{strong}} = \langle V \rangle$ as well as the Q and Q_5 eigenvalues for each state.

States for three sites	E_{strong}	Q	Q_5
$\begin{pmatrix}0,0,0\\0,0,0\end{pmatrix}$	$-3\Lambda g$	0	−3
$\begin{pmatrix}1,0,0\\1,0,0\end{pmatrix}, \begin{pmatrix}0\,1\,0\\0\,1\,0\end{pmatrix}, \begin{pmatrix}0,0,1\\0,0,1\end{pmatrix}$	$-3\Lambda g$	0	−1
$\begin{pmatrix}1\,0\,0\\0\,1\,0\end{pmatrix}, \begin{pmatrix}1\,0\,0\\0\,0\,1\end{pmatrix}$	$-\Lambda g$	0	−1
$\begin{pmatrix}0\,1\,0\\1\,0\,0\end{pmatrix}, \begin{pmatrix}0\,1\,0\\0\,0\,1\end{pmatrix}$	$-\Lambda g$	0	−1
$\begin{pmatrix}0\,0\,1\\1\,0\,0\end{pmatrix}, \begin{pmatrix}0\,0\,1\\0\,1\,0\end{pmatrix}$	$-\Lambda g$	0	−1
$\begin{pmatrix}1\,1\,0\\1\,1\,0\end{pmatrix}, \begin{pmatrix}1\,0\,1\\1\,0\,1\end{pmatrix}, \begin{pmatrix}0\,1\,1\\0\,1\,1\end{pmatrix}$	$-3\Lambda g$	0	+1
$\begin{pmatrix}0\,1\,1\\1\,0\,1\end{pmatrix}, \begin{pmatrix}0\,1\,1\\1\,1\,0\end{pmatrix}$	$-\Lambda g$	0	+1
$\begin{pmatrix}1\,0\,1\\0\,1\,1\end{pmatrix}, \begin{pmatrix}1\,0\,1\\1\,1\,0\end{pmatrix}$	$-\Lambda g$	0	+1
$\begin{pmatrix}1\,1\,0\\0\,1\,1\end{pmatrix}, \begin{pmatrix}1\,1\,0\\1\,0\,1\end{pmatrix}$	$-\Lambda g$	0	+1
$\begin{pmatrix}1\,1\,1\\1\,1\,1\end{pmatrix}$	$-3\Lambda g$	0	+3

$$\begin{pmatrix} n_b(-N), \ldots, n_b(j), \ldots, n_b(N) \\ n_d(-N), \ldots, n_d(j), \ldots, n_d(N) \end{pmatrix} \tag{3.10}$$

to label a given state.

Our key interest here is to analyze how K, the kinetic energy term in (3.4), splits the degeneracies among the low-lying $Q = 0$ states. Since it commutes with Q and Q_5, K connects states within each Q and Q_5 sector only. Thus, we will treat it as a perturbation for $g \gg 1$, and work within the $Q = 0$ sector to construct the low-lying energy spectrum. It is clear from the form of (3.4) that K, which moves a single fermion or antifermion from one lattice site to another, gives no first-order energy shift to the low-lying states. The state splitting therefore requires that we do second-order degenerate perturbation theory in the ground-state sector of (3.10) with $n_b(j) = n_d(j)$ for each site j. Since all energy denominators between the ground state and an excited state with one unbound pair are the same, $E_x - E_g = 2g\Lambda$, the intermediate-state sum can be performed and we obtain an effective second-order Hamiltonian for the ground-state sector

$$H_{\text{eff}} = -\frac{\Lambda}{2g} \sum_{j_1, j_2} [\delta'(j_1 - j_2)]^2 \{2n_b(j_1)[1 - n_b(j_2)] + 2d^\dagger(j_1) b^\dagger(j_1) b(j_2) d(j_2)\}. \tag{3.11}$$

Equation (3.11) expresses the fact that both a fermion and an antifermion at the same initial site must be transferred to a common final site.

A simple and suggestive "spin" formalism can now be introduced, since at each lattice site only two eigenstates, $|0\rangle$ and $|\pm\rangle$, which correspond to "spin-down" and "spin-up," respectively, occur in the $Q = 0$ sector. We identify "spin" raising and lowering operators

$$\begin{aligned} S_+(j) &\equiv d^\dagger(j) b^\dagger(j), \\ S_-(j) &\equiv b(j) d(j) = [S_+(j)]^\dagger \end{aligned} \tag{3.12}$$

such that

$$|\pm(j)\rangle = S_+(j)|0(j)\rangle ,$$

and introduce

$$n_b(j) = n_a(j) = S_3(j) + \tfrac{1}{2}$$

in terms of which (3.11) can be rewritten

$$H_{\text{eff}} = -\frac{\Lambda}{g}\sum_{j_1,j_2}[\delta'(j_1 - j_2)]^2[\tfrac{1}{4} + S_+(j_1)S_-(j_2) - S_3(j_1)S_3(j_2)]. \tag{3.13}$$

Except for the relative minus sign between the spin-spin terms H_{eff} describes the Heisenberg antiferromagnetic chain, about which a great deal is known.[12]

In order to understand this analogy let us, for the moment only, abandon our definition (1.4) of the gradient and return to its definition in (1.1) as a difference, in which case (3.13) contains only nearest-neighbor interactions. If we now make a unitary transformation changing the representation (3.12) by rotating through angle π about the three-axis at every other lattice site, i.e.,

$$S_\pm(j) \to (-)^j S_\pm(j),$$

$$S_3(j) \to S_3(j),$$

the effective spin Hamiltonian becomes

$$\tilde{H}_{\text{eff}} = \frac{\Lambda}{g}\sum_j [\vec{S}(j)\cdot\vec{S}(j+1) - \tfrac{1}{4}]. \tag{3.14}$$

Equation (3.14) now describes the well-studied linear Heisenberg antiferromagnetic chain with nearest-neighbor interactions. The eigenstates of $\tilde{H}_{\text{eff}}$ can be classified into degenerate multiplets of the total spin as well as of its three-component, $\frac{1}{2}Q_5 = +\sum_j S_3(j)$. If we further assume that the sum over j extends over a linear chain with an even number of sites and impose cyclic boundary conditions, we can refer to two exact theorems for important information on the ground state and excitation spectrum of (3.14):

Theorem 1. When $\Lambda/g > 0$, corresponding to an antiferromagnetic interaction, the ground state of (3.14) has total spin $S = 0$ and is unique.[13]

Theorem 2. The theory has no mass gap in the limit as the length of the linear chain becomes infinite; i.e., there is a state orthogonal to the ground state having vanishingly small excitation energy.[14]

There is a minor difference in theorem 1 for a lattice with an odd number of sites arising from the fact that it is impossible to form a state of $S_3 = 0$. In this case $S_3 = \pm\frac{1}{2}$ is the lowest possible value and the ground state is two-fold degenerate corresponding to the invariance of the massless theory under $Q_5 \to -Q_5$. This doubling of ground state is also suggested by the careful study of the infinitely long linear chain.[15]

The original solution for the ground state and excitation spectrum of (3.14) is due to Bethe[16] in 1931. In agreement with the general theorems his analysis of the lowest-lying excitations in each sector of definite S_3 showed that the ground state of the system is unique (for even numbers of sites) and corresponds to a state having $S_3 = 0$ (i.e., in the language of the Thirring model one has a linear superposition of states having half of the lattice sites empty and half occupied by fermion-antifermion pairs). His methods also show that the excitation spectrum starts off linearly in k, corresponding to a massless particle spectrum. If these results carry over to the solution of (3.13), which we constructed using (1.4) to avoid the doubling of the free fermion states on a lattice, we see from the above theorems that to leading order in $1/g$ there exists a low-lying spectrum of massless excitations of the Thirring model in addition to the arbitrarily massive ($\sim g\Lambda$) normal fermionic excitations. This low-lying excitation spectrum corresponds to bound fermion pairs but, like a fermion, obeys the exclusion-principle limit of no more than one pair per lattice site. This spectrum is built upon a unique or doubly degenerate vacuum (depending upon whether we use an even or an odd number of sites), and there is no spontaneous breaking of γ_5 invariance leading to Goldstone bosons. Recall that recently Coleman has obtained a similar result for the strong-coupling limit of the massive Thirring model.[17]

The above insights into solutions of the nearest-neighbor problem (3.14) are useful guides, although we have been unable to generalize Bethe's technique to solve completely the problem at hand in (3.13). In particular the sectors with $S_3 = \mp\frac{1}{2}(2N+1)$, corresponding to all sites empty, or occupied by a pair, are eigenstates of H_{eff} with eigenvalue 0 and are evidently the nondegenerate ground states in their respective sectors of $Q_5 = \mp(2N+1)$. These states are eigenstates of the total Hamiltonian of energy $-g\Lambda(2N+1)$ according to Table II and (3.7).

A less trivial case is the exact solution of (3.13) in the sector $Q_5 = \mp(2N-1)$ which corresponds to constructing appropriate superpositions of (3.10) with a single bound pair present (or absent). The ground state in this sector lies below the above result for the $Q_5 = \mp(2N+1)$ sector, and the excitation spectrum is found (see Appendix B) to start off linearly in the total momentum k: i.e.,

$$E(k)_{Q_5=\pm(2N-1)} = \Lambda\left[-g(2N+1) - \frac{\pi^2}{6g} - \frac{1}{2g}\left(\frac{|k|}{\Lambda} - \pi\right)^2\right]$$
$$= \Lambda\left[-g(2N+1) - \frac{2\pi^2}{3g}\right] + \frac{\pi}{g}|k| - \frac{|k|^2}{2g\Lambda}, \tag{3.15}$$

where $k_{\min} = 2\pi/L < |k| < \pi\Lambda = k_{\max}$. With suitable wave-function renormalization (3.15) describes a relativistic massless particle spectrum since in the limit $\Lambda \to \infty$ the k^2 term vanishes. Except for a numerical factor resulting from the definition (1.4), the spectrum (3.15) is identical to that found by Bethe[16] for (3.14).

In all other sectors containing (or lacking) two or more bound pairs the analysis becomes much more formidable since it becomes necessary to solve a quasiparticle scattering problem for two or more spin waves. This is the problem solved only for nearest-neighbor interactions by Bethe in his remarkable 1931 analysis of (3.14). In Appendix C we describe the formulation of this scattering problem in the $Q_5 = -(2N-3)$ two-pair sector.

On the basis of the analysis of (3.14) we expect the ground-state solution of (3.13) with the lowest energy to lie in the $Q_5 = \pm 1 = +2S_3$ sector. We have explicitly verified this on a three-site lattice, but for the general case must rely on a variational calculation to construct upper bounds on the ground-state energy in each Q_5 sector. This bound can then be compared with the known Bethe solutions for $\tilde{H}_{\text{eff}}$. In particular for the sector, $Q_5 = -(2N+1)+2p$, with pairs at $0 \le p \le (2N+1)$ sites we use a fully symmetrized trial state

$$|\bar{\Psi}(p)\rangle = \frac{1}{p!}\binom{2N+1}{p}^{-1/2} \sum_{i_1 \cdots i_p} S^\dagger_{i_1} S^\dagger_{i_2} \cdots S^\dagger_{i_p} |0\rangle$$
$$= \lim_{\alpha \to 0} \frac{1}{p!\binom{2N+1}{p}^{1/2}} \left(\frac{d}{d\alpha}\right)^p \exp\left(\alpha \sum_{i=-N}^{N} S^\dagger_i\right)|0\rangle . \tag{3.16}$$

The latter form automatically does the bookkeeping of summing over all ways of choosing p different lattice sites since $(S^\dagger_i)^2 = 0$. The upper bound on the total energy as a function of the number of occupied or spin-up lattice sites is evaluated directly as a function of p,

$$E_{\text{var}}(p) = -(2N+1)\Lambda g - \frac{2\pi^2}{3g}\frac{p(2N+1-p)}{2N}\Lambda, \qquad p = 0, 1, \ldots, 2N+1. \tag{3.17}$$

The bound in (3.17) is also the exact result for $p = 0$ and for $p = 1$, coinciding with (3.15) for the ground state with $k = 0$. Equation (3.17) describes a parabola as a function of p with a doubly degenerate minimum at $p = N$ and $N+1$ corresponding to $S_3 = \mp\frac{1}{2}$. This suggests that (3.13) has the same general structure as the theory defined by (3.14): namely, the ground state is a γ_5 doublet, and the spectrum has no mass gap.[18]

As a further calibration of the guess (3.16) for the ground state we can use it to calculate an upper bound on the ground-state energy of (3.14) and compare the answer with Bethe's exact result. Since (3.16) is symmetric in all $2N+1$ sites the expectation value of the sums over sites in (3.13) and (3.14) can be performed explicitly, and their ratio

$$\frac{\sum_{j_1, j_2} [\delta'(j_1 - j_2)]^2}{\sum_j 1} = \frac{\pi^2}{3} \tag{3.18}$$

gives the factor by which the bound on the $1/g$ terms in (3.17) has to be reduced to compare with the exact energy. In particular the ground state of (3.17) when $p = N$ gives for the nearest-neighbor case

$$E_{\text{var}}(N) = -(2N+1)\Lambda g - \frac{\Lambda}{g}(N+1), \tag{3.19}$$

which is to be compared with the exact ground-state energy of

$$E_{\text{Bethe}}(N) = -(2N+1)\Lambda g - \frac{\Lambda}{g}(N+1)2\ln 2 . \tag{3.20}$$

This comparison suggests that the guess (3.16) of a symmetric spin function without correlations is a reasonable representation of the low-lying states, and also that the general structure of the spectra for (3.13) and (3.14) is similar.

B. Weak-coupling analysis

As we have defined it on the lattice the Thirring model realizes its γ_5 symmetry in the strong-coupling region in the normal way—i.e., a singly or doubly degenerate vacuum but not an infinitely degenerate one with Goldstone bosons. The bound pairs obey the exclusion principle and, at least for the case of nearest-neighbor interactions described by (3.14), are massless. We want to show that is entirely consistent with what is known about the weak-coupling limit of the theory. As an explicit example we also construct the exact solution of the simple three-site example and track the low-lying states from the $g \ll 1$ to the $g \gg 1$ region.

Since our definition of the gradient reproduces the relativistic free-particle energy-momentum relation, it is not surprising that our formulation

joins smoothly to the known results of the continuum model for small g, and that we arrive at the usual weak-coupling Feynman rules. In contrast, the procedure based on the definition of the gradient operator by (1.1) and the splitting of fermions from antifermions onto alternate lattice sites leads to a very different low-lying spectrum as a result of violating local γ_5 invariance. We shall discuss this and its implications for gauge models in the next section.

To study the weak-coupling behavior of (3.4) it is convenient (as usual) to diagonalize the kinetic energy term in momentum space. We do this by introducing

$$b(k)=\sum_j \frac{1}{(2N+1)^{1/2}} b(j)e^{-ikj/\Lambda},$$
$$d(k)=\sum_j \frac{1}{(2N+1)^{1/2}} d(j)e^{-ikj/\Lambda}, \tag{3.21}$$

which satisfy the familar anticommutation relations, $\{b^\dagger(k'), b(k)\}=\delta_{kk'}$, etc. We obtain

$$K=\sum_{k=-\pi\Lambda}^{+\pi\Lambda} k(b^\dagger(k)b(k)-d^\dagger(k)d(k))$$
$$=\sum_{-k_{\max}}^{k_{\max}} k[n_b(k)-n_d(k)], \tag{3.22}$$

$$Q=\sum_k [n_b(k)-n_d(k)], \tag{3.23}$$

$$Q_5=\sum_k [n_b(k)+n_d(k)-1],$$

$$V=g\Lambda Q_5-\frac{2g\Lambda}{(2N+1)}\sum_{k_1\cdots k_4}\delta_p(k_1+k_2-k_3-k_4)$$
$$\times b^\dagger(k_1)d^\dagger(k_2)d(k_3)b(k_4), \tag{3.24}$$

where

$$\delta_p(k_1+k_2-k_3-k_4)\equiv\frac{1}{2N+1}\sum_j e^{i(k_1+k_2-k_3-k_4)j/\Lambda}$$

is the periodic δ function which vanishes unless the sum of the momentum vectors is zero or a multiple of 2π. It is apparent from (3.22) and (3.23) that for $g\to 0$, the lowest eigenstate of k corresponds to filling all $k<0$ states with fermions and all positive-energy states with antifermions, i.e.,

$$n_b(k)=\theta(-k),$$
$$n_d(k)=\theta(k). \tag{3.25}$$

This leads to a doubly degenerate ground state in the neutral $Q=0$ sector, depending on whether the $k=0$ state is empty or occupied by a pair, with energy

$$E_{Q=0,\ Q_5=\pm 1}=-2\sum_{k=0}^{k_{\max}} k, \tag{3.26}$$

and to two states of the same energy and with charge $Q=\pm 1$, $Q_5=0$ corresponding to a fermion *or* an antifermion present in the state $k=0$. The neutral $Q=0$ ground states in the different Q_5 sectors can also be deduced from (3.22). For $Q_5=2p-(2N+1)$ the ground state corresponds to a state having p fermion-antifermion pairs chosen successively to have the largest negative (for fermions) and positive (for antifermions) available k values, i.e.,

$$|\psi_G\{Q=0,\ Q_5=2p-(2N+1)\}\rangle=\prod_{k=(N+1-p)k_{\min}}^{Nk_{\min}=k_{\max}}(b^\dagger_{-k}d^\dagger_k)|0\rangle, \tag{3.27}$$

$$E_G(p)=-2\sum_{k=(N+1-p)k_{\min}}^{k_{\max}} k,\quad k_{\min}=\frac{2\pi}{L}=\frac{2\pi\Lambda}{2N+1}. \tag{3.28}$$

So far we have just been labeling the eigenstates of definite Q and Q_5 for the theory of a free massless fermion. An iterative weak-coupling treatment of (3.24) modifies this spectrum but does not change the nature of the Q_5 symmetry. Thus we find that the massless free fermions of the $g=0$ limit bind in pairs and become the massless states of strong-coupling limit. Although there is no spin degree of freedom in the linear lattice chain in one space and one time dimension, both the weak-coupling $g\to 0$ and the strong-coupling $1/g\to 0$ limits, the lowest-lying objects are "fermions" in that they obey an exclusion principle.

It is instructive to trace the low-lying levels from the weak- into the strong-coupling region by exactly diagonalizing H in (3.4) for a lattice of a few sites only. In particular choosing a three-site lattice with $N=1$, there are four possible states at each lattice site as in Table II or a total of 64 states in terms of which to diagonalize H. However, only the nine-dimensional sector with $Q=0$ and $Q_5=-1$ needs to be considered. Since (3.4) is invariant under a parity transformation

$$b_j\to d^\dagger_{-j},\quad Q\to Q,$$
$$d_j\to b^\dagger_{-j},\quad Q_5\to -Q_5, \tag{3.29}$$

the spectrum in the $Q=0$ and $Q_5=+1$ sector is identical. The sectors with $Q\neq 0$ all lie higher in energy. We found this to be the case in the study of the strong-coupling behavior in the preceding section and can readily confirm it here by direct calculation. In particular the low-lying states,

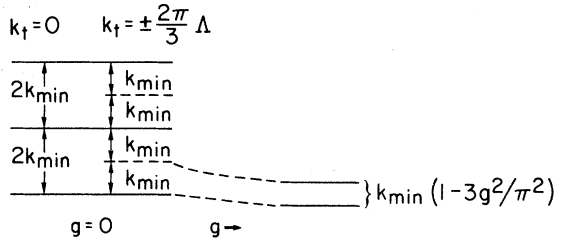

FIG. 2. Low-lying states of the Thirring model for weak coupling.

with $Q_5=0$ and $Q=\pm 1$ formed as described below (3.26) by putting a single fermion or antifermion in the $k=0$ momentum state, are all pushed up in energy relative to the ground state for $g>0$. Within the $Q=0$ sector itself the nine-dimensional problem is further reduced to three tractable three-dimensional ones by classifying the states according to their total momentum $k/\Lambda=0$, $2\pi/3$, or $-2\pi/3$ [modulo 2π due to the umklapp processes in (3.24), i.e., $4\pi/3=-2\pi/3+2\pi\Rightarrow-2\pi/3$]. The corresponding nine eigenvalues are

$$\begin{aligned}E&=-\tfrac{10}{3}g\Lambda-\frac{8\Lambda}{3}\left(g^2+\frac{\pi^2}{3}a^2\right)^{1/2}\cos\frac{\phi}{3}\\&=-\tfrac{10}{3}g\Lambda+\frac{8\Lambda}{3}\left(g^2+\frac{\pi^2}{3}a^2\right)^{1/2}\cos\left(\frac{\phi}{3}+\frac{\pi}{3}\right)\\&=-\tfrac{10}{3}g\Lambda+\frac{8\Lambda}{3}\left(g^2+\frac{\pi^2}{3}a^2\right)^{1/2}\cos\left(\frac{\phi}{3}-\frac{\pi}{3}\right),\end{aligned}\tag{3.30}$$

where

$$\cos\phi\equiv\left(1+\frac{\pi^2a^2}{3g^2}\right)^{-3/2};$$

$a=1$ in the zero-momentum sector, and $a=\pm\frac{1}{2}$ in the sector of momentum $\pm(2\pi/3)\Lambda$. The small- and large-g limits reduce to previous results. For small g the spectrum is as shown in Fig. 2. In the limit $g=0$ there is a gap of $k_{min}=(2\pi/3)\Lambda$ between the energy of the lowest-lying zero-momentum ground state ($a=1$) and the lowest state with momentum $(2\pi/3)\Lambda$ ($a=\frac{1}{2}$). As g increases these states are shifted downward in energy, tracking one another linearly corresponding to a massless excitation spectrum, but there is a wave-function renormalization—i.e.,

$$E_0(a=\tfrac{1}{2})-E_0(a=1)=k_{min}\left[\left(1-\frac{3g^2}{\pi^2}\right)+\cdots\right].\tag{3.31}$$

A similar analysis of the five-site problem has been carried out for $g\ll 1$ as a perturbation expansion and shows that the ground state lies in the $Q=0$, $Q_5=-1$ (or $+1$) sector, and that within this sector the lowest-energy state of momentum $k_m=(2\pi/5)\Lambda$ follows the $k=0$ ground state down in energy as g increases staying a distance $\sim k_m$ away. As in the exact three-site analysis (3.31), it starts off for small $g\ll 1$ like a massless spectrum and acquires a wave-function renormalization. All other states are pushed up in mass so that the gap between them and the ground state is $\sim g\Lambda$.

In summary we have found on the basis of our lattice formalism that the free massless fermions of the $g\to 0$ limit form into bound pairs obeying an exclusion principle for each lattice site for strong coupling $g\gg 1$. For the case of nearest-neighbor interactions in (3.14) there exists a proof that these bound pairs are also massless. We have no proof of the massless nature of the low-lying excitations in the actual case with long-range interactions described by (3.13). However, we found the excitations to be massless for a lattice with a few sites (3 or 5), and the similarity in the structure of the states as discussed earlier leads us to conjecture that the low-lying excitations of the Thirring model (3.13) are indeed massless.

IV. THE SCHWINGER MODEL

We turn now to the simplest model of interacting fermions plus gauge fields—i.e., the Schwinger model,[7] or QED in one space and one time dimension. The new ingredient here is the gauge field, which we treat in the same way as prescribed by Wilson[2] and Kogut and Susskind[3]; namely, we associate the gauge field with the links between lattice points. Hence, each link corresponds to an independent degree of freedom (i.e., the gauge field) in this formalism. The Schwinger model is a "warm-up exercise" preparatory to tackling the full four-dimensional "color-gauge theory." We will show that it is soluble for strong coupling since then we can reduce it to a variant of the linear Heisenberg antiferromagnetic chain studied in Sec. III.

The Hamiltonian of the Schwinger model on a lattice in one space and one time dimension is written as

$$H=\Lambda\left[\sum_{j_1,j_2}\chi^\dagger_{j_1}\alpha(+i\delta'(j_1-j_2))U(j_1-j_2)\chi_{j_2}+\tfrac{1}{2}\sum_l E^2(l)\right]\tag{4.1}$$

in terms of the charged fermion χ_j and the gauge field $E(l)=-\dot{A}_l$. We define[3] $U(j_1-j_2)$ as

$$U(j_1-j_2)\equiv\prod_{j_1<l<j_2}U(l)$$
$$=U(l_{j_1,j_1+1})U(l_{j_1+1,j_1+2})\cdots U(l_{j_2-1,j_2}), \tag{4.2}$$

where the product is to be taken over all links, l, between the lattice points j_1 and j_2. For a unit link

$$U(l)=e^{igaA(l)},\quad a\equiv 1/\Lambda \tag{4.3}$$

with the convention

$$A(l_{1,2})=-A(l_{2,1})=-A(-l_{1,2}), \tag{4.4}$$

so that

$$U(l_{1,2})=U^\dagger(l_{2,1})=U^\dagger(-l_{1,2}). \tag{4.5}$$

Note that the electric flux has a direction associated with it so that $U(l_{1,2})$ is to be thought of as creating a unit flux (of magnitude $\frac{1}{2}g^2$) oriented from site 1 to site 2.

As indicated the first term in Eq. (4.1) is summed over lattice points, while the second term is over all links. We have scaled all degrees of freedom by the appropriate powers of Λ so as to work with dimensionless fields. The coupling constant g has dimension Λ, and we introduce the dimensionless constant $g_0=ga$. Formally (4.1) reduces to the usual continuum Hamiltonian of the Schwinger model in the limit $a\to 0$. Finally we assume that canonical commutation relations for the fermion field are given by (2.14), and for the gauge field by

$$[A(n),E(m)]=i\delta_{n,m}. \tag{4.6}$$

This completes our specification of the theory. One useful fact which follows from (4.3) and (4.6) is that the operator U acts as a ladder operator on the eigenstates of E shifting them by the value g_0, since

$$[E(n),U(m)]=[E(n),e^{ig_0A(m)}]$$
$$=g_0e^{ig_0A(n)}\delta_{n,m}$$
$$=g_0U(n)\delta_{n,m}. \tag{4.7}$$

We are interested in setting up an approximation scheme that is reliable for studying the strong-coupling behavior of (4.1) when $g_0\to\infty$. Note that this limit means that $g=g_0/a\gg 1/a$ as $a\to 0$. It turns out to be useful to rescale[3] the gauge fields to

$$\mathfrak{a}(l)\equiv g_0A(l),$$
$$\mathcal{E}(l)\equiv\frac{1}{g_0}E(l), \tag{4.8}$$

so that in place of (4.7) we have

$$[\mathcal{E}(n),U(m)]=U(n)\delta_{n,m}. \tag{4.9}$$

Now, if $U(n)$ operates on an eigenstate of $\mathcal{E}$ containing S units of gauge flux on link n, i.e.,

$$\mathcal{E}(n)|S(n)\rangle=S|S(n)\rangle, \tag{4.10}$$

it increases the flux by one unit, i.e., by (4.9)

$$U(n)|S(n)\rangle=|(S+1)(n)\rangle. \tag{4.11}$$

The virtue of this rescaling is that we recognize immediately that the free gauge field energy is the dominant term in H for large $g_0\gg 1$:

$$H\equiv\Lambda\left[\tfrac{1}{2}g_0^2\sum_l\mathcal{E}^2(l)\right.$$
$$\left.+\sum_{j_1j_2}\chi^\dagger_{j_1}\alpha(i\delta'(j_1-j_2))\prod_{j_1<l<j_2}e^{i\mathfrak{a}(l)}\chi_{j_2}\right]$$
$$\equiv H_0+K. \tag{4.12}$$

It is therefore natural to divide H into the large energy

$$H_0\equiv\frac{\Lambda}{2}g_0^2\sum_l\mathcal{E}^2(l) \tag{4.13}$$

associated with the flux links, plus the perturbing effects of the fermion sources in the kinetic term, K. The ground state of H_0 is evidently the state of zero flux with eigenvalue 0, and all other states containing one or more flux links lie higher by at least $\frac{1}{2}g_0^2\Lambda\gg\Lambda$.

As H_0 contains no reference to the fermionic configuration, all zero electric flux states $|\bar\psi,0\rangle$, where $\bar\psi$ is any fermionic configuration, are degenerate. The kinetic term K will lift this degeneracy; however, since $U(j_1-j_2)$ creates flux links, by (4.11) connecting $|\bar\psi,0\rangle$ to states with flux whose energy is $\frac{1}{2}|j_1-j_2|g_0^2\Lambda$, there is no energy shift in first order. Second order is the lowest order in which the flux links cancel by (4.5). Using the matrix representations of Sec. III we find the effective second-order Hamiltonian to be

$$H_{\rm eff}=-\Lambda\sum_{j_1j_2j_3j_4}[-i\delta'(j_1-j_2)][-i\delta'(j_3-j_4)]\sum_n\left[(b^\dagger_{j_1}b_{j_2}-d_{j_1}d^\dagger_{j_2})U(j_1-j_2)|n\rangle\right.$$
$$\left.\times\frac{1}{E_n}\langle n|(b^\dagger_{j_3}b_{j_4}-d_{j_3}d^\dagger_{j_4})U(j_3-j_4)\right]. \tag{4.14}$$

The only intermediate states which contribute in (4.14) are those in which the electric flux created from $|0\rangle$ by $U(j_3 - j_4)$ is annihilated by $U(j_1 - j_2)$. Hence the sums are restricted to $j_2 = j_3$, $j_1 = j_4$, and the energy denominator, E_n, is determined by the fact that

$$H_0[U(j_1 - j_2)|0\rangle] = H_0 \prod_{j_1 \le l \le j_2} U(l)|0\rangle$$

$$= \frac{\Lambda}{2} g_0^2 |j_1 - j_2| U(j_1 - j_2)|0\rangle . \tag{4.15}$$

The intermediate-state sum can now be performed leading to an effective Hamiltonian for the fluxless gauge-invariant sector which is very similar to (3.11) for the Thirring model

$$H_{\text{eff}} = -\frac{2\Lambda}{g_0^2} \sum_{j_1, j_2} \frac{[\delta'(j_1 - j_2)]^2}{|j_1 - j_2|} \{n_b(j_1)[1 - n_b(j_2)] + n_d(j_1)[1 - n_d(j_2)] + 2C_{j_1}^\dagger C_{j_2}\}, \tag{4.16}$$

where

$$C_j \equiv d_j b_j .$$

As for the Thirring model we can also define conserved fermion charges Q and Q_5. Since we limit out discussion to "gauge-invariant" *zero-flux states*, there must be either a fermion-antifermion pair or nothing at each site. Once again these are neutral states, $Q = 0$, which differ in their Q_5 eigenvalues depending on the number of sites, p, occupied by pairs, i.e., $Q_5 = -(2N+1) + 2p$. Within this subspace $n_b(j) = n_d(j)$ and (4.16) further simplifies to

$$H_{\text{eff}} = -\frac{4\Lambda}{g_0^2} \sum_{j_1 j_2} \frac{[\delta'(j_1 - j_2)]^2}{|j_1 - j_2|} [n_{j_1}(1 - n_{j_2}) + C_{j_1}^\dagger C_{j_2}]$$

$$= -\frac{4\Lambda}{g_0^2} \sum_{j_1, j_2} \frac{[\delta'(j_1 - j_2)]^2}{|j_1 - j_2|} (\tfrac{1}{4} + S_{j_1}^+ S_{j_2}^- - S_{j_1}^3 S_{j_2}^3), \tag{4.17}$$

where we have introduced the spin operator in the same way as for the Thirring model. Therefore, for gauge-invariant states in which no flux links are excited, the Schwinger model is equivalent to the linear Heisenberg antiferromagnetic chain and the analysis of Sec. III can be applied.[19]

The continuum Schwinger model has been solved exactly, so it is of interest to compare the spectrum of low-lying states obtained in the lattice formulation with the spectrum of the continuum theory. It is a feature of the Schwinger model that for the sector of gauge-invariant states only fermion-antifermion bound states exist which cannot be pulled apart. This result is of special interest in connection with efforts based on field theory to understand color or quark confinement. We also find this result in the strong-coupling limit of the lattice model: All gauge-invariant states formed from the vacuum must have zero total charge, $Q = 0$, and the lattice sites containing particles must be joined to those with antiparticles by flux links, each of which costs an energy $\frac{1}{2} g_0^2 |j_1 - j_2| \Lambda$. Therefore, no separated individual charges exist at finite energy for $\Lambda \to \infty$. In that (4.12) and (4.17) are equivalent to the strong-coupling Thirring model (3.13), aside from the extra factor $1/|j_1 - j_2|$ in H_{eff}, plus extra massive "photons" our conclusions are similar to those described in Sec. III.

The major difference between the low-lying spectrum in our lattice formulation and the usual description of states created by the "gauge-invariant observables" from the vacuum in the continuum theory is that we discuss as physical the *massless neutral states* of *zero flux links*. We do this because these bound particle-antiparticle pairs at a lattice site (as listed in Table II for example) can be created by a well-defined gauge-invariant operator. In contrast to this one does not consider the massless solutions with zero flux links[20] in the continuum spectrum because the bilinear operator is singular and not included among the "gauge-invariant observables" of the Schwinger model. In the continuum, gauge-invariant operators are defined by a point-separation method, which in the lattice language corresponds to having particles and antiparticles to different lattice sites with gauge field links between them. Therefore, a neutral pair created in this way contains an excited flux link and so is massive.

States made by joining a quark and an antiquark via a massive flux link are preserved by Kogut and Susskind[4] in their lattice formulation. In particular Banks, Kogut, and Susskind[21] show that the gauge-invariant state

$$|\gamma\rangle = \sum_j b_j^\dagger U(l_{j,j+1})\, d_{j+1}^\dagger |0\rangle$$

represents a massive "photon" with zero momentum; however, since they split particle and antiparticle unlike our formulation, they find no additional low-lying states of interest. To summarize, having formulated the fermion theory so that particles and antiparticles can exist at the same lattice point without doubling the number of degrees of freedom, we show that in addition to massive states there exist low-lying massless and gauge-invariant states formed of bound pairs without flux links in the strong-coupling limit. Moreover, these correspond to the massless states of the Thirring model.[20]

Throughout this discussion we have restricted our attention to gauge-invariant states. While it is consistent to work within only this sector since H in (4.12) is gauge invariant, we cannot prove in general that this sector contains the state of lowest energy. This is because there exist low-lying gauge-noninvariant states containing particles and/or antiparticles on different sites, but no flux links. However, these "unshielded particles" cannot move according to (4.12) without the kinetic-term K creating massive flux links. As a result of this restriction it is reasonable to conjecture that the gauge-invariant states will have their energy lowered by the action of K relative to the non-gauge-invariant states. This conjecture has been shown to be true for the case of three sites by explicit calculation, but no general proof has been found.

Depending on how one defines the fermion gradient, one is evidently led to two different formulations of what one means by the Schwinger and Thirring models for strong couplings. The question of which is the "right" one cannot be posed in terms of observation since these are no more than mathematical models. We have presented a formulation that preserves the local chiral invariance of the model and which in addition meets the following criteria:

(1) It is well defined and self-consistent. There is a well-defined procedure for defining the conserved charges associated with local symmetries such as chiral invariance, and, moreover, the free-field limit of any theory formulated in our way is guaranteed to be sensible.

(2) The fact that our free-field Hamiltonian leads to usual Feynman propagators says that for weak coupling our version of the theory looks, in perturbation theory, like ordinary renormalizable field theory.

(3) There is a well-defined procedure for deriving equations of motion and commutation relations for conserved currents and, for example, the commutators of time and space compounds of conserved currents contain the appropriate "Schwinger terms."

The real question which remains to be answered is whether a four-dimensional non-Abelian color-gauge theory interacting with quarks can reproduce the observations of quark and color confinement as well as of hadronic spectra. At the same time the formalism applied to Abelian QED must lead to Coulomb's law and the observed properties of "unconfining" Maxwell-Dirac theory.

V. THE NON-ABELIAN COLOR GAUGE THEORIES IN THREE SPACE AND ONE TIME DIMENSION

The results obtained in the preceding discussion can be directly generalized to non-Abelian gauge models in higher dimensions. As in the preceding discussion, because our formalism includes gauge-invariant states with fermions and antifermions at the same lattice site and no flux links, there are low-mass states in the strong-coupling limit. We follow the same prescription of Wilson and Kogut and Susskind for the gauge field in three space and one time dimension. For the fermion field the gradient operator as defined in (1.6) leads to the Hamiltonian

$$H = H_0(\text{electric part of gauge field}) + \Lambda \sum_{\vec{j}_1 \vec{j}_2} \chi^\dagger(\vec{j}_1)\{i\vec{\alpha}\}\chi(j_2)\cdot[\vec{\delta}'(\vec{j}_1 - \vec{j}_2)U(\vec{j}_1 - \vec{j}_2)] + O\left(\frac{1}{g^2}\right)$$

$$\equiv H_0 + K + O(1/g^2), \tag{5.1}$$

where

$$\delta_1'(\vec{j} - \vec{l}) \equiv \delta'(j_1 - l_1)\,\delta_{j_2,l_2}\delta_{j_3,l_3}, \text{ etc.}, \tag{5.2}$$

and $U(\vec{j}_1 - \vec{j}_2)$ is a product of terms of the form

$$U(\vec{l}) = e^{ig\lambda\cdot\vec{A}(\vec{l})\cdot\vec{l}}, \tag{5.3}$$

where $A(l)$ are the canonical link fields, and λ are c-number matrices belonging to a specific $(N, \bar{N})$ representation of the gauge group as determined by the choice of representation for the fermion fields. Equation (5.2) defines the obvious straight line path on the lattice for the flux links joining $\vec{j}_1$ to $\vec{j}_2$.

In the strong-coupling region of large g_0 the important properties of (5.1) are as follows:

(1) The low-mass states are those with zero

flux links and an arbitrary configuration of fermions. All others are pushed up in energy above $\sim g_0^2\Lambda$.

(2) When $U(l)$ hits an unoccupied link—i.e., one for which no gauge field has been excited—it excites the link and increases the energy of the state by $\sim g_0^2\Lambda$.

At this point we proceed in close parallel to the discussion of the Schwinger model with strong coupling. Focusing our attention on the sector of gauge-invariant states, we study the way in which the fermionic part of H mixes all the zero-energy eigenstates of H_0 (gauge) that have no flux links to split their degeneracy. In a theory with the SU(3) $\times[\mathrm{SU}(3)]_{\mathrm{color}}$ symmetry of the quark model all states with $(q\bar{q})$ or (qqq) at a lattice site in color-singlet states are included in the low-lying sector of gauge-invariant states. These are the states having the quantum numbers of ordinary hadrons.

If we choose the same spinor representation introduced in (2.25) and (2.26) we can rewrite the fermionic part of H as

$$K=\sum_{\vec{j}_1\vec{j}_2} i\vec{\delta}'(\vec{j}_1-\vec{j}_2)U(\vec{j}_1-\vec{j}_2)\cdot[B^\dagger(j_1)\vec{\sigma}B(j_2) - D(j_1)\vec{\sigma}D^\dagger(j_2)]. \tag{5.4}$$

As before K moves a quark in a straight line from $\vec{j}_2$ to $\vec{j}_1\neq\vec{j}_2$ (or an antiquark from $\vec{j}_1$ to $\vec{j}_2$) and at the same time excites a unit of gauge flux on each intervening link. Therefore, we must go to second or higher order in K in order to mix the degenerate color-singlet fluxless states.

Furthermore, since $\delta'(0)=0$, the action of K in second order allows scattering and interaction among these states, but it introduces no self-mass term involving only quarks all at the same lattice site. Hence our effective Hamiltonian for the low-lying gauge-invariant states of "bare colorless hadrons" corresponds to a theory of bare massless strongly interacting particles. Our starting point is a strong-coupling theory with the full chiral SU(3) $\times$ SU(3) symmetry if we choose a fundamental quark triplet. Instead of having to drive the pion mass down to zero to ensure partial conservation of axial-vector current (PCAC) we have a zero-mass starting point and must solve the problem of generating the hadronic masses either by a dynamical breakdown mechanism yet to be explored or by explicitly introducing chiral-breaking interactions into H *ab initio*.

The real work of solving for the hadronic spectra and interactions still remains to be done. What we have formulated here is a starting point in terms of a chirally invariant gauge theory (of color) which reduces in the strong-coupling region to a system of interacting "bare" particles with hadronic quantum numbers. In the gauge-invariant sector the quark and gluon degrees of freedom are frozen out since such states with excited flux links are pushed up to very high energy above $\approx g_0^2\Lambda>\Lambda$. This is a very different starting point from earlier formulations that destroy local chiral invariance by splitting fermion field components onto different lattice sites.

In conclusion we make some general observations:

(1) According to (5.4) K, acting on a fluxless gauge-invariant state, moves a quark or an antiquark, creating the associated flux link. To second order it can either move a quark (or antiquark) from an initial site to an intermediate one, and then move it back again to where it started, thereby canceling the flux link, since $U(j_1-j_2)U(j_2-j_1)=1$, or it can move both a quark and an antiquark from site j_1 to j_2 without creating flux links in the final state. This is illustrated in Figs. 3(a) and 3(b). This amounts to a kinetic energy term as we saw in the analysis of the Thirring and Schwinger models.

(2) If there are two hadrons present on different sites, the second-order application of K can lead to a quark-interchange interaction between them, as illustrated in Fig. 4. Starting from color-singlet states the hadrons will also end up as individual color singlets if no flux links are created in the final state. However, SU(3) quantum numbers can

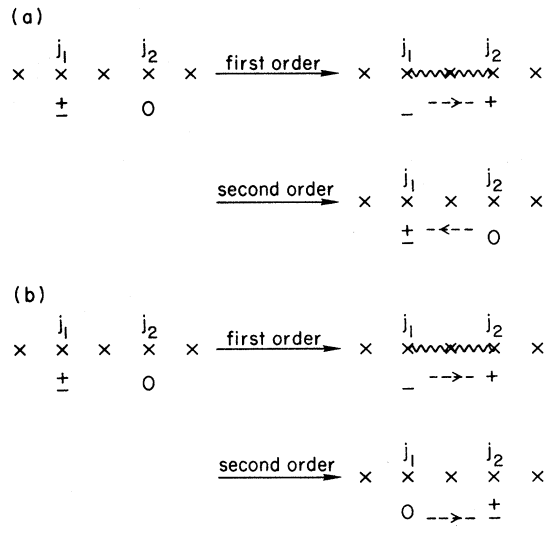

FIG. 3. Motion of a $q\bar{q}$ state on a lattice to second order $1/g^2$. (a) q (or $\bar{q}$) moves to a different lattice site exciting the intervening flux links and then returns; (b) q moves to a new site and is then followed by the $\bar{q}$.

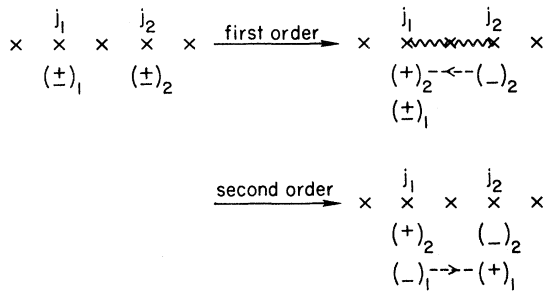

FIG. 4. Quark-interchange interaction between mesons.

be changed.

(3) A single three-quark baryon can move from one site to another on the lattice only as a result of third- and higher-order applications of K. This is because each order of K can move but one quark at a time. This means that baryonic masses are displaced relative to the zero-order degenerate eigenvalue of H_0 by factors of order $1/g_0^4$ in contrast to the $1/g_0^2$ shift from second-order application of K to the meson states. The significance of this for hadronic mass spectra and for the choice of coupling strengths g_0^2 remains to be studied. So does the entire question of how our bare massless mesonic states become dressed to form the true physical states containing $(q\bar{q})$ clouds with which they can interact via the quark-interchange mechanism.

(4) In the gauge-invariant sector, all exotic states of nonzero triality contain flux links and are therefore pushed very high up in energy above $g_0^2\Lambda$. Exotic states of the second kind—namely, states such as $[(q\bar{q})_{\text{octet}}(q\bar{q})_{\text{octet}}]_{\text{singlet}}$ with quarks and antiquarks finally coupled to color-singlet configurations, but not contained in the normal quark model—do occur. However, whereas the vacuum and ordinary $q\bar{q}$ mesonic states will have their degeneracy split and can be pushed down in energy with second-order application of K, these exotics of the second kind are shifted only in higher order since it takes fourth-order application of K to move them on the lattice. Hence, if they were stable, we would expect to find them lying higher in the energy spectrum. In fact, it is easy to see that such states can decay, in second order, to ordinary separated $q\bar{q}$ states.

(5) States of pure gluon, or flux link, configurations lie very high in energy above our low-mass gauge-invariant sector since they will have the energy of at least four flux links, $2g_0^2\Lambda$.

APPENDIX A

We have introduced the gradient operator on the lattice in the following way:

$$(\nabla f)_{(j)} = \sum_k ik\, e^{ikj/\Lambda} f(k). \tag{A1}$$

This definition of the gradient does not satisfy the Leibnitz product rule. As a matter of fact we can prove the following theorem.

Theorem. No definition of a gradient operator on the lattice satisfies the Leibnitz product rule.

Proof. Assume conversely that one can define a derivative operator which does satisfy

$$d(fg) = f\,dg + df\,g. \tag{A2}$$

In particular Eq. (A2) implies that (choose $f = g$ = const)

$$d(\text{const}) = 0. \tag{A3}$$

From Eq. (A2) it follows that for integer n

$$d(e^{inkj/\Lambda}) = n(e^{i(n-1)kj/\Lambda}) d(e^{ikj/\Lambda}), \tag{A4}$$

where k is one of the allowed momenta on the lattice $k = (2\pi/L)m$. Choosing $n = 2N+1 = L\Lambda$ in Eq. (A4) and using (A3)

$$d(e^{iLkj}) = d(1) = 0 = L(e^{i(L\Lambda-1)kj/\Lambda}) d(e^{ikj/\Lambda}). \tag{A5}$$

Hence

$$d(e^{ikj/\Lambda}) = 0, \tag{A6}$$

which cannot be true.

Next we would like to derive the correct form of the chain rule with our definition of the gradient. By definition

$$\begin{aligned}\nabla(fg) = \sum_k ik\, e^{ikj/\Lambda}(fg)(k) &= \sum_k ik\, e^{ikj/\Lambda} \sum_j \frac{e^{-ikj'/\Lambda}}{L\Lambda} f(j')g(j') \\ &= \sum_{k,j'} ik \frac{e^{ikj/\Lambda} e^{-ikj'/\Lambda}}{L\Lambda} \sum_{l_1 l_2} e^{il_1 j'/\Lambda} f_{l_1} e^{il_2 j'/\Lambda} g_{l_2} \\ &= \sum_{k;l_1 l_2} ik\, e^{ikj/\Lambda} \sum_{j'} \frac{e^{-i(k-l_1-l_2)j'/\Lambda}}{L\Lambda} f(l_1) g(l_2).\end{aligned} \tag{A7}$$

It is clear that

$$\frac{1}{L\Lambda} \sum_{j'} e^{-(k-l_1-l_2)j'/\Lambda} = \delta(k - l_1 - l_2) + \delta(k - l_1 - l_2 - 2\pi\Lambda) + \delta(k - l_1 - l_2 + 2\pi\Lambda).$$

Hence,

$$\nabla(fg)=\sum_{l_1l_2} i(l_1+l_2)e^{i(l_1+l_2)j/\Lambda}f_{l_1}g_{l_2}\theta\left(\frac{2\pi N}{2N+1}-|l_1+l_2|\right)$$

$$+\sum_{l_1l_2} i(l_1+l_2-2\pi\Lambda)e^{i(l_1+l_2)j/\Lambda}f_{l_1}g_{l_2}\theta\left(\frac{2\pi N}{2N+1}-\left|\frac{l_1+l_2}{\Lambda}-2\pi\right|\right)$$

$$+\sum_{l_1l_2} i(l_1+l_2+2\pi\Lambda)e^{i(l_1+l_2)j/\Lambda}f_{l_1}g_{l_2}\theta\left(\frac{2\pi N}{2N+1}-\left|\frac{l_1+l_2}{\Lambda}+2\pi\right|\right), \tag{A8}$$

which can be rewritten as

$$\nabla(fg)=\sum_{l_1l_2} i(l_1+l_2)e^{i(l_1+l_2)j/\Lambda}f(l_1)g(l_2)$$

$$+2\pi i\Lambda\Bigg[\sum_{l_1l_2} e^{i(l_1+l_2)j/\Lambda}f(l_1)g(l_2)\theta\left(\frac{2\pi N}{2N+1}-\left|\frac{l_1+l_2}{\Lambda}+2\pi\right|\right)$$

$$-\sum_{l_1l_2} e^{i(l_1+l_2)j/\Lambda}f(l_1)g(l_2)\theta\left(\frac{2\pi N}{2N+1}-\left|\frac{l_1+l_2}{\Lambda}-2\pi\right|\right)\Bigg]. \tag{A9}$$

Note that in the first term on the right-hand side of Eq. (A9) the sum over l_1 and l_2 is unrestricted due to the sum of contributions coming from all three terms on the right-hand side of Eq. (A8). It is easy to see that the first term in Eq. (A9) gives the usual Leibnitz product rule. The other two terms are the modification to the usual chain rule

$$\nabla(fg)=f\nabla g+(\nabla f)g+\sum_{j_1j_2} S(j;l_1,l_2)f(l_1)g(l_2), \tag{A10}$$

where

$$S(j;l_1,l_2)=2\pi i\Lambda\left[\sum_{l_1l_2} e^{i(l_1+l_2)j/\Lambda}\theta\left(\frac{2\pi N}{2N+1}-\left|\frac{l_1+l_2}{\Lambda}+2\pi\right|\right)-\sum_{l_1l_2} e^{i(l_1+l_2)j/\Lambda}\theta\left(\frac{2\pi N}{2N+1}-\left|\frac{l_1+l_2}{\Lambda}-2\pi\right|\right)\right]. \tag{A11}$$

It is important to note that the support of S comes from the regions (for large N) $l_1+l_2>\pi\Lambda$ and $l_1+l_2<-\pi\Lambda$. This immediately implies that S *does not have a* $k=0$ *component* in its Fourier decomposition

$$S(k=0)=\frac{1}{L}\sum_j S(j)=0. \tag{A12}$$

Hence we can define a function $I(j;l_1l_2)$ through the relation

$$\nabla I(j)=S(j). \tag{A13}$$

In momentum space Eq. (A13) implies that

$$ikI(k)=S(k).$$

Thus,

$$I(j;l_1l_2)=\sum_k \frac{e^{ikj/\Lambda}}{ik}S(k;l_1,l_2)$$

$$=\sum_k \frac{e^{ikj/\Lambda}}{ik}\frac{1}{L}\sum_{j'} e^{-ikj'/\Lambda}S(j;l_1,l_2)$$

$$=\sum_k \frac{1}{L}\sum_{j'} \frac{e^{ik(j-j')/\Lambda}}{ik}S(j;l_1,l_2),$$

and so

$$\nabla\left(fg(j)-\sum_{l_1 l_2} I(j;l_1,l_2)f(l_1)g(l_2)\right)=(\nabla f)g+f\nabla g$$

as desired.

APPENDIX B

The object of this appendix is to find the spectrum, $t(k)$, of the second-order Thirring Hamiltonian in the sector $Q_5=2S_3=-L+2$, and compute the $L\equiv 2N+1\to\infty$ limit of the expression for $E(k)$. Recapitulating, we have (3.13)

$$H_{\text{eff}}=\frac{\Lambda}{g}\sum_{j_1 j_2}[\delta'(j_1-j_2)]^2[S_3(j_1)S_3(j_2)-S_+(j_1)S_-(j_2)-\tfrac{1}{4}] \tag{B1}$$

and

$$H_0=-\Lambda Lg\underline{1}=-\Lambda g\sum[n_b(j)+n_a(j)-1]^2. \tag{B2}$$

If we let $|\psi_0\rangle$ be the state of all spins down, so that

$$S_-(j)|\psi_0\rangle=0 \tag{B3}$$

for all "j", then we can define $L=(2N+1)$ linearly independent states

$$|\psi(k)\rangle\equiv\frac{1}{\sqrt{L}}\sum_l e^{ikl/\Lambda}S_+(l)|\psi_0\rangle, \tag{B4}$$

which span the subspace $Q_5=-L+2$.

It remains to show that the $|\psi(k)\rangle$'s are eigenstates of H_{eff}. To do this, evaluate

$$H_{\text{eff}}|\psi(k)\rangle=\left[H_{\text{eff}},\frac{1}{\sqrt{L}}\sum_l e^{ikl/\Lambda}S_+(l)\right]|\psi_0\rangle$$

$$=-\frac{\Lambda}{g}\sum_{l,j_1,j_2}\frac{e^{ikl}}{\sqrt{L}}[\delta'(j_1-j_2)]^2$$

$$\times[\delta_{j_1 l}S_+(j_2)+(l)+\delta_{j_2 l}S_+(j_1)]|\psi_0\rangle. \tag{B5}$$

The proof is completed if we then observe the two facts

$$\sum_{j_2}[\delta'(l-j_2)]^2=\sum_{j_2,k_1,k_2}\frac{-k_1k_2}{L^2}e^{i(k_1+k_2)(l-j_2)/\Lambda}$$

$$=\frac{1}{L}\sum k^2 \tag{B6}$$

and

$$\sum\frac{e^{ikl/\Lambda}}{\sqrt{L}}[\delta'(j_1-l)]^2=\frac{e^{ikj_1/\Lambda}}{\sqrt{L}}X(k), \tag{B7}$$

where, for $L\to\infty$, $X(k)$ takes the form

$$X(k)=\frac{1}{2}\left(\frac{|k|}{\Lambda}-\pi\right)^2-\frac{\pi^2}{6}. \tag{B8}$$

Before proving (B7) and (B8) we see that direct substitution of (B6) and (B7) into (B5) yields

$$H_{\text{eff}}|\psi(k)\rangle=-\frac{\Lambda}{g}\left[\frac{1}{L}\sum k^2+X(k)\right]|\psi(k)\rangle$$

$$\xrightarrow[L\to\infty]{}+\frac{\Lambda}{g}\left(-\frac{2\pi^2}{3}+\frac{\pi|k|}{\Lambda}-\frac{|k|^2}{2\Lambda^2}\right),$$

which agrees with (3.15) if you add back the common energy of $-gL\Lambda$ coming from H_0.

The proof of (B7) proceeds as follows:

$$\sum_l\frac{e^{ikl/\Lambda}}{\sqrt{L}}\sum_l\delta'(j_1-l)^2=\sum_{l,k_1,k_2}-\frac{k_1k_2}{L^2}e^{i(k_1+k_2)(j_1-l)/\Lambda}e^{+ikl/\Lambda}$$

$$=-\sum_{k_1k_2}\frac{k_1k_2}{(L)^{3/2}}\delta_p(k-k_1-k_2)e^{i(k_1+k_2)j_1/\Lambda}, \tag{B9}$$

where $\delta_p(k_1+k_2-k)$ is the periodic δ function defined to be zero unless $k=k_1+k_2+$ multiple of $2\pi\Lambda$. Since $k_1+k_2=k+(2\pi\Lambda)r$, Eq. (B9) becomes

$$\sum_l\frac{e^{ikl/\Lambda}}{\sqrt{L}}[\delta'(j_1-l)]^2=\frac{e^{ikj_1/\Lambda}}{\sqrt{L}}X(k)$$

where

$$X(k)\equiv-\frac{1}{L}\sum_{k_1,k_2}k_1k_2\delta_p(k_1+k_2-k).$$

To evaluate the explicit form of $X(k)$ we observe that due to the periodic δ function there are three regions which contribute to the double sum, namely:

$k_2 = k - k_1, \quad \theta(\pi - |k - k_1|),$

$k_2 = k - k_1 - 2\pi, \quad \theta(k - k_1 - \pi),$

and

$k_2 = k - k_1 + 2\pi, \quad \theta(-\pi - (k - k_1)).$

Hence,

$$\frac{1}{L}\sum_{k_1 k_2} - k_1 k_2 \delta_p(k - k_1 - k_2) = \frac{1}{L}\sum_q q^2 + \frac{2\pi}{L}\left\{\sum_{k_1} k[\theta(k - k_1 - \pi)\theta(k) - \theta(-\pi - (k - k_1))\theta(-k)]\right\},$$

and so as $L \to \infty$ this becomes

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} q^2 dq + \frac{1}{2\pi}\left[\int_{-\pi}^{k-\pi} q\,dq\,\theta(k) - \int_{\pi-k}^{\pi} q\,dq\,\theta(-k)\right] = \frac{\pi^2}{3} + \tfrac{1}{2}(k^2 - 2k\pi)$$

$$= \tfrac{1}{2}(|k| - \pi)^2 - \frac{\pi^2}{6}.$$

APPENDIX C

In this appendix we formulate the problem of diagonalizing the effective second-order Thirring Hamiltonian in the sector $Q_5 = 2S_3 = -L + 4$. As before, since $H_0 = -L\Lambda g\underline{1}$ for all of the Q_5 sectors we will measure all energies with respect to $E_0 = -Lg\Lambda$ and so we want to solve the equation

$$H_{\text{eff}}|\psi\rangle = E|\psi\rangle \tag{C1}$$

for

$$H_{\text{eff}}\frac{\Lambda}{g}\sum_{j_1, j_2}[\delta'(j_1 - j_2)]^2[S_3(j_1)S_3(j_2) - S_+(j_1)S_-(j_2) - \tfrac{1}{4}]. \tag{C2}$$

It is clear that the most general state with $Q_5 = -L + 4$ can be represented as

$$|\psi\rangle = \sum_{n,m;n\neq m} \mathcal{C}_{n,m} S_+(n)S_+(m)|\psi_0\rangle,$$

where $|\psi_0\rangle$ is the unique state of $Q_5 = 2S_3 = -L$. As in Appendix B we evaluate

$$H_{\text{eff}}|\psi\rangle = E|\psi\rangle = \sum_{n,m;n\neq m} \mathcal{C}_{n,m}[H_{\text{eff}}, S_+(n)S_+(m)]|\psi_0\rangle. \tag{C3}$$

Using the usual commutation relations for spin matrices we obtain

$$[S_+(j_1)S_-(j_2), S_+(n)S_+(m)]|\psi_0\rangle = [-2S_+(j_1)S_+(m)\delta_{nm}\delta_{j_2 n} + S_+(j_1)S_+(m)\delta_{j_2 n} + S_+(j_1)S_+(n)\delta_{j_2 m}]|\psi_0\rangle \tag{C4}$$

and

$$[S_3(j_1)S_3(j_2), S_+(n)S_+(m)]|\psi_0\rangle = [S_+(n)S_+(m)(\delta_{nj_1}\delta_{nj_2} + \delta_{mj_1}\delta_{mj_2} + \delta_{nj_1}\delta_{nj_2} + \delta_{mj_1}\delta_{mj_2}) - \tfrac{1}{2}(\delta_{nj_1} + \delta_{nj_2} + \delta_{mj_1} + \delta_{mj_2})]|\psi_0\rangle. \tag{C5}$$

Substituting (C4) and (C5) into (C3) and equating coefficients of $S_+(l)S_+(p)$ we obtain

$$(E - 2c)\mathcal{C}_{lp} = \frac{\Lambda}{g}\delta'(l-p)^2\mathcal{C}_{lp} - \frac{\Lambda}{g}\sum_n [\delta'(l-n)]^2\mathcal{C}_{n,p} - \frac{\Lambda}{g}\sum_m [\delta'(p-m)]^2\mathcal{C}_{l,m}, \tag{C6}$$

where

$$c \equiv \frac{\Lambda}{g}\sum_{l,p}[\delta'(l-p)]^2 = \frac{\Lambda}{g}\frac{1}{L}\sum k^2 \xrightarrow[L\to\infty]{} \frac{\Lambda}{g}\frac{\pi^2}{3}. \tag{C7}$$

If we think of $\mathcal{C}_{l,p}$ as a two-variable quasiparticle wave function (C6) describes a scattering problem in a potential $(\Lambda/g)[\delta'(l-p)]^2$. Going to momentum space

$$\mathcal{C}_{l,p} \equiv \sum_{k_1 k_2} a(k_1, k_2)|k_1 k_2\rangle_{l,p} \equiv \sum \frac{a(k_1, k_2)}{L} e^{ik_1 l} e^{ik_2 p}, \tag{C8}$$

and Eq. (C6) becomes

$$\left\{E-2c+\frac{\Lambda}{g}\left[\left(X\frac{k_t+k_\pi}{2}\right)+\left(X\frac{k_t-k_\pi}{2}\right)\right]\right\}a(k_t,k_\pi)=\sum_{k_t'k_\pi'}V(k_tk_\pi;k_t'k_\pi')a(k_t',k_\pi'), \tag{C9}$$

where

$$\begin{aligned} k_t&=k_1+k_2, \quad k_\pi=k_1-k_2,\\ k_t'&=k_1'+k_2', \quad k_\pi'=k_1'-k_2', \end{aligned} \tag{C10}$$

and

$$(Va)(k_t,k_\pi)=\frac{\Lambda}{g}\sum_{k_t',k_\pi'}\delta_p(k_t-k_t')\left\{\sum_{q_1q_2}\frac{(-q_1q_2)}{L^2}\delta_p(q_1+q_2+\tfrac{1}{2}(k_\pi-k_\pi'))\right\}a(k_t',k_\pi'). \tag{C11}$$

Hence the center-of-mass motion separates, up to an umklapp, and one is left with a nontrivial quasiparticles scattering problem.

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[18]In Ref. 13 it is also proved as a corollary to Theorem 1 that there is a unique ground state within each S_3 sector for the antiferromagnetic chain described by (3.13); i.e., for our case without nearest-neighbor interactions. This corollary still does not tell us that the ground state of the system is unique (or a γ_5 doublet) since the lowest states within each S_3 sector might be degenerate with one another. Such a degeneracy is not found for the bounds according to (3.16) and (3.17).

[19]Owing to the extra factor $1/|j_1-j_2|$ in (4.17) relative to (3.13), we can carry the analysis of the excitation spectrum further. If we neglect boundary effects at the end points of the lattice so that H_{eff} in (4.17) describes a translationally invariant system we can use the method of Ref. 13 to prove that there is no mass gap to the spectrum of excitations for this system.

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