PHYSICAL REVIEW D VOLUME 14, NUMBER 6 15 SEPTEMBER 1976

Collective phenomena in $\lambda\phi^4$ field theory treated in the random-phase approximation*

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(Received 12 February 1976)

We investigate $\lambda\phi^4$ theory in the random-phase approximation as a prototypic model for understanding the formation of bound states (here 2-meson bound states) and the nature of the effective interaction between such states. We give the random-phase approximation a functional-derivative interpretation which allows us to determine the effective expansion parameter of the random-phase approximation to be $(\lambda_R{}^2/N)D_R(q^2)$, where $D_R(q^2)$ is the propagator of the collective mode. The field theory in this approximation has both mass and coupling-constant renormalization, and we derive expressions for unrenormalized and renormalized 2-, 4-, and 6-point functions for the original scalar field, and determine the 4-point function for bound-state–bound-state scattering as well as the effective coupling between bound states. We show the relation between the random-phase approximation and the O(N) model for large N and prove that without single-field symmetry breaking there is a range of renormalized coupling constant where there is no ghost.

INTRODUCTION

In trying to understand the structure of hadrons as seen in deep-inelastic electroproduction, and high-transverse-momenta phenomena in pp collisions, many of the features present are qualitatively described in terms of a parton (say quark) substructure, where quarks are acting independently but are never seen as out-states. The particles seen in the laboratory are presumably bound states of some underlying quark-gluon theory, and in a different regime, such as low-energy hadron scattering, one can describe the physics in terms of phenomenological fields for the physical particles such as pions and nucleons (say, as in the σ model). What is missing is a bridge between "parton" phenomena and particle phenomena. That is, one would like to start with a quark-gluon Lagrangian, find the bound states, see how they interact, and determine a phenomenological Lagrangian to describe that interaction. In this paper we emphasize that there exists a new form (for quantum field theory) of perturbation theory, where, to lowest order, bound states already exist. This approximation allows one to look at bound-state interactions in a rather straightforward fashion. The perturbation theory is defined by assuming that a certain variational derivative is of order ϵ. That is, we will make an expansion in the third fluctuation of the underlying quantum field.

The problem we have in the back of our mind is studying the bound states of a quark-gluon theory with non-Abelian couplings using this new perturbation theory. However, for the sake of clearly illustrating our technique, we confine our considerations in this paper to a self-interacting scalar field. This model, despite its simplicity, will demonstrate the feasibility of the approximation techniques. We will study more realistic application elsewhere. We begin with the Lagrangian for a scalar field with $\lambda\phi^4$ interactions in the presence of external sources $S(x)$ for $\phi^2(x)$ and $J(x)$ for $\phi(x)$. We shall show that the introduction of two sources makes it possible to obtain either of two different approximations. This illustrates one of the important points of this paper, which is how naturally the choice of external sources leads to an approximation scheme. Here we will primarily be concerned with one of these approximations, the random-phase approximation (RPA), which is obtained by taking the "normalized" vacuum expectation value of the field equation and replacing the term of the form

$$\lambda_0 \frac{\langle 0\sigma_1 | \phi^3 | 0\sigma_2\rangle}{\langle 0\sigma_1 | 0\sigma_2\rangle}$$

by

$$\lambda_0 \frac{\langle 0\sigma_1 | \phi^2 | 0\sigma_2\rangle}{\langle 0\sigma_1 | 0\sigma_2\rangle} \frac{\langle 0\sigma_1 | \phi | 0\sigma_2\rangle}{\langle 0\sigma_1 | 0\sigma_2\rangle}.$$

Having made this approximation, all lowest-order Green's functions may be derived by functional differentiations of the resulting equation. It is reasonable to expect that a scheme based on a lowest-order approximation which is simple with respect to the source S of ϕ^2 (as this approximation will be shown to be) would tend to pick out any highly correlated excitations of the field ϕ^2. Corresponding to this expectation we show that this RPA approximation leads to bound states of ϕ^2 and that the scalar-scalar scattering to lowest

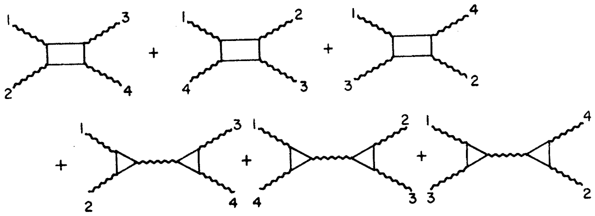

FIG. 1. Pictorial representation of bound-state–bound-state scattering.

order is saturated by the exchange of the bound-state propagator in s, t, u. We also find that bound-state–bound-state scattering to lowest order is not so different from usual naive models in that one has contributions from the box diagram and from bound-state exchange with a specific form for the *BBB* vertex, and thus we get the diagrams of Fig. 1. The straight line is the *full* scalar propagator $G(q^2)$ and the wiggly line is the full propagator $D_B(q^2)$ for $\chi=\phi^2$. We shall find that $\lambda_0^2 D_B(q^2)$ is effectively the expansion parameter of this theory.

We show that our RPA is essentially the O(N) (Refs 1 and 2) expansion although we are not constrained to take N large. We further show as a consequence of this relationship that without symmetry breaking there is a region of the renormalized coupling constant for both theories where the results are not plagued by the ghosts characteristic of the previous calculations.

I. APPROXIMATION SCHEME FOR $\lambda_0\phi^4$

The most familiar approximation scheme for $\lambda_0\phi^4$ theory is the normal perturbation expansion in powers of λ_0. This is by no means the only expansion and, as we shall show, it is particularly simple to generate other expansions by using functional techniques. We will deal with an N-component field ϕ^α, but most of our arguments hold when we have only one component. In order to keep the results simple in appearance we usually will use matrix notation and suppress the indices associated with the different components. Thus the Lagrangian density is

$$\mathcal{L}=\tfrac{1}{2}\partial_\mu\phi\partial_\mu\phi-\tfrac{1}{2}m_0^2\phi^2+\tfrac{1}{4}\lambda_0\chi^2 - \tfrac{1}{2}\lambda_0\chi\phi^2+J\phi+S\chi\,. \qquad (1)$$

Here

$$\phi=\begin{bmatrix}\phi_1\\ \phi_2\\ \vdots\\ \phi_n\end{bmatrix} \quad\text{and}\quad J=\begin{bmatrix}J_1\\ J_2\\ \vdots\\ J_n\end{bmatrix}.$$

The equations of motion are

$$(\Box^2+m_0^2)\phi^\alpha+\lambda_0\chi\phi^\alpha=J^\alpha \qquad (2)$$

and

$$\chi=\phi^\alpha\phi_\alpha-\frac{2S}{\lambda_0}. \qquad (3)$$

The field χ will turn out to excite in a simple way the bound state that appears in our scheme. When Eqs. (2) and (3) are combined we obtain the usual result,

$$(\Box^2+m_0^2)\phi^\alpha+\lambda_0(\phi_b\phi^b)\phi^\alpha-2S\phi^\alpha=J^\alpha\,. \qquad (4)$$

Here, this one time, we have displayed all the internal indices.

The external sources $S(x)$ and $J(x)$ are introduced for convenience in deriving Green's-function equations for the fields $\chi(x)$ and $\phi(x)$. Our choice of source couplings here will lead to Green's-function equations which clearly suggest approximation methods particularly tractable with this type of source coupling and interaction.

The generating functional for the vacuum amplitude is

$$Z(J,S)=e^{iW(J,S)} = {}_+\langle 0|0\rangle_-\,. \qquad (5)$$

Here

$$W(J,S)=\frac{1}{i}\ln Z(J,S) \qquad (6)$$

generates the connected n-point functions for ϕ and χ and

$$\Gamma(\Phi_c)\equiv W(J,S)-\int d^4x\, J(x)\frac{\delta W}{\delta J(x)} \qquad (7)$$

generates the one-particle irreducible graphs.

In what follows we will often use for the n-coordinate Green's functions the notation

$$W(123\cdots)=\frac{\delta^n W}{\delta J(1)\delta J(2)\delta J(3)\cdots},$$

$$W(\hat{1}\hat{2}\cdots)=\frac{\delta^n W}{\delta S(1)\delta S(2)\cdots}, \qquad (8)$$

and

$$W(\hat{1}2\cdots)=\frac{\delta^n W}{\delta S(1)\delta J(2)\cdots}.$$

We will also use the equivalent notation

$$W(1)=\frac{\delta W}{\delta J(1)}=\Phi_c(1)=\frac{\langle\phi(1)\rangle}{\langle|\rangle},$$

$$W(\hat{1})=\frac{\delta W}{\delta S(1)}=\chi_c(1)=\frac{\langle\chi(1)\rangle}{\langle|\rangle}, \qquad (9)$$

$$W(12)=\frac{\delta\Phi_c(1)}{\delta J(2)}=G(12)=i\left[\frac{\langle T(\phi(1)\phi(2))\rangle}{\langle\,\rangle}-\Phi_c(1)\Phi_c(2)\right], \tag{10}$$

and

$$W(\hat{1}\hat{2})=D_B(12)=\frac{\delta^2W}{\delta S(1)\delta S(2)}=i\left[\frac{\langle T(\chi(1)\chi(2))\rangle}{\langle\,\rangle}-\chi_c(1)\chi_c(2)\right]. \tag{11}$$

In these equations $\Phi_c(1)$ means $\Phi_c(x_1)$, $G(12)$ means $G(x_1x_2)$, etc., and $\langle A\rangle$ means ${}_+\langle 0|A|0\rangle_-$. The reason that we use two equivalent notations is that the Green's functions expressed as G and Φ_c are generally more familiar, but the W notation emphasizes symmetry and often makes intermediate steps more transparent. Consequently, throughout the paper we write results in either or both forms or in a mixed form depending on subsequent manipulations.

Taking the normalized vacuum expectation of Eq. (4) yields the equation obeyed by Φ_c,

$$\left(\Box^2+m_0^{\,2}+\frac{\lambda_0}{i}\frac{\delta}{\delta S}+\lambda_0\frac{\delta W}{\delta S}\right)\Phi_c=J. \tag{12a}$$

Since

$$\frac{\delta W}{\delta S(x)}=\frac{\langle\chi(x)\rangle}{\langle\,\rangle}=\frac{\langle\phi^2(x)\rangle}{\langle\,\rangle}=\frac{1}{i}\operatorname{tr}G(xx)+[\Phi_c(x)]^2-\frac{2S(x)}{\lambda_0}. \tag{13a}$$

We can also write (12a) as

$$\left[\Box^2+m_0^{\,2}+\frac{\lambda_0}{i}\operatorname{tr}G(xx)+\lambda_0\Phi_c^{\,2}(x)-2S(x)+\frac{\lambda_0}{i}\frac{\delta}{\delta S(x)}\right]\Phi_c(x)=J(x) \tag{12b}$$

From Eq. (13a) it follows that

$$\frac{\delta W}{\delta S}=\frac{1}{i}\frac{\delta^2W}{(\delta J)^2}+\Phi_c^{\,2}-\frac{2S}{\lambda_0}. \tag{13b}$$

Differentiation of (13b) yields

$$\frac{1}{i}\frac{\delta^2W}{\delta S\delta J}=\left(\frac{1}{i}\right)^2\frac{\delta^3W}{\delta J(\delta J)^2}+2\Phi_c\cdot\frac{1}{i}\frac{\delta^2W}{\delta J\delta J} \tag{14a}$$

or

$$\frac{1}{i}\frac{\delta}{\delta S}\Phi_c=\left(\frac{1}{i}\right)^2\frac{\delta}{\delta J}\operatorname{tr}G+2\Phi_c\cdot\frac{1}{i}G. \tag{14b}$$

Inserting Eq. (14a) into (12a) we find

$$\left[\Box^2+m_0^{\,2}-2S+\frac{\lambda_0}{i}\frac{\delta^2W}{(\delta J)^2}+\frac{2\lambda_0}{i}\frac{\delta}{\delta J}\frac{\delta W}{\delta J}\cdot+\lambda_0\Phi_c^{\,2}\right]\Phi_c+\left(\frac{1}{i}\right)^2\frac{\lambda_0\delta^3W}{\delta J(\delta J)^2}=J \tag{15a}$$

or in terms of G and Φ_c only

$$\left[\Box^2+m_0^{\,2}+\frac{\lambda_0}{i}\operatorname{tr}G(xx)+\frac{2\lambda_0}{i}G(xx)\cdot+\lambda_0\Phi_c^{\,2}(x)-2S(x)+\lambda_0\left(\frac{-i\delta}{\delta J(x)}\right)^2\right]\Phi_c(x)=J(x). \tag{15b}$$

Equations (15) are equivalent to Eqs. (12) but are written entirely in terms of derivatives with respect to the source J. Consequently, when approximations are made which involve dropping derivatives these sets of equations lead to different results. Equations (12) are appropriate for studying the random-phase approximation (RPA) while Eqs. (15) lead to the Hartree approximation.

By functional differentiation, with respect to J, we can find the two field Green's functions. Equations (12) lead to

$$[\Box^2+m_0^{\,2}+\lambda_0W(\hat{1})]W(12)+\frac{\lambda_0}{i}W(\hat{1}12)+\lambda_0W(1)W(\hat{1}2)=\delta(1-2) \tag{16a}$$

or, equivalently,

$$\left[\Box^2+m_0^{\,2}+\frac{\lambda_0}{i}\operatorname{tr}G(11)+\lambda_0\Phi_c^{\,2}(1)+2\lambda_0\Phi_c(1)\Phi_c(1)\cdot-2S(1)+\frac{\lambda_0}{i}\frac{\delta}{\delta S(1)}\right]G(12)+\frac{\lambda_0}{i}\left[\Phi_c(1)\cdot\frac{\delta}{\delta J(1)}\right]G(12)=\delta(1-2), \tag{16b}$$

while Eqs. (15) result in

$$\left[\square^2+m_0{}^2-2S(1)+\frac{\lambda_0}{i}\mathrm{tr}W(11)+\frac{2\lambda_0}{i}W(11)\cdot+2\lambda_0\Phi_c(1)\Phi_c(1)\cdot+\lambda_0\Phi_c{}^2(1)\right]W(12)+\frac{1}{i}\lambda_0\mathrm{tr}W(1112)+\frac{\lambda_0}{(i)^2}[\mathrm{tr}W(112)]\Phi_c(1)$$
$$=\delta(1-2)-\frac{2\lambda_0}{i}W(211)\cdot\Phi_c(1) \tag{17a}$$

or perhaps, more transparently,

$$\left[\square^2+m_0{}^2+\frac{\lambda_0}{i}\mathrm{tr}G(11)+\frac{2\lambda_0}{i}G(11)\cdot+2\lambda_0\Phi_c(1)\Phi_c(1)\cdot+\lambda_0\Phi_c{}^2(1)-2S(1)+\lambda_0\left(\frac{1}{i}\frac{\delta}{\delta J}\right)^2\right]G(12)$$
$$+\frac{2\lambda_0}{i}\left[\frac{\delta}{\delta J(2)}G(11)\right]\cdot\Phi_c(1)+\frac{\lambda_0}{i}\left[\frac{\delta}{\delta J(2)}\mathrm{tr}G(11)\right]\Phi_c(1)=\delta(1-2)\,. \tag{17b}$$

The sets of Eqs. (12) and (16) or (15) and (17) are, of course, as they stand utterly intractable. In order to proceed we must make some sort of truncation approximation so that at least one of the n-field Green's functions is expressible in terms of itself and various other Green's functions of less than n fields.

The above equations are written in a form which is particularly tractable for the truncations we wish to study in this paper, but it is still not difficult to extract from the ones using J as the primary source the usual approximation schemes used in studying the $\lambda\phi^4$ problem. For example, we can obtain the results of the perturbation expansion in powers of λ_0 from Eq. (15b) in a direct manner by substituting the expansions $\Phi_c(x) = \Phi_c^0(x)+\lambda_0\Phi_c^1(x)+\cdots$ and $G(12) = G^0(12)+\lambda_0 G^1(12)+\cdots$ into this expression and collecting terms of the same power in λ_0 and requiring that they vanish order by order. We can also easily generate the loop expansion for the Goldstone or σ model from Eq. (15b). The most direct procedure to do this is to rewrite (15b) in the form[3]

$$\left[\square^2+m_0{}^2-2S(x)+\frac{\lambda_0}{i}\frac{\delta}{\delta J(x)}\cdot\Phi_c(x)+\frac{2\lambda_0}{i}\frac{\delta}{\delta J(x)}\Phi_c(x)\right.$$
$$\left.+\lambda_0\Phi_c{}^2(x)+\lambda_0\left(\frac{-i\delta}{\delta J(x)}\right)^2\right]\Phi_c(x)=J\,. \tag{15c}$$

The tree approximation $\Phi^0(x)$ is obtained by neglecting terms appearing in (15c) that have an explicit variational derivative in them. It follows that

$$[\square^2+m_0{}^2+\lambda_0(\Phi_c^0(x))^2]\Phi_c^0(x)=J(x)\,. \tag{18}$$

With $J(x)=0$ we find $[m_0{}^2+\lambda_0\eta^2]\eta=0$ where $\eta=\langle\phi(0)\rangle|_{J=0}$. This is the usual lowest-order symmetry-breaking condition. The other Green's functions to this order are found by functionally differentiating Eqs. (18). The one-loop approximation is obtained by substituting $\phi_c=\Phi_c^0+\Phi_c^1$ into Eq. (15c) and discarding terms which represent more than one loop [i.e., terms containing $(\Phi_c^1)^2$, $(\delta/\delta J)\Phi_c^1$, $(\delta^2/\delta J^2)\Phi_c^1$, and $(\delta^2/\delta J^2)\Phi_c^0$]. l-loop approximations Φ_c^l to Φ_c are found by making the observation that $\Phi_c^{l_1}-\Phi_c^{l_p}(\delta/\delta J)^m\phi^l$ is an n-loop object if $l_1+l_p+m+l=n$. There are no common approximation schemes which come as naturally from the Eqs. (12) and (16) which involve derivatives with respect to the source S. We shall show that natural approximations on these equations are related to O(N) expansions. The approximations used on equations expanded in this way are also closely related to similar approximations used on the Nambu model.[4]

The technique used with the loop expansion suggests that dropping variational derivative terms is a straightforward and potentially profitable approximation technique. This leads us to write for the zeroth approximation to Eq. (12a)

$$\left(\square^2+m_0{}^2+\lambda_0\frac{\delta W^0}{\delta S}\right)W^0=J \tag{19a}$$

or equivalently

$$\left[\square^2+m_0{}^2+\frac{\lambda_0}{i}\mathrm{tr}G^0(xx)+\lambda_0(\Phi_c^0(x))^2-2S(x)\right]\Phi_c^0(x)=J(x)\,. \tag{19b}$$

With the sources off, this equation shows that

$$\left[m_0{}^2+\frac{\lambda_0}{i}\mathrm{tr}G^0(xx)+\lambda_0\eta^2\right]\eta=0\,,$$

where

$$\eta=\left.\frac{\langle\phi\rangle^0}{\langle|\rangle}\right|_{J=S=0}. \tag{20a}$$

If there is no linear spontaneous symmetry breaking, $\eta = 0$ and this equation is automatically satisfied. If $\eta \neq 0$, then the theory is constrained by the requirement

$$m_0^2 + \frac{\lambda_0}{i} \operatorname{tr} G(xx) + \lambda_0 \eta^2 = 0 . \tag{20b}$$

Thus we have assumed in this lowest approximation that $(\lambda_0/i)(\delta/\delta S)\Phi_c \equiv \epsilon_1(x)$ is ignorable so that we may factorize the vacuum expectation value of the field equation by the replacement

$$\lambda_0 \langle \phi^2 \phi \rangle = \lambda_0 \frac{\langle \Phi^2 \rangle}{\langle | \rangle} \frac{\langle \phi \rangle}{\langle | \rangle} + \epsilon_1(x) . \tag{21}$$

This approximation treats $\chi = \phi^2 - 2S/\lambda_0$ as a field in its own right on the same footing as ϕ for truncation expansions. The approximation tells us the theory is dominated by the disconnected in ϕ and χ piece of the interaction. We would thus expect this procedure to be appropriate where χ excites strongly bound states. Note that we have now defined a definite calculational procedure and that even if $\epsilon_1(x)$ is not a small quantity it might be that this approximation scheme defines some sort of asymptotic expansion for the theory. We will discuss this point at greater length later.

The lowest Hartree approximation is obtained from Eq. (15b) by neglecting the term of the form $[-i\delta/\delta J(x)]^2 \Phi_c(x)$. This leads to the equation

$$\left[\Box^2 + m_0^2 + \frac{3\lambda_0}{i} \operatorname{tr} G^0(xx) - 2S(x) + \lambda_0 (\Phi_c^0(x))^2 \right] \Phi_c^0(x) = J(x) \tag{22}$$

The dual notation is of no great aid for this approximation at this point so we have dropped it. With the sources off, Eq. (22) requires that $[m_0^2 + (3\lambda_0/i)\operatorname{tr} G^0(xx) + \lambda_0 \eta^2]\eta = 0$, which if $\eta \neq 0$ constrains the theory by the condition

$$\left[m_0^2 + \frac{3\lambda_0}{i} \operatorname{tr} G^0(xx) + \lambda_0 \eta^2 \right] = 0 . \tag{23}$$

Note that in terms of the fields the term we have ignored is

$$\epsilon_2(x) = \lambda_0 \left[\frac{-i\delta}{\delta J(x)} \right]^2 \Phi_c(x)$$

$$= \lambda_0 \frac{\langle (\phi - \Phi_c)^2 (\phi - \Phi_c) \rangle}{\langle | \rangle} . \tag{24}$$

This emphasizes that as in the case of the RPA the lowest Hartree approximation replaces the vacuum expectation of ϕ^3 by combinations involving the vacuum expectation of ϕ^2 and ϕ, thus, as we shall see, greatly simplifying the structure of the Green's-function equations. The Hartree approximation says that to lowest order the cube of the fluctuation of the field away from its classical value is negligible, while in terms of fluctuations away from Φ_c RPA is the statement, using Eqs. (21) and (14a), that

$$\lambda_0 \left\langle \left(\phi^2 - \frac{\langle \phi^2 \rangle}{\langle | \rangle} \right) (\phi - \Phi_c) \right\rangle = \epsilon_1 . \tag{25}$$

II. MESON PROPAGATOR IN RPA

Either by differentiation of Eqs. (19) or by recognizing the implications of our approximation prescription on Eqs. (16) we have to lowest order in RPA that

$$[\Box^2 + m_0^2 + \lambda_0 W^0(\hat{1})] W^0(12) + \lambda_0 W^0(1) W^0(\hat{1}2) = \delta(1-2) \tag{26a}$$

or

$$\left[\Box^2 + m_0^2 + \frac{\lambda_0}{i} \operatorname{tr} G^0(11) + \lambda_0 (\Phi_c^0)^2 + 2\lambda_0 \Phi_c^0(1) \Phi_c^0(1) \cdot - 2S(1) + \frac{\lambda_0}{i} \Phi_c^0(1) \cdot \frac{\delta}{\delta J(1)} \right] G^0(12) = \delta(1-2) . \tag{26b}$$

Until we start examining higher-order approximations, we now drop the superscript 0 which indicates we are dealing with the lowest approximation. If the sources are turned off, Eq. (26b) becomes

$$\left[\Box^2 + m_0^2 + \frac{\lambda_0}{i} \operatorname{tr} G(11) + 2\lambda_0 \eta \eta \cdot + \lambda_0 \eta^2 \right] G(12) + \frac{\lambda_0}{i} \eta \cdot \left[\frac{\delta}{\delta J(1)} G(12) \right] \Bigg|_{J=S=0} = \delta(1-2) . \tag{27}$$

In the case of primary interest throughout this paper we choose to have no spontaneous symmetry breaking and take $\eta = 0$ so that Eq. (27) becomes

$$\left[\Box^2 + m_0^2 + \frac{\lambda_0}{i} \operatorname{tr} G(11) \right] G(12) = \delta(1-2) . \tag{28}$$

Note that

$$\frac{\lambda_0}{i} \operatorname{tr} G(11) = \frac{\lambda_0}{i} \operatorname{tr} G(00) \equiv \delta m^2 \tag{29}$$

using translational invariance. Thus we take

$$m_0^2 + \delta m^2 = m_R^2 , \tag{30}$$

where clearly m_R^2 should be identified as the renormalized meson mass to this order of approximation. In terms of Fourier transforms we have with n the dimensionality of spacetime

$$G(12) = \frac{1}{(2\pi)^n} \int d^n k \, G(k) e^{-ik \cdot (x_1 - x_2)} ,$$

so that from (28) we find

$$G_{\alpha\beta}(k)=\frac{\delta_{\alpha\beta}}{m_R{}^2-k^2-i\epsilon}, \tag{31}$$

where

$$m_R{}^2=m_0{}^2+\frac{\lambda_0}{i}\int\frac{d^nk}{m_R{}^2-k^2-i\epsilon}.$$

Thus to lowest order the only effect of our truncation procedure on the scalar-meson propagator is to renormalize the mass.

It is also useful to note at this point that if we assume $\eta=0$, then we can make a stronger statement than Eq. (28). In that case if $J(x)=0$, the theory is invariant under the reflection $\phi\to-\phi$ and consequently all odd matrix elements of ϕ vanish. Then for $J(x)=0$, $\eta=0$,

$$\left[\Box^2+m_0{}^2+\frac{\lambda_0}{i}\operatorname{tr}G(11)-2S(1)\right]G(12)=\delta(1-2). \tag{32}$$

In the case where $\eta\neq 0$ we do not have enough information yet to solve Eq. (26) for all components of G, because of the term involving $(\delta/\delta J)G$. However, observing that the projection operator representation

$$G_{\alpha\beta}(12)=\left(\delta_{\alpha\beta}-\frac{\eta_\alpha\eta_\beta}{\eta^2}\right)G_\pi(12)+\frac{\eta_\alpha\eta_\beta}{\eta^2}G_\alpha(12)$$

is valid and if we anticipate that

$$\eta_\gamma\frac{\lambda_0}{i}\left[\frac{\delta}{\delta J_\gamma(1)}G_{\alpha\beta}(12)\right]_{J=S=0}=\eta_\gamma\frac{\lambda_0}{i}\left[\frac{\delta}{\delta J_\gamma(1)}\frac{\delta}{\delta J_\alpha(1)}\frac{\delta}{\delta J_\beta(2)}W\right]_{J=S=0}=\frac{\eta_\alpha\eta_\beta}{\eta^2}F(12),$$

Eq. (27) becomes two equations

$$(\Box^2+m_R{}^2+\lambda_0\eta^2)G_\pi(12)=\delta(1-2) \tag{33}$$

and

$$(\Box^2+m_R{}^2+3\lambda_0\eta^2)G_\alpha(12)+F(12)=\delta(1-2). \tag{34}$$

Equation (34) cannot be solved until we know $F(12)$ but using (20b) Eq. (33) is

$$\Box^2G_\pi(12)=\delta(12)$$

or (35)

$$G_\pi(k)=\frac{1}{-k^2-i\epsilon},$$

which explicitly shows the Goldstone boson resulting from $\eta\neq 0$. We will prove this result in the next section.

III. BOUND-STATE PROPAGATION

We have completely analyzed meson propagation to lowest order for $\eta=0$ and found it to be essentially free. However, it is clear from Eq. (26) that the higher-connected Green's functions do not vanish and that this theory has considerably more structure than a free theory. We will now show that, in fact, to this order the theory has a bound state excited by the composite operator $\phi^2(x)$. To do this we functionally differentiate Eq. (26a) and look for closed sets of Green's-function equations.

We define

$$W_0^{-1}(11')=[\Box^2+m_0{}^2+\lambda_0W^0(\hat{1})]\delta(1-1'). \tag{36}$$

Note that $W_0(11')$ defined here is not the same as $W^0(11')$ when J is not zero or when there is symmetry breaking. (26a) may be written as

$$\int d1'W_0^{-1}(11')W(1'2)+\lambda_0W(1)W(\hat{1}2)=\delta(1-2). \tag{26c}$$

Here we continue our practice of not labeling the Green's function to denote that we are working in the lowest approximation. It will often be useful to write (26) in the form

$$W(1'1)=W_0(1'1)-\lambda_0\int d2\,W_0(1'2)W(\hat{2}1)W(2). \tag{26d}$$

Note that the equations are not soluble with symmetry breaking until we have an expression for $W(\hat{2}1)$. Differentiation with respect to $S(2)$ of (19a) results in

$$W(\hat{2}1)=-\lambda_0\int d1'W_0(11')W(\hat{1}'\hat{2})W(1'). \tag{37}$$

To solve this we need the four-field Green's function $W(\hat{1}'\hat{2})$. By differentiating (26c) with respect to $S(3)$ we get a related function

$$W(12\hat{3})+\lambda_0\int d1'W_0(11')W(\hat{1}'\hat{3})W(1'2)+\lambda_0\int d1'W_0(11')\frac{\delta}{\delta S(3)}[W(\hat{1}'2)W(1')]=0.$$

By setting $1=2$, taking the trace, and using (13b) to derive

$$W(\hat{1}\hat{2})=\frac{1}{i}\operatorname{tr}W(11\hat{2})+2W(1)\cdot W(1\hat{2})-\frac{2}{\lambda_0}\delta(1-2),$$

Eq. (36) becomes

$$W(\hat{1}\hat{3}) - 2W(1)W(1\hat{3}) + \frac{2}{\lambda_0}\delta(1-3) + \frac{\lambda_0}{i}\int d1'\mathrm{tr}W_0(11')W(\hat{1}\hat{3})W_0(1'1) + \frac{\lambda_0}{i}\int d1'\mathrm{tr}W_0(11')\frac{\delta}{\delta S(3)}[W(\hat{1}'1)W(1')] = 0. \quad (38)$$

This equation can be simplified, using Eq. (26d) and

$$\frac{\delta W_0(11')}{\delta S(3)} = -\lambda_0 \int W(\hat{3}\hat{2})W_0(12)W_0(21')d2, \quad (39)$$

to become

$$W(\hat{1}\hat{3}) - 2W(1)W(1\hat{3}) + \frac{2}{\lambda_0}\delta(1-3) + \frac{\lambda_0}{i}\int d1'\mathrm{tr}[W_0(11')W_0(1'1)W(\hat{1}'\hat{3})] + \frac{\lambda_0}{i}\int d1'\frac{\delta}{\delta S(3)}\mathrm{tr}[W_0(11')W(\hat{1}'1)W(1')] = 0. \quad (40)$$

The last term using (37) is

$$\int d1'd2\frac{-\lambda_0^2}{i}\frac{\delta}{\delta S(3)}\mathrm{tr}[W_0(11')W(1')W_0(12)W(\hat{2}\hat{1}')W(2)]. \quad (41)$$

We clearly cannot solve Eq. (40) with the term of the form (41) present, but if it were gone, (40) with the source off becomes a fairly simple integral equation for $W(\hat{1}\hat{3})$ in terms of the Green's functions of fewer fields. One easy way to avoid having to worry about (41) is to take $\eta = 0$, because then clearly (41) vanishes if the sources are off. This is, in fact, the method that we use in this paper. However, there are other ways which allow us to argue that (41) is negligible. If we were doing perturbation theory in λ_0 we could have just argued that the last term of (40) had one more power of λ_0 in it than the other terms and hence is small compared to them. We cannot do that directly here, because W_0 and $W(1)$ are implicit functions of λ_0. However, we do something that has exactly the same effect and which is very similar in many ways to what was done to achieve the loop expansion of the σ model.[3] To this end, note that as an exact relationship

$$\frac{\delta}{\delta S(3)} = \int d4\left[\frac{\delta\lambda_0 W(\hat{4})}{\delta S(3)}\frac{\delta}{\delta\lambda_0 W(\hat{4})} + \frac{\delta W(4)}{\delta S(3)}\frac{\delta}{\delta W(4)}\right]$$
$$= \lambda_0\int W(\hat{4}\hat{3})\frac{\delta}{\delta\lambda_0 W(\hat{4})}d4 + \int W(4\hat{3})\frac{\delta}{\delta W(4)}d4. \quad (42)$$

Thus Eq. (41) (continuing to use lowest-order quantities) may be rewritten using (36) as

$$-\frac{\lambda_0}{i}\int d1'd2\,d4[\lambda_0 W(\hat{3}\hat{4})]$$
$$\times\left[\frac{\delta}{\delta\lambda_0 W(\hat{4})} - \int d5 W_0(54)W(4)\frac{\delta}{\delta W(5)}\right]$$
$$\times\,\mathrm{tr}[W_0(11')W(1')W_0(12)\lambda_0 W(\hat{2}\hat{1}')W(21)]. \quad (43)$$

Since the variational derivative does not generate any more isolated powers of λ_0 we see that the term (43) is different from the other terms of Eq. (40) by having one more term of form $\lambda_0^2 W(\hat{3}\hat{4})$. We will develop this point further later in the paper but since the term we neglected in (19a) is $(\lambda_0/i)[\delta/\delta S(1)]W(1) \propto \lambda_0^2 W(\hat{1}\hat{3})$ we see that $\lambda_0^2 W(\hat{3}\hat{4})$ is the basic structure in terms of which this theory is expanded. An essentially equivalent, but perhaps more familiar and explicit, argument can be made if the number of fields N is allowed to get very large. To see how this works we assume that $\lambda_0 = \lambda_0'/2N$, where λ_0' does not depend on N. Equation (22) shows that S, m_0^2, $\Box^2$, $\lambda_0\mathrm{tr}G^0$, and $\lambda_0(\Phi_c^0)^2$ are of order $(N)^0$ and hence $\mathrm{tr}G^0$ and $(\Phi_c^0)^2 \propto N$. From

$$W(\hat{1}\hat{2}) = \frac{\delta}{\delta S(\hat{1})}\left[\frac{\langle\phi^2(2)\rangle}{\langle 1\rangle} - \frac{2S(2)}{\lambda_0}\right]$$
$$= \frac{\delta}{\delta S(\hat{1})}\left[\frac{\mathrm{tr}G(22)}{i} + \Phi_c^2(2) - \frac{2S(2)}{\lambda_0}\right] \quad (44)$$

we see that $W^0(\hat{1}\hat{2}) \propto N$. Similar reasoning shows that $W^0(1)W^0(1\hat{3}) \propto N$ and also that the third and fourth terms of Eq. (40) are $\propto N$. The last term in (40), as is perhaps made clearer by representation (41), is of order $(N)^0 = 1$ and is consequently negligible compared to the other terms.

Finally, note that these results for large N agree with our previous observation that $\lambda_0^2 W(\hat{1}\hat{2})$ is the expansion parameter of this theory since it follows that this quantity is of order $1/N$. We emphasize, however, that although the overlap between our methods and the large-N expansion is now apparent, we do not need N large to generate a definite clearly defined expansion procedure. The question of convergence of such a series with or without N large is a much more complex question which we cannot seriously attempt to answer.

We conclude that with any of the arguments above, the correct expression for $W(\hat{1}\hat{3})=D_B(13)$ to this order with sources off is given by (40) as

$$D_B(13)+\frac{\lambda_0}{i}N\int d1' g_0(11')g_0(1'1)D_B(1'3) + 2\lambda_0\eta^2\left[\int d1' g_0(11')D_B(1'2)\right] = -\frac{2}{\lambda_0}\delta(1-3). \quad (45)$$

Here we have introduced the notation

$$G_0^{\alpha\beta}(12)\equiv\delta^{\alpha\beta}g_0(12). \quad (46a)$$

Clearly $g_0(12)=(1/N)\mathrm{tr}G_0(12)$. In Fourier space we have from (31) that

$$g_0(k)=\frac{1}{m_R^2-k^2-i\epsilon}, \quad (46b)$$

which reduces in the case of symmetry breaking to

$$g_0(k)_{\eta\neq 0}=G_\pi(k)=\frac{1}{-k^2-i\epsilon}, \quad (46c)$$

using (35).

Equation (45) is easily solved in Fourier space. We make the definitions

$$D_B(12)=\frac{1}{(2\pi)^n}\int d^nk\, D_B(k)e^{-ik(x_1-x_2)} \quad (47)$$

and

$$\Sigma(q^2)=\frac{i}{(2\pi)^n}\int g_0(q+k)g_0(k)d^nk\,; \quad (48)$$

Equation (45) becomes

$$D_B(k)=-\frac{2}{\lambda_0}\left[\frac{1}{1-\lambda_0N\Sigma(q^2)+2\lambda_0\eta^2g_0(k)}\right]. \quad (49)$$

In the case which is of primary interest to us in this paper $\eta=0$ and

$$D_B(k)=-\frac{2}{\lambda_0}\left[\frac{1}{1-\lambda_0N\Sigma(q^2)}\right]. \quad (50)$$

We can now solve for $G(k)$ in terms of $D_B(k)$. Equation (26d) may be written as

$$G_{\alpha\beta}(1'1)=\delta_{\alpha\beta}g_0(1'1) + \lambda_0^2\eta_\alpha\eta_\beta\int d2\,d3\,g_0(1'2)D_B(23)g_0(31)$$

or as a Fourier transform

$$G_{\alpha\beta}(k)=\delta_{\alpha\beta}g_0(k)+\lambda_0^2\eta_\alpha\eta_\beta[g_0(k)]^2D_B(k). \quad (51)$$

The form we will be primarily interested in with $\eta=0$ is

$$G_{\alpha\beta}(k)\equiv W_{\alpha\beta}(k)=\delta_{\alpha\beta}g_0(k). \quad (52)$$

We have carried out the development of the RPA approximation in this great detail in part to make it clear to the reader how it corresponds to the O(N) approximations in the case of linear spontaneous symmetry breaking. To complete the comparison we make the following observations:

(1) The Lagrangian here becomes the Lagrangian of Ref. 2 with sources added if we make the substitutions $\lambda_0\to\lambda_0/2N$, and $\chi\to(\chi-m_0^2)/\lambda_0$.

(2) Corresponding to these substitutions we see that

$$W(\hat{1}\hat{2})\to\left(\frac{2N}{\lambda_0'}\right)^2W(\hat{1}\hat{2}), \qquad W(\hat{1}2)\to\frac{2N}{\lambda_0'}W(\hat{1}2). \quad (53)$$

With the above identifications it is easy to see that the lowest-order Green's functions given here with symmetry breaking are exactly those of Ref. 2.

Finally, we should go through all the analogous calculations in the Hartree approximation. Because of complications with renormalization the details will be discussed elsewhere, but it is straightforward to show, using the same techniques as above, that $W(11,\hat{2})$ in the Hartree approximation has structure very similar to the RPA results.

IV. COUPLING-CONSTANT RENORMALIZATION

Equations (29) and (30) showed how the physical meson mass to this lowest order is related to the bare parameters λ_0 and m_0. To complete the renormalization to this order we must rewrite (49) and (50) in renormalized form. Since the symmetry-breaking case has been fully discussed in the literature we will not discuss Eq. (49) further. We point out that in the interest of simplicity and clarity we do not handle the renormalization for (50) so as to keep the arbitrariness of the point of subtraction of divergent integrals manifest. It is not difficult to renormalize using the same prescription as applied to (49) in Ref. 2 and to arrive at the same physical results as with the scheme used here.

Because $D_B(k)$ arises from terms of the form $\lambda_0\delta/\delta S$ in Green's-function equations it always appears in the form $\lambda_0^2D_B(k)$, as is exemplified by Eq. (44). Since $\Sigma(q^2)$ is logarithmically divergent, one subtraction makes it finite if we make the identification

$$\Sigma(q^2)-\Sigma(0)\equiv\tilde{\Sigma}_R(q^2).$$

We have using (50),

$$\lambda_0{}^2 D_B(k) = -\frac{2}{N}\lambda_R\left[\frac{1}{1-\lambda_R\tilde{\Sigma}_R(q^2)}\right] \equiv \frac{\lambda_R{}^2}{N} D_R(k)\,. \tag{54}$$

Here the natural identification

$$\lambda_R \equiv \frac{\lambda_0 N}{1-\lambda_0 N\Sigma(0)} \tag{55}$$

has been made. We have defined λ_R so that the dependence of $\lambda_0{}^2 D_B$ on N is explicit. In order to keep the N dependence of the renormalized composite field χ_R the same as χ we define χ_R by $D_R \equiv (1/N)\langle T(\chi_R\chi_R)\rangle$. Consequently it follows that $\chi_R = (\lambda_0 N/\lambda_R)\chi$. All questions regarding collective excitation can be answered by an analysis of the denominator of (54). This has been discussed in detail in Ref. 5 and the discussion here is essentially identical to the discussion there. The natural place to subtract $\Sigma(q^2)$ to do an analytic evaluation of the integral involved is not at $q^2=0$ but at the point $q^2=4m^2$. Defining

$$\tilde{\Sigma}'_R(q^2) = \Sigma(q^2) - \Sigma(4m^2)\,,$$

it follows that

$$\begin{aligned}\tilde{\Sigma}'_R(q) = {} & \theta(q^2-4m_R{}^2)\frac{1}{8\pi^2}\left(\frac{1-4m_R{}^2}{q^2}\right)^{1/2}\tanh^{-1}\left(\frac{1-4m_R{}^2}{q^2}\right)^{1/2} - \frac{i\pi}{2}[\theta(q^0)] \\ & + \theta(q^2)\theta(4m_R{}^2-q^2)\frac{1}{8\pi^2}\left(\frac{4m_R{}^2}{q^2}-1\right)^{1/2}\left[\frac{\pi}{2}-\tan^{-1}\left(\frac{4m_R{}^2}{q^2}-1\right)^{1/2}\right] \\ & + \theta(-q^2)\frac{1}{8\pi^2}\left(1-\frac{4m_R{}^2}{q^2}\right)^{1/2}\coth^{-1}\left(1-\frac{4m_R{}^2}{q^2}\right)^{1/2}.\end{aligned} \tag{56}$$

From the above it follows that $\tilde{\Sigma}'_R(0) = 1/8\pi^2$ and hence, since $\tilde{\Sigma}_R(0)=0$, that

$$\tilde{\Sigma}_R(q^2) = \tilde{\Sigma}'_R(q^2) - \frac{1}{8\pi^2}\,. \tag{57}$$

The condition for ϕ^2 exciting a bound state is

$$\frac{8\pi^2}{\lambda_R} = 8\pi^2\tilde{\Sigma}_R(q^2) \tag{58}$$

The right-hand side of (58) is plotted in Fig. 2.

For more details of this see the Appendix of Ref. 5.

From Fig. 2 and (58) we can arrive at the following conclusions:

(1) For $\lambda_R < -8\pi^2$ a true bound state of mass $M_B{}^2$, with $M_B{}^2 < 4m^2$, develops. As $\lambda_R \to -\infty$, $M^2 \to 0$. Also, for this range of λ_R, $1-\lambda_R \mathrm{Re}\tilde{\Sigma}_R(q) = 0$ possesses a single root, so that a resonance of mass $u^2 > 4m^2$ develops. As $\lambda_R \to -\infty$, $u^2 \to 3.2767 \times 4m^2$.

(2) For $-8\pi^2 < \lambda_R < 0$, scattering is enhanced for $q^2 \sim 4m^2$ but there is no true collective behavior.

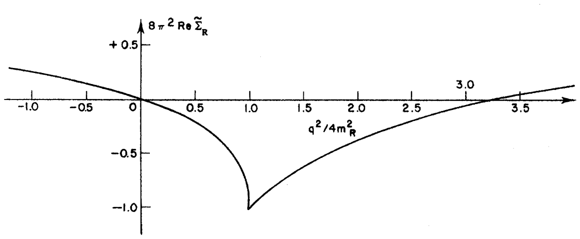

FIG. 2. Plot of $8\pi^2\mathrm{Re}\tilde{\Sigma}_R$.

(3) For $\lambda_R > 0$ (58) is satisfied for $q^2 < 0$, denoting the presence of a ghost in the theory if λ_R is positive. Also, a resonance exists with $u^2 > 3.2767 \times 4m^2$. This third conclusion shows that for reasons of stability $\lambda_R > 0$ must be excluded. We also note that the asymptotic behavior of $D_B(q^2)$ for large q^2 is $D_B(q^2) \sim 1/\ln q^2$ and not $1/q^2$.

We conclude that our approximation scheme to this order is not ghost-ridden, and that the theory thus far is seen to be consistent. Note that the symmetry-breaking case of the $1/N$ literature has a ghost in $D_B(k)$ which cannot be avoided regardless of the sign of λ_R. In the next few sections of this paper we will work out the various order Green's functions of the theory to lowest approximation and demonstrate that the contact interaction between four-meson fields no longer appears. Consequently, all interactions will be mediated by bound-state propagation much as if the interaction was of the form $\phi^2\chi$.

V. BOUND-STATE WAVE FUNCTION

We define the bound-state wave function in terms of

$$\frac{1}{i^2}\frac{\delta^3 W}{\delta S(1)\delta J(2)\delta J(3)} = -W(\hat{1}23) \tag{59}$$

since

$$\begin{aligned}-W(\hat{1}23)\big|_{J=S=0} = {} & \langle T(\phi^2(1)\phi(2)\phi(3))\rangle \\ & - \langle\phi^2(1)\rangle\langle T(\phi(2)\phi(3))\rangle\,,\end{aligned} \tag{60}$$

from which it follows that

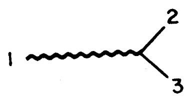

FIG. 3. Bound-state wave function.

$$\langle B(p)|T\phi(2)\phi(3)|0\rangle \propto \int e^{ip\cdot 1} d^4 1(\Box_1^2 - m_B^2) W(\hat{1}23)\,, \tag{61}$$

where m_B^2 is the solution of Eq. (58) for the bound-state mass. In the case of $J=0$ and no symmetry breaking, Eq. (39) can be rewritten as

$$W(\hat{1}23) = -\lambda_0 \int W(27)W(37)W(\hat{7}\hat{1})d7\,. \tag{62}$$

We can pictorially represent this by Fig. 3. Letting $S=0$ and letting

$$W(\hat{k}_1 k_2 k_3) = \int e^{i(k_1x_1 + k_2x_2 + k_3x_3)} W(\hat{1}23)dx_1dx_2dx_3$$

we obtain

$$W(\hat{k}_1 k_2 k_3) = -(2\pi)^4 \lambda_0 \delta(k_1 + k_2 + k_3) D_B(k_1) W(k_2) W(k_3)\,, \tag{63}$$

where $D_B(k_1)$ and $W(k_2)$ are the bound state and meson propagators given by Eqs. (50) and (31). Here and in what follows we have confined explicit consideration to four-dimensional spacetime. Results in other dimensions are found by trivial modification of the formulas presented. With the identification of $\chi_R = (\lambda_0 N/\lambda_R)\chi_R$ the above leads to the renormalized equation

$$W_R(k_1 k_2 k) = -(2\pi)^4 \lambda_R \delta(k_1 + k_2 + k_3) D_R(k_1) W(k_2) W(k_3)\,.$$

VI. MESON-MESON SCATTERING

The simplicity of the random-phase approximation is that the unrenormalized meson-meson scattering amplitude is just $2\lambda_0^2[D_B(s) + D_B(t) + D_B(u)]$, and thus renormalization is easily achieved by a coupling-constant renormalization (and simultaneously a wave renormalization for the bound state). In the Hartree approximation, one also gets a contribution from the original contact interactions of four meson fields which is completely absent in the RPA. This additional interaction completely obscures the renormalization procedure. Consequently, we leave further discussion of the Hartree approximation to a separate publication.

The simplest way of evaluating the scattering amplitude $W(1234)$ and just confirming the above statement is by differentiating the identity

$$\int W(12)W^{-1}(23)d2 = \delta(1-3)\,. \tag{64}$$

When $J=0$ we find

$$W(1234) = -\int W(17)\frac{\delta W^{-1}(78)}{\delta J(3)\delta J(4)} W(82)d7\,d8\,. \tag{65}$$

But we find from the Appendix, where we have tabulated easily derived results, that

$$\begin{aligned}\frac{\delta W^{-1}(78)}{\delta J(3)\delta J(4)} &= \lambda_0 W(\hat{7}34)\delta(7-8) \\ &\quad + \lambda_0 W(73)\int W(\hat{7}49)W^{-1}(98)d9 \\ &\quad + \lambda_0 W(74)\int W(\hat{7}39)W^{-1}(98)d9\,.\end{aligned} \tag{66}$$

Thus using Eqs. (62) and (66) we find

$$\begin{aligned}W(1234) = \lambda_0^2 \int d7\,d8\, W(\hat{7}\hat{8})[&W(17)W(72)W(18)W(84) \\ &+ W(17)W(73)W(28)W(84) \\ &+ W(17)W(74)W(38)W(82)]\,.\end{aligned} \tag{67}$$

This can be represented schematically by Fig. 4. The above can be rewritten as

$$W(1234) = \int W(15)W(26)W(37)W(48) \times \Gamma(5678)d5\,d6\,d7\,d8\,,$$

where $\Gamma(5678)$ is the one-particle irreducible four-point function. Consequently,

$$\begin{aligned}\Gamma(1234) &= \lambda_0^2\delta(1-2)\delta(3-4)W(\hat{3}\hat{1}) \\ &\quad + \lambda_0^2\delta(1-4)\delta(2-3)W(\hat{2}\hat{1}) \\ &\quad + \lambda_0^2\delta(1-3)\delta(4-2)W(\hat{4}\hat{1})\,.\end{aligned} \tag{68}$$

Introducing the Fourier transform $\Gamma(p_1p_2p_3p_4)$ via

$$\Gamma(1234) = \frac{\int dp_1dp_2dp_3dp_4}{(2\pi)^{16}} e^{-i(p_1x_1 + p_2x_2 + p_3x_3 + p_4x_4)} \times \Gamma(p_1p_2p_3p_4)\,, \tag{69}$$

we find

$$\begin{aligned}-\Gamma_{abcd}(p_1p_2p_3p_4) &= (2\pi)^4\delta(p_1+p_2+p_3+p_4)\left\{\frac{2\lambda_c^2\delta_{ab}\delta_{cd}}{\lambda_0[1-\lambda_0 N\Sigma(s)]} + \frac{2\lambda_0^2\delta_{ad}\delta_{bc}}{\lambda_0[1-\lambda_0 N\Sigma(u)]} + \frac{2\lambda_0^2\delta_{ac}\delta_{bd}}{\lambda_0[1-\lambda_0 N\Sigma(t)]}\right\} \\ &= (2\pi)^4\delta(p_1+p_2+p_3+p_4)A(s,t,u)\,.\end{aligned} \tag{70}$$

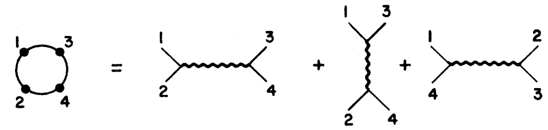

FIG. 4. Scalar-meson 4-point function.

Defining the renormalized coupling constant as in Eq. (55)

$$A(s=0,t=0,u=0)=\frac{6\lambda_R}{N}.$$

We obtain

$$A(s,t,u)=\frac{-\lambda_R{}^2}{N}[D_R(s)\delta_{ab}\delta_{cd}+D_R(t)\delta_{ad}\delta_{bc}+D_R(u)\delta_{ac}\delta_{bd}].$$

VII. VERTEX FUNCTION FOR THREE BOUND STATES

We are interested in knowing the vertex for three bound-state mesons since this is most important for phenomenological field theory. This information is contained in

$$W(\hat{1}\hat{2}\hat{3})=\frac{\delta^3 W}{\delta S(1)\delta S(2)\delta S(3)}$$

and, more appropriately for scattering, in its Fourier transform

$$W(p_1p_2p_3)=\int e^{+i(p_1x_1+p_2x_2+p_3x_3)}W(\hat{1}\hat{2}\hat{3})dx_1dx_2dx_3.$$

For bound-state–bound-state scattering one needs for single-bound-state exchange the quantity

$$\lim_{p^2\to M_B{}^2}(p_1{}^2-M_B{}^2)(p_2{}^2-M_B{}^2)W(M_B{}^2M_B{}^3p_3{}^2). \tag{71}$$

From $\chi=\phi^2-2S/\lambda_0$ we formally obtain

$$W(\hat{1}\hat{2}\hat{3})=\frac{1}{i}\operatorname{tr}W(11\hat{2}\hat{3})+2W(\hat{3}1)W(1\hat{2})+2\Phi_c(1)W(1\hat{2}\hat{3}), \tag{72}$$

so that for $J=0$ we have

$$W(\hat{1}\hat{2}\hat{3})=\frac{1}{i}\operatorname{tr}W(11\hat{2}\hat{3}).$$

For $W(11\hat{2}\hat{3})$ we find by differentiating Eq. (64) that

$$W(11\hat{2}\hat{3})=-\int W(17)\frac{\delta W^{-1}(78)}{\delta S(2)\delta S(3)}W(81)d7\,d8-2\int\frac{\delta W^{-1}(78)}{\delta S(2)}W(17\hat{3})W(81)d7\,d8. \tag{73}$$

Since from the Appendix we have

$$\frac{\delta W^{-1}(78)}{\delta S(2)}=\lambda_0 W(\hat{7}\hat{2})\delta(7-8),$$
$$\frac{\delta W^{-1}(78)}{\delta S(2)\delta S(3)}=\lambda_0 W(\hat{7}\hat{2}\hat{3})\delta(7-8), \tag{74}$$

we find that $W(\hat{1}\hat{2}\hat{3})$ obeys the integral equation

$$iW(\hat{1}\hat{2}\hat{3})=-\lambda_0\operatorname{tr}\int W(15)W(\hat{5}\hat{2}\hat{3})W(51)d5+2\lambda_0{}^2\operatorname{tr}\int W(15)W(56)W(61)W(\hat{5}\hat{2})W(\hat{6}\hat{3})d5\,d6. \tag{75}$$

Pictorially we get Fig. 5, which sums to Fig. 6. This is the basic vertex function for the composite field χ. Equation (75) is solved by the Fourier transform. Letting

$$W(\hat{1}\hat{2}\hat{3})=\frac{1}{(2\pi)^{12}}\int e^{-i(p_1x_1+p_2x_2+p_3x_3)}W(p_1p_2p_3)dp_1dp_2dp_3, \tag{76}$$

we obtain

$$W(q_1q_2q_3)=2\lambda_0{}^2\delta(q_1+q_2+q_3)D(q_2)D(q_3)\Lambda(q_1{}^2,q_2{}^2)+i\frac{\lambda_0}{(2\pi)^4}\left[\int W(q_1-k)W(k)dk\right]W(q_1q_2q_3) \tag{77}$$

or

$$\begin{aligned}W(q_1q_2q_3)&=\frac{2\lambda_0{}^2}{1-\lambda_0N\Sigma(q_1)}\times\delta(q_1+q_2+q_3)D_B(q_2)D_B(q_3)\Lambda(q_1{}^2q_2{}^2)\\&=\lambda_0{}^3\delta(q_1+q_2+q_3)\times D_B(q_1)D_B(q_2)D_B(q_3)N\Lambda(q_1{}^2q_2{}^2),\end{aligned} \tag{78}$$

where

$$N\Lambda(q_1{}^2q_2{}^2)=\frac{1}{(2\pi)^8 i}\operatorname{tr}\int W(q_1-k_2)W(q_2+k_2)W(k_2)dk_2$$

is the triangle graph. Renormalization follows from $\lambda_0{}^2D=(\lambda_R{}^2/N)D_R$, and $\chi_R=(\lambda_0N/\lambda_R)\chi$, which shows that we should form the three bound-state propagators from χ_R so that $W_R(\hat{1}\hat{2}\hat{3})$

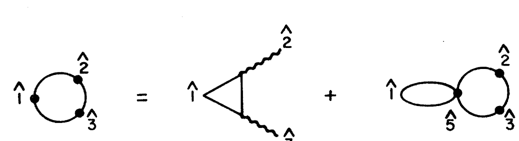

FIG. 5. Integral equation for three-bound-state vertex.

$= (\lambda_0 N/\lambda_R{}^3) W(\hat{1}\hat{2}\hat{3})$. With this (78) may be written as

$$W_R(q_1 q_2 q_3) = -\lambda_R{}^3 N \delta^4(q_1+q_2+q_3) \times D_R(q_1) D_R(q_3) D_R(q_3) \delta(q_1{}^2 q_2{}^2)\,. \tag{79}$$

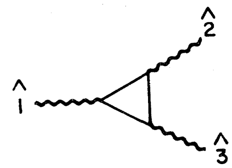

FIG. 6. Three-bound-state vertex.

VIII. MESON-SIX-POINT FUNCTION AND MESON-BOUND-STATE SCATTERING

Meson-bound-state scattering can be found from

$$W(12\hat{3}\hat{4}) = \int \frac{\delta W^{-1}(89)}{\delta S(3)\delta S(4)} W(18)W(92)d8\,d9 - \int \frac{\delta W^{-1}(89)}{\delta S(3)}[W(18\hat{4})W(92) + W(18)W(92\hat{4})]d8\,d9\,. \tag{80}$$

Using Eqs. (74) and (62) we obtain from (80)

$$W(12\hat{3}\hat{4}) = -\lambda_0 \int W(\hat{8}\hat{3}\hat{4})W(18)W(82)d8 + \lambda_0{}^2 \int W(\hat{8}\hat{3})W(\hat{9}\hat{4})W(89) \times [W(18)W(29) + W(82)W(19)]d9\,d8\,. \tag{81}$$

This can be renormalized as before by using

$$\lambda_0 \chi = \frac{\lambda_R}{N}\chi_R$$

to give

$$W_R(12\hat{3}\hat{4}) = \frac{N^2\lambda_0{}^2}{\lambda_R{}^2} W(12\hat{3}\hat{4})$$

so that

$$W_R(12\hat{3}\hat{4}) = -\frac{\lambda_R}{N}\int W_R(\hat{8}\hat{3}\hat{4})W(18)W(82)d8 + \lambda_R{}^2 \int W_R(\hat{8}\hat{3})W_R(\hat{9}\hat{4})W(89) \times [W(18)W(29) + W(82)W(19)]d9\,d8\,. \tag{82}$$

Since $W_R(\hat{8}\hat{3}\hat{4}) \propto N$, the whole amplitude is proportional to N^0. Pictorially we get the diagrams of Fig. 7.

By once again taking the functional derivatives of Eq. (64) we find that the meson-six-point function obeys the integral equation

$$\begin{aligned} W(123456) = {} & \int W(17)\frac{\delta W^{-1}(78)}{\delta J(3)\delta J(4)\delta J(5)\delta J(6)} W(82)d7\,d8 \\ & - \int \frac{\delta W^{-1}(78)}{\delta J(3)\delta J(5)}[W(1746)W(82) + W(17)W(8246)]d7\,d8 \\ & - \int \frac{\delta W^{-1}(78)}{\delta J(3)\delta J(5)}[W(1745)W(82) + W(17)W(8245]d7\,d8\,. \end{aligned} \tag{83}$$

Since $\delta W^{-1}(78)/\delta J(3)\delta J(4)\delta J(5)\delta J(6)$ from (A11) contains $\delta(7-8)W(\hat{7}3456)$ plus known terms, we need only this expression to calculate the 6-point function. Now

$$W(1234\hat{5}) = -\int W(17)\frac{\delta W^{-1}(78)}{\delta J(2)\delta J(5)\delta S(5)} W(82)d7\,d8 - \int \frac{\delta W^{-1}(78)}{\delta J(3)\delta J(4)}[W(17\hat{5})W(82) + W(82\hat{5})W(17)]d7\,d8\,. \tag{84}$$

Using (A9) and (A10) we get

$$\begin{aligned} W(1234\hat{5}) = {} & -\lambda_0 \int W(17)W(27)W(\hat{7}\hat{5}34)d7 - \lambda_0 \int W(17)W(73\hat{5})W(\hat{7}24)d7 - \lambda_0 \int W(17)W(74\hat{5})W(\hat{7}23)d7 \\ & + \lambda_0{}^2 \int W(3,10)W(4,10)W(\hat{10},\hat{7})[W(17\hat{5})W(72) + W(71)W(72\hat{5})]d10\,d7 \\ & + \lambda_0{}^2 \int W(\hat{7}\hat{8})[W(73)W(48) + W(74)W(38)][W(17\hat{5})W(82) + W(82\hat{5})W(71)]d7\,d8\,, \end{aligned} \tag{85}$$

which pictorially is given by Fig. 8. It is easy to see that $W(1234\hat{5}) \propto 1/N$.

Continuing to calculate (83) we find, using the expressions (A9) and (A11) found in the Appendix for $\delta W^{-1}(78)/\delta J(3)\delta J(4)\delta J(5)\delta J(6)$ and $\delta W^{-1}(78)/\delta J(3)\delta J(5)$, that there are terms of the form

$$-\lambda_0 \int W(17)W(27)W(\hat{7}3456)d7 \tag{86}$$

and

$$+\lambda_0^2 \int W(1746)W(82)W(3,10)W(4,10)W(1\hat{0},\hat{7})d7\,d10\,. \tag{87}$$

Thus pictorially we get terms of Fig. 9 for $W(123456)$. Renormalization is achieved as before. It is easy to see that $W(123456) \propto 1/N^2$.

IX. BOUND-STATE–BOUND-STATE SCATTERING

To determine bound-state–bound-state scattering we need to know

$$W(\hat{1}\hat{2}\hat{3}\hat{4}) = \frac{1}{i}\,\mathrm{tr}\,W(11\hat{2}\hat{3}\hat{4})\,.$$

In terms of the inverse Green's function we have

$$W(11,\hat{2}\hat{3}\hat{4}) = -\int W(17)W(81)\frac{\delta W^{-1}(78)}{\delta S(2)\delta S(3)\delta s(4)}d7\,d8 - \int W(81)\left[W(17\hat{3})\frac{\delta W^{-1}(78)}{\delta S(2)\delta S(4)} + W(17\hat{4})\frac{\delta W^{-1}(78)}{\delta S(2)\delta S(3)}\right]$$
$$-2\int \frac{\delta W^{-1}(78)}{\delta S(2)}[W(17\hat{3}\hat{4})W(81) + W(17\hat{3})W(81\hat{4})]d7\,d8\,. \tag{88}$$

Since

$$W(17\hat{3}) = - \quad W(18)\frac{\delta W^{-1}(18)}{\delta S(3)}W(97)d8\,d9\,,$$

$$W(17\hat{3}\hat{4}) = -\int \frac{\delta W^{-1}(89)}{\delta S(3)\delta S(4)}W(18)W(97)d8\,d9$$

$$= -\int \frac{\delta W^{-1}(89)}{\delta S(3)}[W(18\hat{4})W(97) + W(18)W(97\hat{4})]d8\,d9\,,$$

and

$$\frac{\delta W^{-1}(78)}{\delta S(2)} = \lambda_0 W(\hat{7}\hat{2})\delta(7-8)\,,$$

$$\frac{\delta W^{-1}(78)}{\delta S(2)\delta S(3)} = \lambda_0 W(\hat{7}\hat{2}\hat{3})\delta(7-8)\,,$$

$$\frac{\delta W^{-1}(78)}{\delta S(2)\delta S(3)\delta S(4)} = \lambda_0 W(\hat{7}\hat{2}\hat{3}\hat{4})\delta(7-8)\,, \tag{89}$$

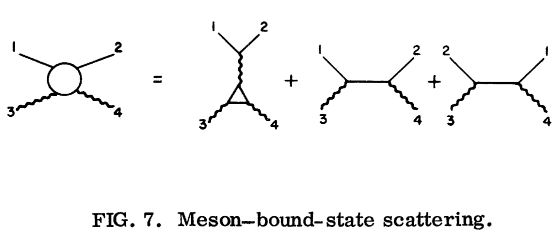

FIG. 7. Meson–bound-state scattering.

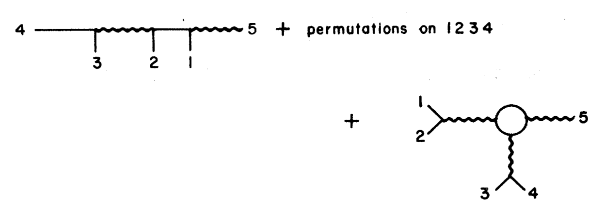

FIG. 8. $W(1234\hat{5})$.

we get the following integral equation for $W(\hat{1}\hat{2}\hat{3}\hat{4})$:

$$
\begin{aligned}
iW(\hat{1}\hat{2}\hat{3}\hat{4}) = \lambda_0 &\int [\operatorname{tr} W(17)W(71)]W(\hat{7}\hat{2}\hat{3}\hat{4})d7\,d8 \\
&+ 2\lambda_0^2 \int \operatorname{tr}[W(17)W(78)W(81)]W(\hat{8}\hat{3})W(\hat{7}\hat{2}\hat{4})d7\,d8 \\
&+ 2\lambda_0^2 \int \operatorname{tr}[W(18)W(87)W(71)]W(\hat{8}\hat{4})W(\hat{7}\hat{2}\hat{3})d7\,d8 \\
&- 2\lambda_0^3 \int \operatorname{tr}[W(18)W(87)W(79)W(91)]W(\hat{7}\hat{2})W(\hat{8}\hat{3})W(\hat{9}\hat{4})d7\,d8\,d9 \\
&+ 2\lambda_0^2 \int W(\hat{7}\hat{2})\operatorname{tr}[W(17)W(78)W(81)]W(\hat{8}\hat{3}\hat{4})d7\,d8 \\
&- 2\lambda_0^3 \int W(\hat{7}\hat{2})W(\hat{8}\hat{3})W(\hat{9}\hat{4})\{\operatorname{tr}[W(17)W(78)W(89)W(19)] + \operatorname{tr}[W(17)W(79)W(98)W(81)]\}d7\,d8\,d9
\end{aligned}
\tag{90}
$$

so

$$
\begin{aligned}
iW(\hat{1}\hat{2}\hat{3}\hat{4}) &= W_0(1234) \\
&\quad - \lambda_0 \int \operatorname{tr}[W(17)W(71)W(\hat{7}\hat{2}\hat{3}\hat{4})]d7\,d8 .
\end{aligned}
$$

Letting

$$
\begin{aligned}
W(\hat{1}\hat{2}\hat{3}\hat{4}) = \frac{1}{(2\pi)^{12}} &\int e^{-i(p_1x_1+p_2x_2+p_3x_3+p_4x_4)} \\
&\times W(p_1p_2p_3p_4)dp_1dp_2dp_3dp_4 .
\end{aligned}
$$

We have

$$
\begin{aligned}
W(q_1q_2q_3q_4) = W_0(q_1q_2q_2q_4) &+ \frac{i\lambda_0}{(2\pi)^4}\operatorname{tr}\left[\int W(q_1-k)W(k)dk\right] \\
&\times W(q_1q_2q_3q_4)
\end{aligned}
\tag{91}
$$

or

$$
\begin{aligned}
W(q_1q_2q_3q_4) &= \frac{1}{1-\lambda_0 N\Sigma(q_1^{\,2})}\,\frac{W_0(q_1q_2q_3q_4)}{i} \\
&= -\frac{\lambda_0}{2}D_B(q_1^{\,2})\frac{W_0(q_1q_2q_3q_4)}{i} .
\end{aligned}
\tag{92}
$$

This renormalizes in the usual way. $W_0(\hat{1}\hat{2}\hat{3}\hat{4})$ is schematically given by Fig. 1.

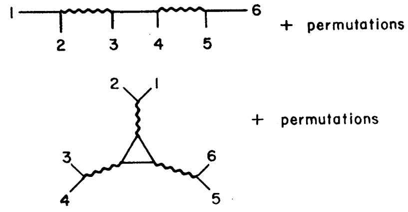

FIG. 9. $W(123456)$.

We have thus shown that bound states scatter via the box diagram or via bound-state exchange or production. The second process tells us that the collective mode is also a collective mode of the four-point scalar-meson system, since one expects that we get contributions to 4 mesons → 4 mesons as illustrated in Fig. 10.

The preceding discussions should have more than amply served to convince the reader that we can handle in a straightforward way (at least to this order) in a closed form all the Green's functions of this theory and that renormalization proceeds without difficulty, while the theory is characterized by having no explicit remnants of the original contact 4-field interaction.

X. CORRECTIONS TO THE RPA

All results to this point have come from considering the truncation (19a) of Eq. (12) or equivalently the truncation (26a) of Eq. (16). In this section we will explicitly find the first-order correction to the RPA and in so doing indicate how to find the corrections of any order to the theory.[6]

A straightforward way to proceed is by the following prescription. Expand all Green's functions order by order so that we have for example that

$$
\Phi = \Phi^0 + \Phi^1 + \Phi^2 + \cdots ,
$$

$$
G = G^0 + G^1 + G^2 + \cdots .
$$

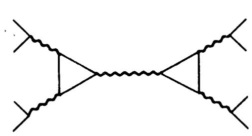

FIG. 10. Bound-state contributions of 4 mesons–4 mesons.

We assume in addition that for any Green's functions that $W^b W^c \cdots W^m$ is of order $b+c+\cdots+m$ and, further, that $(\lambda_0\delta/\delta S)^m W^n$ is of order $m+n$. We, of course, assume that Φ^0 and G^0 satisfy Eqs. (19) and (26). Using this prescription it is possible to calculate any order perturbation correction to RPA. For example, Eq. (12b) becomes

$$\left[\Box^2+m_0^{\,2}+\frac{\lambda_0}{i}\mathrm{tr}(G^0+G^1+\cdots)+\lambda_0(\Phi_c^0+\Phi_c^1+\cdots)(\Phi_c^0+\Phi_c^1+\cdots)-2S+\frac{\lambda_0}{i}\frac{\delta}{\delta S}\right](\Phi_c^0+\Phi_c^1+\cdots)=J. \tag{93}$$

From this we find to first order that

$$\left[\Box^2+m_0^{\,2}+\frac{\lambda_0}{i}\mathrm{tr}G^0+\lambda_0(\Phi_c^0)^2+2\lambda_0\Phi_c^0\Phi_c^0\cdot-2S\right]\Phi_c^1+\left(\frac{\lambda_0}{i}\mathrm{tr}G^1\right)\Phi_c^0+\frac{\lambda_0}{i}\frac{\delta}{\delta S}\Phi_c^0=0. \tag{94}$$

Using Eq. (35) this becomes

$$\left[\Box^2+m_0^{\,2}+\frac{\lambda_0}{i}\mathrm{tr}G^0(11)-2S+\lambda_0(\Phi_c^0(1))^2+2\lambda_0\Phi_c^0(1)\Phi_c^0(1)-2S(1)+\frac{\lambda_0}{i}\Phi_c^0(x)\frac{\delta}{\delta J(x)}\cdot\right]\Phi_c^1(1)=\frac{\lambda_0^{\,2}}{i}\int W^0(\hat{1}\hat{1}')\Phi^0(1')W_0(1'1). \tag{95}$$

This equation is not closed and hence not solvable, any more than Eqs. (19) were, until we find G^1. The important feature that is already clear here is that the right-hand-side driving term of (95) is proportional to the characteristic expansion factor $\lambda_0^{\,2}W^0(\hat{1}\hat{1}')\propto 1/N$.

We could proceed to calculate the meson and bound-state propagators by differentiating Eq. (95), but it is somewhat more convenient to exploit in a more explicit manner the fact that the approximation is essentially an expansion in the bound-state propagator. For simplicity in the following we consider only the case where $J(x)$ and $\langle\phi(x)\rangle=0$. There is no intrinsic difference in the techniques used for the more general case. Under these conditions it follows that

$$\frac{\delta}{\delta S(1)}=\int d2\frac{\delta\lambda_0 W(\hat{2})}{\delta S(1)}\frac{\delta}{\delta\lambda_0 W(\hat{2})}$$
$$=\lambda_0\int d2\, W(\hat{1}\hat{2})\frac{\delta}{\delta\lambda_0 W(\hat{2})}. \tag{96}$$

As an exact equation we can now rewrite Eq. (16a) as

$$\left[\Box^2+m_0^{\,2}+\lambda_0 W(\hat{1})+\frac{\lambda_0^{\,2}}{i}\int d3\, W(\hat{1}\hat{3})\frac{\delta}{\delta\lambda_0 W(\hat{3})}\right]W(12)=\delta(1-2). \tag{97}$$

From Eq. (44), which is an exact equation, we see with $J=\eta=0$ that

$$W(\hat{1}\hat{2})=\frac{\delta}{\delta S(1)}\left[\frac{1}{i}\mathrm{tr}W(22)-\frac{2S(2)}{\lambda_0}\right],$$

which using (96) becomes

$$\int d3\, W(\hat{1}\hat{3})\left[\delta(3-2)-\lambda_0\frac{\delta}{\delta\lambda_0 W(\hat{3})}\frac{1}{i}\mathrm{tr}W(22)\right]=-\frac{2}{\lambda_0}\delta(1-2) \tag{98}$$

or, equivalently,

$$W^{-1}(\hat{1}\hat{2})=-\frac{\lambda_0}{2}\left[\delta(1-2)-\lambda_0\frac{\delta}{\delta\lambda_0 W(\hat{1})}\frac{1}{i}\mathrm{tr}W(22)\right]. \tag{99}$$

Equation (97) may be further developed by noting that

$$\frac{\delta}{\delta\lambda_0 W(\hat{5})}W(12)=-\int W(13)\frac{\delta W^{-1}(34)}{\delta\lambda_0 W(\hat{5})}W(42)d3\,d4 \tag{100}$$

and defining a vertex function

$$\Gamma(5;34)\equiv\frac{\delta W^{-1}(34)}{\delta\lambda_0 W(\hat{5})}, \tag{101}$$

so that an exact equation (97) now becomes

$$\int d4\left\{[\Box+m_0^{\,2}+\lambda_0 W(\hat{4})]\delta(1-4)-\frac{\lambda_0^{\,2}}{i}\int d3\,d5\, W(\hat{1}\hat{5})W(13)\Gamma(5;34)\right\}W(42)=\delta(1-2) \tag{102}$$

or

$$W^{-1}(14)=\left\{[\Box^2+m_0^{\,2}+\lambda_0 W(\hat{4})]\delta(1-4)-\frac{\lambda_0^{\,2}}{i}\int d3\,d5\, W(\hat{1}\hat{5})W(13)\Gamma(5;34)\right\}. \tag{103}$$

Going through the same procedure we find as an exact equation that (99) becomes

$$W^{-1}(\hat{1}\hat{2}) = -\frac{\lambda_0}{2}\left\{\delta(1-2) + \frac{\lambda_0}{i}\mathrm{tr}\left[\int d3\, d6\, W(23)\Gamma(1;36)W(62)\right]\right\}. \quad (104)$$

These techniques should be familiar to the reader who knows how to calculate electrodynamic Green's functions using source techniques. Now we will do perturbation theory on Eqs. (103), (104), and (101), expanding by this method the inverse propagators instead of the propagators.

As before, we take

$$W^{0-1}(14) = [\Box^2 + \lambda_0 W^0(4) + m_0{}^2]\delta(1-4).$$

Thus, we see from (101) that

$$\Gamma^0_{cd}(5;34) = \delta(4-5)\delta(3-4)\delta_{cd}. \quad (105)$$

Consequently iterating to first order we have

$$(\tilde{W}^1)^{-1} \equiv [W^0(14) + W^1(14)]^{-1} = [\Box^2 + m_0{}^2 + \lambda_0 W(4)]\delta(1-4) - \frac{\lambda_0{}^2}{i} W^0(\hat{1}\hat{4})W^0(14). \quad (106)$$

We have used the fact that $\lambda_0{}^2 W^0(\hat{1}\hat{4})$ is a first-order quantity. This equation is not yet solved since $W(1) = (1/i)\mathrm{tr}G(11) - 2S/\lambda_0$ and we must keep $\mathrm{tr}G^1$ as well as $\mathrm{tr}G^0$. To solve for $W(1)$ take the inverse of (106) and keep only first-order terms to explicitly identify W^1. This yields

$$W^1(12) = -\frac{\lambda_0}{i}\int d3[\mathrm{tr}W^1(3\hat{3})]W^0(13)W^0(32) + \frac{\lambda_0{}^2}{i}\int d3\, d4\, dW^0(13)W^0(\hat{3}\hat{4})W^0(24)W^0(43). \quad (107)$$

Setting $1=2$ and taking the trace of (107) and then setting $S=0$ leads to the evaluation of the number $(\lambda_0/i)\mathrm{tr}W^1(11) \equiv \Delta$. We find

$$\Delta = \frac{\lambda_0{}^4}{2}\int W^0(\hat{3}\hat{1})[\mathrm{tr}W^0(15)W^0(14)W^0(45)] \times W^0(\hat{5}\hat{4})\, d3\, d4\, d5. \quad (108)$$

At several times we have stated that $\lambda_0{}^2 D_B(12)$ is essentially the expansion parameter of this theory. As we demonstrated, this quantity is of lowest order, is proportional to $1/N$, which is the large-N limit, and explicitly displays a small parameter. We see from (108), however, that this statement is not accurate if applied in the simplisitc sense of determining the order of a graph by just counting the number of bound-state propagators and claiming, for example, that a graph with two bound-state propagators is of higher order than a graph with one bound-state propagator. This is because $D^0_B(\hat{1}\hat{2})$ is in its entirety constructed of the zeroth-order meson propagator $G^0(12)$, through zeroth-order quantities of the form $\lambda_0\mathrm{tr}(G^0\cdots G^0)$ and λ_0. Consequently, a term like

$$\lambda_0{}^2 D_B\mathrm{tr}(G^0\cdots G^0) \propto \frac{1}{1-\lambda_0\mathrm{tr}(G^0G^0)}\lambda_0\mathrm{tr}(G^0\cdots G^0)$$

is not first order but zeroth order. Of course, all of this is clear in the $1/N$ language for large N, but we do not restrict ourselves to this case. We can see such a situation developing in the preceding equations. The two terms on the right-hand side of (107) are of first order since $\lambda_0\mathrm{tr}G^1$ and $\lambda_0{}^2 W^0(\hat{3}\hat{4})$ are first order. In deriving (108) from (107) we have taken a trace and observed that $D_B \propto 1/[1-\lambda_0\mathrm{tr}(G^0G^0)]$ so as to combine terms in a simple way. The result is (108), which is first order but has two bound-state propagators. Of course, when we generate the perturbation series in the prescribed manner we do not have to ever worry about this sort of explicit identification, because once we have put in the lowest order, the higher orders just fall out by simple iteration as they, in fact, did in Eq. (108).

With (108) we have now explicitly determined (106) to first order. Figure (11) graphically illustrates $G^1(13)$. We will not explicitly renormalize (106) by displaying the integrals, but it is clear that with two subtractions (106) converges so that in momentum space

$$\tilde{W}^1(p) = \frac{Z_1}{p^2 + (m^1_R)^2 + (\lambda^1_R/N)^2 Y(p^2;(m^1_R)^2)}. \quad (109)$$

Here we choose $Y(p^2 = (m^1_R)^2) = 0$, with $(m^1_R)^2$ the renormalized mass of the pole approximate for the first-order approximation and $(\lambda^1_R)^2$ is also the renormalized coupling constant appropriate for first order. We have also used the fact that the corrections are already of first order so $(\lambda^1_R)^2$ and $(m^1_R)^2$ have been written in place of the zeroth-order renormalized quantities for the isolated $(m^1_R)^2$ term which occurs automatically in this form.

We next determine the vertex function (101) to first order by using (106) and noting that

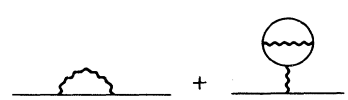

FIG. 11. First-order correction to meson propagator.

$$\frac{\delta}{\delta\lambda_0 W(\hat{3})} W^0(12) = -\frac{\lambda_0{}^2}{i}\int W^0(\hat{1}\hat{4})\mathrm{tr}[W^0(43)W^0(35)W^0(45)]W^0(\hat{5}\hat{2})d4\,d5 \tag{110}$$

so that to first order

$$\Gamma(3;12) = \delta(1-2)\delta(3-1) - \lambda_0{}^4\int d4\,d5\, W^0(\hat{1}\hat{4})\mathrm{tr}[W^0(43)W^0(35)W^0(45)]W^0(\hat{5}\hat{2})W^0(12) + \frac{\lambda_0{}^2}{i} W^0(\hat{1}\hat{2})W^0(13)W^0(32). \tag{111}$$

It is clear that all the corrections are of first order or $1/N$. Thus accurate to terms of first order we have from (104)

$$\begin{aligned}[\tilde{W}^1(\hat{1}\hat{2})]^{-1} &\equiv [W^0(\hat{1}\hat{2}) + W^1(\hat{1}\hat{2})]^{-1} \\ &= -\frac{\lambda_0}{2}\Big[\delta(1-2) + \frac{\lambda_0}{i}\mathrm{tr}[\tilde{W}^1(12)\tilde{W}^1(21)] \\ &\quad - \frac{\lambda_0{}^5}{i}\int d3\,d4\,d5\,d6\, W^0(\hat{3}\hat{4})W^0(\hat{5}\hat{6})\mathrm{tr}[W^0(23)W^0(36)W^0(62)]\mathrm{tr}[W^0(41)W^0(15)W^0(45)] \\ &\quad - \lambda_0{}^3\int d3\,d6\, W^0(\hat{3}\hat{6})\mathrm{tr}[W^0(23)W^0(31)W^0(16)W^0(62)]\Big].\end{aligned} \tag{112}$$

The terms that contribute to this bound propagator in first order are represented pictorially in Fig. 12.

We will not go through the details of explicitly writing out the integrals involved but it is, nevertheless, not hard to see how the coupling-constant renormalization works to first order. It follows from Eq. (112) that $\tilde{D}_B^1$ may be written in Fourier space in the form

$$\lambda_0{}^2\tilde{D}_B^1(p) = -2\lambda_0 \frac{1}{1 - \lambda_0 N\Sigma(p^2, m_R^1) - \lambda_0[\lambda_R^0 A(p^2, m_R^1) + (\lambda_R^0)^2 B(p^2, m_R^1)]}, \tag{113}$$

where λ_R^0 and m_R^0 are the zeroth-order renormalized quantities. m_R^1 has first-order corrections relative to m_R^0, so to first-order accuracy we have freely interchanged these masses in terms already of first order. One subtraction makes the denominator finite, so we have

$$\begin{aligned}\lambda_0{}^2\tilde{D}_B^1(p) &= \frac{(\lambda_R^1)^2}{N}\frac{-2}{\lambda_R^1}\frac{1}{1 - \lambda_R^1\Sigma_R(p^2, m_R^1) - (\lambda_R^1)/N)[\lambda_R^1 A_{\mathrm{sub}}(p^2, m_R^1) + (\lambda_R^1)^2 B_{\mathrm{sub}}(p^2, m_R^1)]} \\ &\equiv \frac{(\lambda_R^1)^2}{N}\tilde{D}_R^1(p).\end{aligned} \tag{114}$$

Here we define A_{sub} and B_{sub} as A and B subtracted at $p^2 = 0$ and

$$\lambda_R^1 \equiv \frac{\lambda_0 N}{1 - \lambda_0 N\Sigma(0) - \lambda_0[(\lambda_R^0)^2 A(0) + (\lambda_R^0)^4 B(0)]}.$$

We have also changed λ_R^0 to λ_R^1 wherever it occurs in a quantity that already is of first order.

The above is not a complete analysis of this theory even to first order, but it should now be clear to the reader that the renormalization program can be carried through a reasonably straightforward fashion.

As the conclusion to this discussion of higher corrections, we will examine the internal consistency of the lowest approximation. The fundamental assumption of the RPA was that to lowest order the term $\langle\phi^2\phi\rangle/\langle|\rangle$ in the vacuum expectation of the field equations could be replaced by $(\langle\phi^2\rangle/\langle|\rangle)(\langle\phi\rangle/\langle|\rangle)$. On the other hand, as an exact relation of the theory we proved (14a), which is equivalently written as

$$\left[\frac{i\delta}{\delta J(1)}\right]^2 W(2) = -\frac{i\delta}{\delta S(1)}W(2) + 2iW(21)\cdot W(1). \tag{115}$$

Since the left-hand side of this equation contains a contribution of the form $\langle[\phi(1)^2\phi(2)]\rangle/\langle|\rangle$, which is the term on which the basic approximation is made, we would not expect (115) to be exact for the lowest-order RPA Green's functions. Consequently, instead we write

$$\mathrm{tr}_1 W^0(211) = +iW^0(2\hat{1}) - 2iW^0(21)\cdot W^0(1) + iC(12), \tag{116}$$

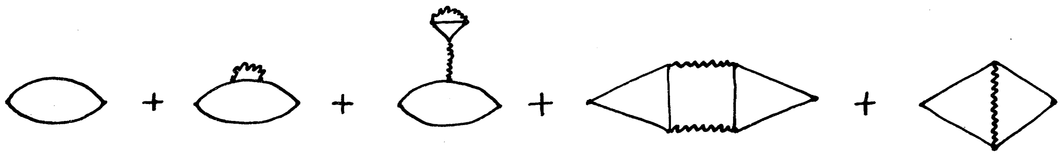

FIG. 12. Graphs contributing to D_B^{-1} to first order.

where $C(12)$ is the correction to the exact relationship (115) due to the approximation. We used (115) for many manipulations in the lowest-order theory. Consequently, the term $C(12)$ must be a higher-order correction in our approximation scheme or these manipulations were meaningless. The proof of this is already implicit in the preceding, but to make it explicit we will determine $C(12)$ directly. We will use the equation obtained by differentiating (116) with respect to $J(2)$, which is

$$\mathrm{tr}_1 W^0(2211) = iW^0(22\hat{1}) - 2iW^0(221)\cdot W^0(1) - 2iW^0(21)\cdot W^0(12) + iC(12,2). \quad (117)$$

If we write equation (19b) in the form

$$O(1)W^0(1) = J(1), \quad (118)$$

we find taking the derivative of this with respect to $S(2)$ that

$$O(1)W^0(1\hat{2}) + \frac{\lambda_0}{i}[\mathrm{tr}_1 W^0(11\hat{2})]W^0(1) + 2\lambda_0[W^0(\hat{2}1)\cdot W^0(1)]W^0(1) - 2W^0(1)\delta(1-2) = 0. \quad (119)$$

Taking the derivative of (118) with respect to $J(2)$ yields

$$O(1)W^0(12) + \frac{\lambda_0}{i}[\mathrm{tr}_1 W^0(112)]W^0(1) + 2\lambda_0[W^0(21)\cdot W^0(1)]W^0(1) = \delta(1-2). \quad (120)$$

Differentiating (120) once again and taking the trace results in

$$O\,\mathrm{tr}_2 W^0(122) + \frac{2\lambda_0}{i}[\mathrm{tr}_1 W^0(112)]W^0(21) + 4\lambda_0 W^0(12)W^0(21)\cdot W^0(1) + \frac{\lambda_0}{i}[\mathrm{tr}_{1,2}W^0(1122)]W^0(1) + 2\lambda_0[\mathrm{tr}_2 W^0(221)]\cdot W^0(1)W^0(1) + 2\lambda_0\{\mathrm{tr}_2[W^0(21)W^0(12)]\}W^0(1) = 0. \quad (121)$$

We simplify Eq. (121) by using Eqs. (116), (117), (119), and (120) and setting $1=2$ to find

$$\left[O(1) + \frac{2\lambda_0}{i}W^0(11) + 2\lambda_0 W^0(1)W^0(1)\cdot + \frac{\lambda_0}{i}W^0(1)\frac{\delta}{\delta J(1)}\right]C(11) = -\frac{2\lambda_0}{i}\left[\frac{\delta}{\delta S(1)}W^0(1)\right]\cdot W^0(11) = \frac{2\lambda_0^2}{i}\left[\int d2\, W_0(12)W^0(\hat{2}\hat{1})W^0(2)\right]\cdot W^0(11). \quad (122)$$

The last equality in (122) is obtained by the use of Eq. (35). We thus conclude that $C(11)$ is a first-order correction as conjectured.

XI. SUMMARY AND CONCLUSIONS

We have extensively examined the RPA in lowest order and developed a method for generating higher-order corrections through iterations of the lowest-order results. The results of this method strikingly differ from ordinary perturbation theory in that they are dominated by a two-field induced bound state in such a way that the results have a similar topological structure to normal theories of interaction form $\phi^2\chi$, where ϕ and χ are independent fields. We have demonstrated to lowest and first order (with some fillable gaps) that the theory is renormalizable. Although we have not studied this problem, it should be possible to demonstrate renormalizability to all orders by use of Eqs. (101), (103), and (104), and an additional equation for $W(1234)$. We have shown that this theory is related to the $O(N)$ expansion in the literature although it is not necessary to take N large to generate our results. We have not seriously examined the order-by-order convergence of the theory for N large or small. We have, however, demonstrated that with no symmetry breaking the theory to lowest order is ghost-free for a range of renormalized coupling constant. Incidentally, it is also easy to show that in lowest order this theory is asymptotically free.

Finally, we point out that the techniques used here are applicable to a large range of Lagrangians and that we have just examined the simplest and hence probably the least physical problem.

Note added. Some time after this work was completed, a paper by L. F. Abbott, J. S. Kang, and H. J. Schnitzer [Phys. Rev. D. 13, 2212 (1976)] was called to our attention which through effective potential methods reached similar conclusions about O(N) having a region without tachyons when there is no symmetry breaking. Papers by M. Kobayashi and T. Kugo, [Prog. Theor. Phys. 54, 1537 (1975)] and R. W. Haymaker [Phys. Rev. D 13, 968 (1976)] also study aspects of this problem.

Note added. The reader should not be alarmed by the discussion following Eq. (58) in which it is observed that λ_R must be negative for a tachyon-free theory. The sign of λ_R required for a consistent theory is entirely an artifact of our subtraction procedure ($k^2=0$) and can be made positive by another choice of subtraction point. It is the quantity $\lambda_R{}^2\tilde{D}$ that is renormalization-group invariant and hence independent of any subtraction procedure. In fact, λ_R is really an inappropriate parameter with which to characterize the theory produced here which we emphasize is inequivalent to that produced by a normal perturbation expansion in λ_0 (or a normal loop expansion). This is made clear by the fact that no vestige of the original ϕ^4 contact interaction remains in the renormalized scattering amplitude or any other process to any order.

All interactions appearing in the renormalized theory are trilinear ($\phi^2\chi$) and a more appropriate expansion parameter is clearly the coupling constant associated with this trilinear interaction. This is easily displayed by subtracting the quantity $\lambda_0{}^2D$ at the physical bound-state mass, μ^2. We then find that $\lambda_0{}^2D$ can be cast into the form

$$\lambda_0{}^2D(k^2)=g^2H(k^2),$$

where

$$16\pi^2g^2\equiv\frac{1}{[\pi^2/(2\pi)^4]\int d\kappa^2\rho(\kappa^2)/(\kappa^2-M_B{}^2)^2}$$

and

$$H(k^2)\equiv\frac{1}{(M_B{}^2-k^2)[1-g^2\pi(k^2)]}.$$

Clearly g may be identified as the renormalized $\phi^2\chi$ coupling constant, and we note that g^2 has the structure

$$g^2=M_B{}^2d(x),$$

where

$$d(x)=\frac{1}{1-\{1/[x(1-x)]^{1/2}\}\tan^{-1}[x/(1-x)]^{1/2}}$$

and $d(x)$ is a dimensionless function of $x=M_B{}^2/4m^2$. It is easy to verify that $d(x)$ varies monotonically from 0 to $+\infty$ as x varies from +1 to 0. Thus for the weak-binding case ($M_B{}^2\sim4m^2$, $x\sim1$) we see that this theory may be characterized by a small dimensionless parameter quite independently of the size of N.

Note added. A simple and elegant way to determine the order of any graph in this expansion without counting powers of N directly or following the detailed prescription of Sec. X is to just count the minimum number of integrations needed to integrate over the bound-state propagators. The order of the graph is the number of such integrations, and the expansion may therefore be termed a bound-state loop expansion. Such a loop expansion has already been implied by Eq. (25), which tells us that we are, in fact, making an expansion in fluctuations of ϕ^2, i.e., ϕ^2 loops. This is implemented, functionally, by the operation $\delta/\delta S$, which generates ϕ^2 loops.

ACKNOWLEDGMENTS

Most of the research reported here was completed while one of us (G.S.G.) was the guest of Los Alamos Scientific Laboratory. One of us (F.C.) enjoyed conversations on the Hartree model with R. W. Haymaker. We would also like to thank Neal Snyderman for useful conversations.

APPENDIX: FORMAL RELATIONS

From $\chi=\phi^2-2S/\lambda_0$ we obtain

$$W(\hat{1})=\frac{1}{i}W(11)+\Phi_c{}^2(1)-2S/\lambda_0, \tag{A1}$$

$$W(\hat{1}2)=\frac{1}{i}W(112)+2\Phi_c(1)W(12), \tag{A2}$$

$$W(\hat{1}\hat{2})=\frac{1}{i}W(11\hat{2})+2\Phi_c(1)W(1\hat{2})-\frac{2\delta(1-2)}{\lambda_0}, \tag{A3}$$

$$W(\hat{1}\hat{2}\hat{3})=\frac{1}{i}W(11\hat{2}\hat{3})+2W(1\hat{3})W(1\hat{2})+2\Phi_c(1)W(1\hat{2}\hat{3}), \tag{A4}$$

$$W(\hat{1}\hat{2}\hat{3}\hat{4})=\frac{1}{i}W(11\hat{2}\hat{3}\hat{4})+2[W(1\hat{3}\hat{4})W(1\hat{2})+W(1\hat{3})W(1\hat{2}\hat{4})]+2W(1\hat{4})W(1\hat{2}\hat{3})+2\Phi_c(1)W(1\hat{2}\hat{3}\hat{4}). \tag{A5}$$

From

$$W^{-1}(78)=[\Box_7{}^2+m_0{}^2+\lambda_0 W(\hat{7})]\delta(7-8)+\lambda_0\Phi_c(7)\int W(\hat{7}9)W^{-1}(98)d9,$$

we obtain when $J=S=0$ and $\langle\Phi\rangle=0$

$$\frac{\delta W^{-1}(78)}{\delta S(2)}=\lambda_0 W(\hat{7}\hat{2})\delta(7-8), \qquad (A6)$$

$$\frac{\delta W^{-1}(78)}{\delta S(2)\delta S(3)}=\lambda_0 W(\hat{7}\hat{2}\hat{3})\delta(7-8), \qquad (A7)$$

$$\frac{\delta W^{-1}(78)}{\delta S(2)\delta S(3)\delta S(4)}=\lambda_0 W(\hat{7}\hat{2}\hat{3}\hat{4})\delta(7-8), \qquad (A8)$$

$$\frac{\delta W^{-1}(78)}{\delta J(3)\delta J(4)}=\lambda_0 W(\hat{7}34)\delta(7-8)+\lambda_0 W(73)\int W(\hat{7}94)W^{-1}(98)d9+\lambda_0 W(74)\int W(\hat{7}93)W^{-1}(98)d9, \qquad (A9)$$

$$\frac{\delta W^{-1}(78)}{\delta J(3)\delta J(4)\delta J(5)}=\lambda_0 W(\hat{7}34\hat{5})\delta(7-8)+\lambda_0 W(73\hat{5})\int W(\hat{7}94)W^{-1}(98)d9+\lambda_0 W(745)\int W(\hat{7}93)W^{-1}(98)d9, \qquad (A10)$$

$$\begin{aligned}\frac{\delta W^{-1}(78)}{\delta J(3)\delta J(4)\delta J(5)\delta J(6)}=&\lambda_0\delta(7-8)W(\hat{7}3456)+\lambda_0 W(7345)\int W(\hat{7}96)W^{-1}(98)d9+\lambda_0 W(7356)\int W(\hat{7}94)W^{-1}(98)d9\\&+\lambda_0 W(7456)\int W(\hat{7}93)W^{-1}(98)d9\\&+\lambda_0 W(76)\int d9\left\{W(\hat{7}9345)W^{-1}(98)+W(\hat{7}93)\left[\frac{\delta^2 W^{-1}(98)}{\delta J(4)\delta J(5)}\right]+W(\hat{7}94)\left[\frac{\delta^2 W^{-1}(98)}{\delta J(3)\delta J(4)}\right]\right.\\&\left.+W(\hat{7}95)\left[\frac{\delta^2 W^{-1}(98)}{\delta J(3)\delta J(5)}\right]\right\}\\&+\lambda_0 W(73)\int d9(345\to456)+\lambda_0 W(74)\int d9(345\to356)+\lambda_0 W(75)\int d9(345\to346). \qquad (A11)\end{aligned}$$

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*Work supported by the Energy Research and Development Administration.

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