Existence of a phase transition in the $(\phi^4)_3$ quantum field theory*

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We show that a properly renormalized $(\phi^4)_3$ field theory undergoes a phase transition from a vacuum with broken symmetry to a vacuum with manifest symmetry as the ϕ^4 coupling strength is increased with the mass parameter held fixed. The same behavior is exhibited by the O(N) model with $(\vec{\phi}^2)^2$ coupling in three space-time dimensions.

I. INTRODUCTION

In a recent paper,[1] Chang has studied the stability of the vacuum in the $(\phi^4)_2$ field theory defined by the Lagrangian density

$$\mathfrak{L} = :[\tfrac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{4}g\phi^4]:,$$

where : : denotes normal ordering. He showed that the vacuum undergoes a second-order phase transition from the symmetric to the unsymmetric state as g is increased with m fixed. In his arguments he used the fact that the $(\phi^4)_2$ theory can be renormalized simply by normal-ordering the Hamiltonian.

The purpose of this paper is to extend Chang's proof of a phase transition to include the $(\phi^4)_3$ field theory. We show that the vacuum in this theory goes from the unsymmetric to the symmetric state as g is increased with m fixed. We show that this result remains essentially unchanged if the Lagrangian is given an internal, O(N) symmetry. We are unable to determine the order of the transition because the Simon-Griffiths theorem[2] does not apply.

The paper is organized as follows: In Sec. II we formulate Chang's argument for a phase transition in the $(\phi^4)_2$ theory, using a cutoff-dependent mass counterterm instead of normal ordering. In Sec. III we show how to extend the argument for the case of the $(\phi^4)_3$ field theory, with or without internal symmetry. In Sec. IV we use the results of Sec. III to find expansions for the effective potential which are valid for small and for very large values of the coupling constant. In Sec. V we mention some interesting problems which we have encountered.

II. REFORMULATION OF CHANG'S ARGUMENT

Consider the following Lagrangian densities in one space dimension and one time dimension:

$$\mathfrak{L}_1 = \tfrac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{4}g\phi^4 + \tfrac{1}{2}\delta m^2\phi^2, \quad (2.1)$$

$$\mathfrak{L}_2 = \tfrac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) + \tfrac{1}{4}\mu^2\phi^2 - \tfrac{1}{4}g\phi^4 + \tfrac{1}{2}\delta\mu^2\phi^2. \quad (2.2)$$

The theory defined by $\mathfrak{L}_1$ will be finite in perturbation theory if δm^2 contains all primitively divergent self-energy graphs. In the theory defined by $\mathfrak{L}_1$, there is only one such graph (see Fig. 1). It can be expressed in terms of a momentum cutoff, Λ, by

$$\delta m^2 = 3g\int \frac{d^2p}{(2\pi)^2}\left(\frac{i}{p^2-m^2+i\epsilon} - \frac{i}{p^2-\Lambda^2+i\epsilon}\right) = \frac{3g}{4\pi}\ln\left(\frac{\Lambda^2}{m^2}\right). \quad (2.3)$$

The use of this counterterm is equivalent to normal-ordering $\mathfrak{L}_1$ with respect to the mass, m. To see this, note that

$$:\phi^4: = \phi^4 - 6\langle m|\phi^2|m\rangle\phi^2 + 3(\langle m|\phi^2|m\rangle)^2, \quad (2.4)$$

where $|m\rangle$ is a free-field state of mass m, and m is a mass which defines the normal-ordering operation. The expectation value in Eq. (2.4) can be evaluated explicitly in terms of a momentum cutoff, Λ, giving

$$:\phi^4: = \phi^4 - \frac{6}{4\pi}\ln\left(\frac{\Lambda^2}{m^2}\right)\phi^2 + \text{constant}, \quad (2.5)$$

or

$$\frac{g}{4}:\phi^4: = \frac{g}{4}\phi^4 - \frac{1}{2}\left(\frac{3g}{4\pi}\right)\ln\left(\frac{\Lambda^2}{m^2}\right)\phi^2 + \text{constant}. \quad (2.6)$$

So, normal-ordering the quartic term is equivalent to using the mass counterterm, δm^2. Normal-ordering the quadratic terms is equivalent to subtracting a constant. Therefore, putting the counterterm in $\mathfrak{L}_1$ is equivalent to normal-ordering $\mathfrak{L}_1$ without the counterterm.

Since δm^2 is defined in terms of the mass m, perturbation theory will give an expansion for the effective potential[3] in terms of both g/m^2 and the classical field, ϕ_c. Because of the symmetry

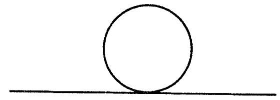

FIG. 1. The only primitively divergent self-energy graph in the $(\phi^4)_2$ field theory.

under the transformation $\phi_c \to -\phi_c$, the effective potential, $V(\phi_c)$, only depends on even powers of ϕ_c. Therefore, near $\phi_c = 0$ we have

$$V(\phi_c) = \tfrac{1}{2}m^2\phi_c{}^2 + O(m^2\phi^4, g\phi_c{}^2). \tag{2.7}$$

Perturbation theory clearly implies that a local minimum of the effective potential occurs at $\phi_c = 0$ for some range of small values of g/m^2. This tells us that, if the absolute minimum is near $\phi_c = 0$, it must be at $\phi_c = 0$. We know that, when $g = 0$, $V(\phi_c)$ has its absolute minimum at $\phi_c = 0$. Under the assumptions that $\langle\phi\rangle$ is near zero for small values of g/m^2 and that perturbation theory is valid when g/m^2 and $\phi_c{}^2$ are small, we can conclude that $\langle\phi\rangle = 0$ for a finite range of small values of g/m^2 in the ground state of the theory defined by $\mathcal{L}_1$.

Before applying perturbation theory to $\mathcal{L}_2$, it is natural to shift the field to minimize the classical potential. We make the substitution

$$\phi = \phi' + \left(\frac{1}{2}\,\frac{\mu^2}{g}\right)^{1/2}. \tag{2.8}$$

In terms of ϕ', $\mathcal{L}_2$ is given by

$$\mathcal{L}_2 = \tfrac{1}{2}(\partial_\mu\phi')(\partial^\mu\phi') - \tfrac{1}{2}\mu^2\phi'^2 - \mu(\tfrac{1}{2}g)^{1/2}\phi'^3 - \tfrac{1}{4}g\phi'^4 - \tfrac{1}{2}\delta\mu^2[\phi'^2 + \mu(\tfrac{1}{2g})^{1/2}\phi']. \tag{2.9}$$

Because of the cubic term, $\mathcal{L}_2(\phi')$ creates divergent self-energy graphs which are not created by $\mathcal{L}_1(\phi)$. However, these new ϕ^3-type divergent graphs are related to the ϕ^4-type graphs. If we choose $\delta\mu^2$ so that the quadratic part of the counterterm cancels the ϕ^4-type divergences, then the linear part of the counterterm will automatically cancel the ϕ^3-type divergences. Therefore, $\mathcal{L}_2(\phi')$ will be finite if $\delta\mu^2$ has the same cutoff dependence as δm^2. We evaluate $\delta\mu^2$ using the classical mass (namely, μ) of the ϕ'field. We have

$$\delta\mu^2 = \frac{3g}{4\pi}\ln\left(\frac{\Lambda^2}{\mu^2}\right). \tag{2.10}$$

If $\delta\mu^2$ is defined in this way, then perturbation theory is applicable for small g/μ^2. By arguments similar to those made for $\mathcal{L}_1$, we conclude that

$$\langle\phi'\rangle = O(g/\mu^2) \tag{2.11}$$

for the ground state of the theory defined by $\mathcal{L}_2(\phi')$. [We cannot say that $\langle\phi'\rangle$ is exactly zero because in this case $V(\phi_c')$ contains terms which are linear in ϕ_c'.] Therefore, if g/u is small, $\mathcal{L}_2(\phi)$ describes a theory with

$$\langle\phi\rangle = (\tfrac{1}{2}\mu^2/g)^{1/2} + O(g/\mu^2) \tag{2.12}$$

in the ground state.

The two Lagrangians are written in a way which suggests that they describe different systems. However, this is not necessarily true, because a change in the mass parameter can be compensated by a finite change in the counterterm. In fact, the two Lagrangians are identical if

$$-m^2 + \delta m^2 = \tfrac{1}{2}\mu^2 + \delta\mu^2. \tag{2.13}$$

Using Eqs. (2.3) and (2.10), we can rewrite this condition as

$$-\frac{m^2}{g} + \frac{3}{4\pi}\ln\left(\frac{\Lambda^2}{m^2}\right) = \frac{1}{2}\frac{\mu^2}{g} + \frac{3}{4\pi}\ln\left(\frac{\Lambda^2}{\mu^2}\right), \tag{2.14}$$

or

$$\frac{m^2}{g} + \frac{3}{4\pi}\ln\left(\frac{m^2}{\mu^2}\right) = -\frac{\mu^2}{2g}. \tag{2.15}$$

This is precisely the equation which Chang obtained in Ref. 1, using a normal-ordering identity. He used this equation to show the existence of a phase transition in the $(\phi^4)_2$ theory.

III. EXTENSION TO THREE-DIMENSIONAL ϕ^4 THEORY

Now it is straightforward to generalize the argument for the existence of a phase transition to the $(\phi^4)_3$ theory. Consider the following Lagrangians in three space-time dimensions:

$$\mathcal{L}_3 = \tfrac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{4}g\phi^4 + \tfrac{1}{2}\delta m^2\phi^2, \tag{3.1}$$

$$\mathcal{L}_4 = \tfrac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) + \tfrac{1}{4}\mu^2\phi^2 - \tfrac{1}{4}g\phi^4 + \tfrac{1}{2}\delta\mu^2\phi^2. \tag{3.2}$$

In the theory defined by $\mathcal{L}_3$, there are two primitively divergent self-energy graphs (see Fig. 2). Call them $\Sigma_a(m^2)$ [Fig. 2(a)] and $\Sigma_b(k^2,m^2)$ [Fig. 2(b)]. They may be expressed in terms of a momentum cutoff, Λ, as follows:

$$\Sigma_a(m^2) = 3g\int\frac{d^3p}{(2\pi)^3}\left(\frac{i}{p^2 - m^2 + i\epsilon} - \frac{i}{p^2 - \Lambda^2 + i\epsilon}\right) = \frac{3g}{4\pi}(\Lambda - m), \tag{3.3}$$

$$\Sigma_b(k^2,m^2) = -6ig^2\int\frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\left[\left(\frac{i}{p^2 - m^2 + i\epsilon}\right)\left(\frac{i}{q^2 - m^2 + i\epsilon}\right)\left(\frac{i}{(k-p-q)^2 - m^2 + i\epsilon}\right) - (m^2 \rightleftharpoons \Lambda^2)\right]. \tag{3.4}$$

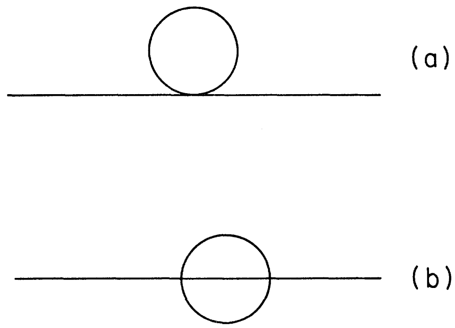

FIG. 2. The primitively divergent self-energy graphs in the $(\phi^4)_3$ field theory.

Since the divergence in $\Sigma_b(k^2,m^2)$ is only logarithmic, one subtraction at any value of k^2 makes it finite. Therefore, the theory will be finite if we use $\Sigma_b(0,m^2)$ in the counterterm. We define δm^2 by

$$\delta m^2 = \Sigma_a(m^2) + \Sigma_b(0,m^2) = \frac{3g}{4\pi}(\Lambda - m) + \frac{3g^2}{8\pi^2}\ln\left(\frac{m}{\Lambda}\right). \quad (3.5)$$

The counterterm in $\mathcal{L}_4$ is defined by

$$\delta\mu^2 = \frac{3g}{4\pi}(\Lambda - \mu) + \frac{3g^2}{8\pi^2}\ln\left(\frac{\mu}{\Lambda}\right). \quad (3.6)$$

We can use the same arguments we used for the $(\phi^4)_2$ theory to derive properties of the theories defined by $\mathcal{L}_3$ and $\mathcal{L}_4$ in their respective weak-coupling limits. For a finite range of small values of g/μ, the vacuum associated with $\mathcal{L}_3$ is normal, with $\langle\phi\rangle = 0$. In the weak-coupling limit of $\mathcal{L}_4$, the vacuum expectation value of the field is given by

$$\langle\phi\rangle = (\tfrac{1}{2}\mu^2/g)^{1/2} + O(g/\mu). \quad (3.7)$$

The two Lagrangians are identical if

$$-m^2 + \delta m^2 = \tfrac{1}{2}\mu^2 + \delta\mu^2. \quad (3.8)$$

Using Eqs. (3.5) and (3.6) we can express (3.8) as

$$\left(\frac{m}{g}\right)^2 + \frac{1}{2}\left(\frac{\mu}{g}\right)^2 + \frac{3}{4\pi}\left(\frac{m}{g} - \frac{\mu}{g}\right) - \frac{3}{8\pi^2}\ln\left(\frac{m}{\mu}\right) = 0. \quad (3.9)$$

The solution to this equation is shown graphically in Fig. 3. It has the following properties:

1. If $g/\mu < 17.70$, there is no solution.
2. If $g/\mu = 17.70$, there is a unique solution with g/m given exactly by

$$\frac{g}{m} = \frac{16\pi}{\sqrt{57} - 3} \approx 11.05. \quad (3.10)$$

3. If $g/\mu > 17.70$, there are two solutions.
4. If $g/\mu \gg 1$, the two solutions are

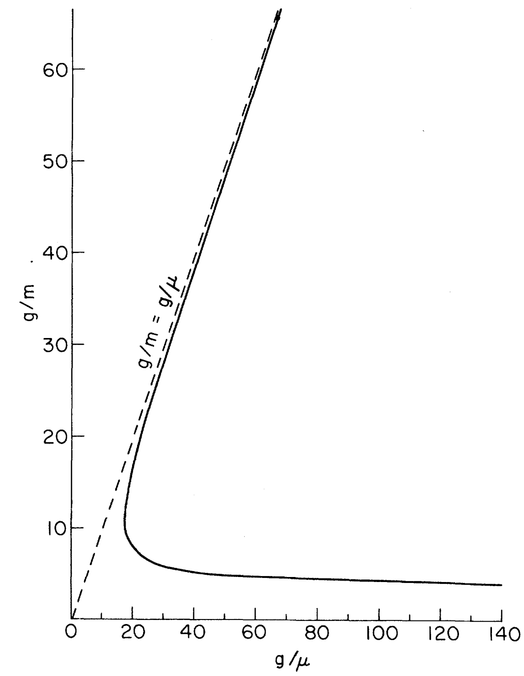

FIG. 3. Invariant coupling g/m as a function of g/μ. The functional relation is defined in Eq. (3.9).

$$\frac{g}{m} = \frac{g}{\mu} + O(1), \quad (3.11)$$

$$\frac{g}{m} = \left[\frac{3}{8\pi^2}\ln\left(\frac{g}{\mu}\right) + O\left(\ln\left(\ln\frac{g}{\mu}\right)\right)\right]^{-1/2}. \quad (3.12)$$

Equation (3.12) tells us that, when $g/\mu \to \infty$, one solution of (3.8) has $g/m \to 0$. This means that a strong-coupling theory defined by $\mathcal{L}_4$ is equivalent to a weak-coupling theory defined by $\mathcal{L}_3$. In particular, the strong-coupling theory defined by $\mathcal{L}_4$ has the same vacuum as the weak-coupling theory defined by $\mathcal{L}_3$. This is the "normal" vacuum, with $\langle\phi\rangle = 0$. On the other hand, when the coupling in $\mathcal{L}_4$ is weak, $\mathcal{L}_4$ defines a theory with an "abnormal" vacuum. In this vacuum

$$\langle\phi\rangle = (\tfrac{1}{2}\mu^2/g)^{1/2} + O(g/\mu) \neq 0.$$

As g/μ is increased from 0 to ∞, there must be a transition from the abnormal to the normal vacuum in the theory defined by $\mathcal{L}_4$.

In one sense, the behavior of the $(\varphi^4)_2$ theory is opposite to the behavior of the $(\phi^4)_3$ theory. In the $(\phi^4)_2$ theory, the vacuum is abnormal (i.e., $\langle\phi\rangle \neq 0$) in the strong-coupling limit, regardless of whether the theory is described by $\mathcal{L}_1$ or $\mathcal{L}_2$. [Equation (2.18) tells us that the two Lagrangians are equivalent when g/m and g/μ both go to infinity.] In the $(\phi^4)_3$ theory, the vacuum is normal when the coupling is large, regardless of whether the theory is described with $\mathcal{L}_3$ or $\mathcal{L}_4$.

In another sense, the $(\phi^4)_2$ theory and the $(\phi^4)_3$ theory behave quite similarly. Each theory has two overlapping descriptions. Either we can describe the theory with m and $\mathcal{L}_1$ or $\mathcal{L}_3$ or we can describe the theory with μ and $\mathcal{L}_2$ or $\mathcal{L}_4$. In the range of couplings where both descriptions apply, the two descriptions are inversely related. Increasing m is equivalent to decreasing μ and vice versa. Roughly speaking, the region in which m is large compared with g is the manifest-symmetry phase, while the region in which μ is large compared with g is the broken-symmetry phase.

The argument for the existence of a phase transition is easily generalized to the O(N) model[4] in two space dimensions and one time dimension. The Lagrangians of interest are the following:

$$\mathcal{L}_5=\tfrac{1}{2}(\partial_\mu\phi_i)(\partial^\mu\phi_i)-\tfrac{1}{2}m^2\phi_i\phi_i -\tfrac{1}{4}g(\phi_i\phi_i)^2+\tfrac{1}{2}\delta m^2\phi_i\phi_i, \tag{3.13}$$

$$\mathcal{L}_6=\tfrac{1}{2}(\partial_\mu\phi_i)(\partial^\mu\phi_i)+\tfrac{1}{4}\mu^2\phi_i\phi_i -\tfrac{1}{4}g(\phi_i\phi_i)^2+\tfrac{1}{2}\delta\mu^2\phi_i\phi_i. \tag{3.14}$$

The repeated Latin indices are meant to be summed from 1 to N. The required counterterms are the following:

$$\delta m^2=\frac{(N+2)g}{4\pi}(\Lambda-m)+\frac{(N+2)g^2}{8\pi^2}\ln\left(\frac{m}{\Lambda}\right), \tag{3.15}$$

$$\delta\mu^2=\frac{(N+2)g}{4\pi}(\Lambda-\mu)+\frac{(N+2)g^2}{8\pi^2}\ln\left(\frac{\mu}{\Lambda}\right). \tag{3.16}$$

The two Lagrangians, $\mathcal{L}_5$ and $\mathcal{L}_6$, are equivalent if the two masses, m and μ, satisfy the following equation:

$$\left(\frac{m}{g}\right)^2+\frac{1}{2}\left(\frac{\mu}{g}\right)^2+\frac{N+2}{4\pi}\left(\frac{m}{g}-\frac{\mu}{g}\right)-\frac{N+2}{8\pi^2}\ln\left(\frac{m}{\mu}\right)=0. \tag{3.17}$$

This equation has a solution with the property that $g/m\to 0$ when $g/\mu\to\infty$. However, this is not enough to show the existence of a phase transition.[5] We also need to show that the following statements are true:

1. If g/m is small enough, then the theory defined by $\mathcal{L}_5$ has the property that the ground-state expectation value of each field, ϕ_i, is exactly zero.

2. If g/μ is small enough, then, for at least one value of i, $\langle\phi_i\rangle\neq 0$ in the ground state of the theory defined by $\mathcal{L}_6$.

Statement 1 is true by the same argument we used for the $(\phi^4)_2$ and $(\phi^4)_3$ theories. Statement 2 cannot be proved by a naive application of perturbation theory. This is because Goldstone bosons appear[6] and create infrared divergences. Coleman has shown that statement 2 is actually false when applied to the O(N) model in two space-time dimensions.[7] Drastic infrared divergences make the appearance of Goldstone bosons impossible in that model. However, statement 2 has been proved to be true for the O(N) model in three space-time dimensions by Fröhlich, Simon, and Spencer.[8] Therefore, the argument for a phase transition works for the O(N) model in three space-time dimensions. In the theory defined by $\mathcal{L}_6$, if g/μ is small enough, the vacuum will not possess O(N) symmetry. [By O(N) symmetry, we mean symmetry under the rotation group of the N fields.] As g/μ is increased, the vacuum will become O(N) symmetric at some finite value of g/μ.

IV. THE EFFECTIVE POTENTIAL

Using the connection between $\mathcal{L}_3$ and $\mathcal{L}_4$, we can now use perturbation theory for the $(\phi^4)_3$ theory in both the weak-coupling and strong-coupling limits.[9] As an example, we will calculate the effective potential for the theory described by Lagrangian $\mathcal{L}_4$, and use it to find $\Delta_F^{-1}(0)$ (the inverse propagator at zero momentum) as a function of μ. We will do this for the strong-coupling as well as the weak-coupling case.

For the weak-coupling case, we simply use the loop expansion. The effective potential is given by

$$V(\phi_c)=\frac{g}{4}\phi_c^4-\frac{\mu^2}{4}\phi_c^2+\frac{3g\mu}{8\pi}\phi_c^2 -\frac{1}{12\pi}\left(3g\phi_c^2-\frac{\mu^2}{2}\right)^{3/2} \tag{4.1}$$

plus two loop corrections. The inverse propagator, $\Delta_F^{-1}(0)$, can be found from $V(\phi_c)$ by

$$-\Delta_F^{-1}(0)=\left.\frac{d^2V(\phi_c)}{d\phi_c^2}\right|_{dV/d\phi_c=0} =\mu^2-\frac{9g\mu}{8\pi}+O(g^2). \tag{4.2}$$

If g/μ is very large, then $\mathcal{L}_4$ is equivalent to $\mathcal{L}_3$ if we make the identification [from Eq. (3.12)]

$$m^2=\frac{3g^2}{8\pi^2}\ln\left(\frac{g}{\mu}\right)+O\left(g^2\ln\left(\ln\frac{g}{\mu}\right)\right). \tag{4.3}$$

When the theory is written in terms of $\mathcal{L}_3$, g/m is very small. Therefore, we can use the loop expansion for $\mathcal{L}_3$ to get

$$V(\phi_c)=\frac{g}{4}\phi_c^4+\frac{m^2}{2}\phi_c^2+\frac{3gm}{8\pi}\phi_c^2 -\frac{1}{12\pi}(m^2+3g\phi_c^2)^{3/2} \tag{4.4}$$

plus two loop corrections. In terms of μ, for large g/μ, this can be written as

$$V(\phi_c)=\frac{g}{4}\phi_c{}^4+\frac{3g^2\phi_c{}^2}{16\pi^2}\ln\left(\frac{g}{\mu}\right)+\frac{3g}{8\pi}\phi_c{}^2\left[\frac{3g^2}{8\pi^2}\ln\left(\frac{g}{\mu}\right)\right]^{1/2}$$
$$-\frac{1}{12\pi}\left[\frac{3g^2}{8\pi^2}\ln\left(\frac{g}{\mu}\right)+3g\phi_c{}^2+O\left(g^2\ln\left(\ln\frac{g}{\mu}\right)\right)\right]^{3/2}$$
$$+O\left(g^2\ln\left(\ln\frac{g}{\mu}\right)\right). \tag{4.5}$$

Therefore, for large g/μ, we have

$$-\Delta_F{}^{-1}(0)=\frac{3g^2}{8\pi^2}\ln\left(\frac{g}{\mu}\right)+O\left(g^2\ln\left(\ln\frac{g}{\mu}\right)\right). \tag{4.6}$$

V. DISCUSSION

The phase transition in the $(\phi^4)_2$ field theory can be demonstrated by a variational calculation in which the trial states are free-field vacuum states.[10] The parameters which are varied are the mass and the expectation value of the free field. A similar variational calculation for the $(\phi^4)_3$ field theory yields nonsensical results. The expectation value of the field is predicted to be zero, regardless of the value of g/μ. The physical mass of the system is predicted to be infinite.

The reason why free-field vacuum trial states do not work in a variational calculation in $(\phi^4)_3$ theory is that carrying out such a variational calculation is equivalent to calculating the effective potential by summing the cactus graphs.[11] In $(\phi^4)_3$ theory, there is a primitively divergent graph [Fig. 2(b)] which is not a cactus graph. This graph must be included in the mass counterterm. Since the simple variational calculation does not include this graph, the cutoff dependence in the counterterm is not completely canceled. As a result, the counterterm completely dominates the effective potential. So far, we have been unable to find trial states which include these divergent graphs and which are simple enough to handle in an exact variational calculation.

It is important to study the existence of a phase transition in the $(\phi^4)_4$ field theory. However, the techniques used in this paper do not have an obvious generalization to three space dimensions and one time dimension. The difficulty is that the $(\phi^4)_4$ theory has an infinite number of primitively divergent self-energy graphs. There are primitively divergent graphs of every order in g. As a result, the mass counterterm cannot easily be written as a closed-form expression involving a momentum cutoff.

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