# Chiral gauge symmetry without anomalies

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(Received 24 December 1975)

The Ward identities derived by use of chiral gauge symmetry, when examined in a perturbation expansion using Lagrangian field theory, are found to contain the Adler-Bell-Jackiw-Schwinger anomalies. The cause of this shortcoming is examined, and the result is that when one uses a formulation of field theory in which only fields already renormalized are introduced these anomalies disappear completely. The specific cases examined here are massless two-dimensional electrodynamics with a vector coupling and massless four-dimensional electrodynamics with an axial-vector coupling. In the first example, it is also found that the polarization tensor disappears in second order, and does not exhibit a pole at zero momentum transfer that would cause the vector field to generate a mass spontaneously.

## INTRODUCTION

If spinor electrodynamics with the Lagrangian

$$L=-\tfrac{1}{4}F_{\mu\nu}F_{\mu\nu}-\overline{\psi}(x)[\gamma\cdot(\partial+ie_0A)+m_0]\psi(x)\,, \quad (1)$$

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$$

is used, the equation of motion for the axial-vector current $j_\mu^5$ is

$$\partial_\mu j_\mu^5=2im_0 j^5\,, \quad (2)$$

with

$$j_\mu^5=\overline{\psi}\gamma_\mu\gamma_5\psi$$

and

$$j^5=\overline{\psi}\gamma_5\psi\,.$$

If the axial-vector and pseudoscalar vertex parts are defined by

$$S_F'(p)\Gamma_\mu^5(p,p')S_F'(p')=-\int d^4x\,d^4y\,e^{ip\cdot x}e^{-ip'\cdot y}\langle 0|T(\psi(x)j_\mu^5(0)\overline{\psi}(y))|0\rangle$$

and

$$S_F'(p)\Gamma^5(p,p')S_F'(p')=-\int d^4x\,d^4y\,e^{ip\cdot x}e^{-ip'\cdot y}\langle 0|T(\psi(x)j^5(0)\overline{\psi}(y))|0\rangle\,,$$

with $S_F'(p)$ being the Fourier transform of the spinor propagator, then the equation of motion of Eq. (2) gives the Ward identity[1]

$$(p-p')_\mu\Gamma_\mu^5(p,p')=2m_0\Gamma^5(p,p')+S_F'(p)^{-1}\gamma_5+\gamma_5S_F'(p')^{-1}\,. \quad (3)$$

However, if the left- and right-hand sides of this equation are examined in perturbation theory, it is found that the graph of Fig. 1 does not satisfy the equality.[1,2] If, following Adler,[1] we define

$$\frac{+ie_0^{\,2}}{(2\pi)^4}R_{\sigma\rho\mu}\equiv 2\int\frac{d^4r}{(2\pi)^4}(-1)\,\mathrm{Tr}\left\{\frac{1}{\gamma\cdot(r+k_1)+m_0}(ie_0\gamma_\sigma)\frac{1}{\gamma\cdot r+m_0}(ie_0\gamma_\rho)\frac{1}{\gamma\cdot(r-k_2)+m_0}\gamma_\mu\gamma_5\right\}, \quad (4)$$

then the regulated value of this integral, upon imposing gauge invariance, satisfies the equation

$$-(k_1+k_2)_\mu R_{\sigma\rho\mu}=2m_0R_{\sigma\rho}+8\pi^2k_{1\alpha}k_{2\beta}\epsilon_{\alpha\beta\sigma\rho}\,. \quad (5)$$

According to the Ward identity of Eq. (3), the last term should not appear; this is the anomaly of the axial-vector Ward identity. This can be accounted for by replacing Eq. (2) by

$$\partial_\mu j_\mu^5=2im_0j^5+\frac{\alpha_0}{4\pi}\epsilon_{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}\,. \quad (6)$$

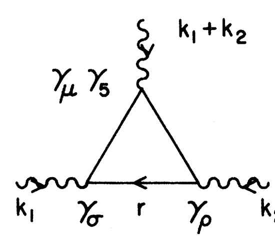


FIG. 1. Anomalous graph in four dimensions.

Higher-order graphs and radiative corrections to the graph of Fig. 1 can be shown to leave Eq. (6) unaltered.[3] However, the anomaly does prevent the resulting theory from being both unitary and renormalizable,[4] placing serious restrictions on the form of axial-vector couplings.

If one were to examine massless spinor electrodynamics with the axial-vector coupling

$$L_I = i e_0 \bar{\psi}\gamma_\mu\gamma_5\psi A_\mu ,$$

then the Lagrangian is symmetric under the chiral gauge transformation

$$\psi(x) \to \exp[-i e_0 \Lambda(x)\gamma_5]\psi(x) , \quad A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x) . \tag{7}$$

The resulting conserved current should be $j_\mu^5$, but once again, an anomaly in the divergence of the axial-vector current occurs,

$$\partial_\mu j_\mu^5 = \frac{\alpha_0}{4\pi}\epsilon_{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta} . \tag{8}$$

Similar problems occur in two dimensions, when the Lagrangian

$$L = -\tfrac{1}{4} F_{\mu\nu} F_{\mu\nu} - \bar{\psi}[\gamma \cdot (\partial + ieA)]\psi , \tag{9}$$

with

$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \gamma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \gamma_5 = -i\gamma_1\gamma_2 ,$$

is used.[5] Here, formally, both the vector current $j_\mu = \bar{\psi}\gamma_\mu\psi$ and the axial-vector current $j_\mu^5 = i\bar{\psi}\gamma_\mu\gamma_5\psi$ should be conserved, but this is impossible as

$$j_\mu^5 = \epsilon_{\mu\nu} j_\nu , \tag{10}$$

where $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$ and $\epsilon_{12} = 1$, unless we also have

$$\partial_\mu \epsilon_{\mu\nu} j_\nu = 0.$$

The anomaly occurs in the divergence of the axial-vector current[6] so that

$$\partial_\mu j_\mu^5 = \frac{e}{2\pi}\epsilon_{\mu\nu} F_{\mu\nu} . \tag{11}$$

This can be used to imply that the vector particle acquires a mass. To see this, from the equation of motion

$$\partial_\nu F_{\mu\nu} = -e j_\mu \tag{12}$$

and Eq. (10), it follows that

$$\partial_\nu \epsilon_{\alpha\mu} F_{\mu\nu} = e j_\alpha^5 . \tag{13}$$

Equations (11) and (13) together imply that

$$\left(\Box - \frac{e^2}{\pi}\right) F = 0 , \tag{14}$$

where $F_{\mu\nu} = \epsilon_{\mu\nu} F$. Consequently, there is a free particle of mass $e\pi^{-1/2}$ in the theory.

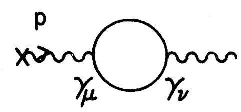


FIG. 2. Vector propagator to second order.

This can also be seen in perturbation theory. If the complete vacuum polarization tensor is to be transverse, it is of the form

$$\Pi_{\mu\nu}(q) = (\delta_{\mu\nu} q^2 - q_\mu q_\nu)\Pi(q^2) . \tag{15}$$

In two dimensions, due to the algebra of the $\gamma$ matrices, the only contribution to $\Pi$ is given by Fig. 2, and it is equal to

$$\Pi(q^2) = \frac{e^2}{\pi q^2} , \tag{16}$$

provided that gauge invariance is imposed. The pole at $q^2 = 0$ in $\Pi$ leads to a singularity in the vector propagator at $q^2 = -e^2/\pi$, again indicating that the vector particle acquires a mass dynamically. The graph of Fig. 2 is also responsible for the perturbation calculation of the anomaly in Eq. (11). The anomalous graph is given in Fig. 3. It is related to the graph of Fig. 2 as

$$\gamma_\mu \gamma_5 = -i\epsilon_{\mu\nu}\gamma_\nu . \tag{17}$$

The source of the anomalies in perturbation theory is easy to pinpoint. In both four and two dimensions, the integrals associated with the graphs of Figs. 1 and 2 are divergent. In order to extract the finite part of these integrals, a regularization procedure has to be devised that respects both the ordinary gauge symmetry and the chiral gauge symmetry that are present in the initial Lagrangian. Such a procedure has not yet been invented; as a result, the regularized integrals cannot be expected to obey both of the Ward identities derived from these two gauge principles.

For example, in the dimensional-regularization scheme,[7] the dimension of the integral being examined is analytically continued to $n$ dimensions in order to define its value. However, the tensors $\epsilon_{\mu\nu\lambda\sigma}$ and $\epsilon_{\mu\nu}$ are defined only in four and two dimensions, respectively, and there is no unambiguous way to extend their definition to $n$ dimensions. As the chiral Ward identities involve these tensors, the dimensional-regularization scheme cannot respect chiral gauge symmetry. This is

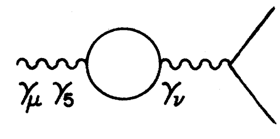


FIG. 3. Anomalous graph in two dimensions.

pursued in Appendix A.

The Pauli-Villars regularization scheme[8] also does not respect chiral gauge symmetry, because of the introduction of a mass into the effective interaction. Bell and Jackiw,[2] in the context of the $\sigma$ model, have modified this regularization method so as to respect chiral gauge symmetry, but only at the expense of breaking ordinary gauge symmetry.

These problems can be overcome in asymptotic field theory.[9-12] In this formulation of quantum field theory, only renormalized fields are introduced; consequently, no ultraviolet divergences ever appear. The results of the renormalization program of Lagrangian field theory can be reproduced without ever appealing to a subtraction procedure involving infinite quantities. This is of particular value in the case of a theory involving chiral gauge symmetry, for, since no divergent integrals ever appear, no regularization scheme that respects this symmetry need be invented. When massless spinor electrodynamics is examined using this formalism in two and four dimensions in the next two sections, it is found that no anomalous corrections to the chiral Ward identities owing to perturbation theory have to be made. Consequently, we must conclude that no such anomalies really occur; the anomalous terms in Eqs. (8) and (11) are merely reflections of an inadequate regularization scheme.

The essentials of asymptotic field theory will be briefly sketched here, using the specific example of a self-interacting scalar field of mass $m$. Two sets of fields $A_{\text{in}}(x)$ and $A_{\text{out}}(x)$, satisfying the free-particle equations of motion

$$(\Box - m^2)A_{\text{in,out}}(x) = 0 \qquad (18)$$

and the commutation relations

$$[A_{\text{in}}(x), A_{\text{in}}(y)] = -i\Delta(x-y)\,,$$
$$[A_{\text{out}}(x), A_{\text{out}}(y)] = -i\Delta(x-y)\,, \qquad (19)$$

are postulated.

These two sets of fields are related by a unitary operator $S$,

$$A_{\text{out}}(x) = S^{\dagger}A_{\text{in}}(x)A\,, \qquad (20)$$

that also satisfies the condition

$$S|0\rangle = |0\rangle\,. \qquad (21)$$

An interpolating field $A(x)$ is defined by the equation

$$A(x) = S^{\dagger}(A_{\text{in}}(x)S)_{+}\,, \qquad (22)$$

where $(\,)_{+}$ is defined as in Ref. 9. If $S$ is expanded in terms of $A_{\text{in}}(x)$,

$$S = \sum_n \frac{(-i)^n}{n!}\int d^4x_1\cdots d^4x_n\,\omega(x_1\cdots x_n) \times :A_{\text{in}}(x_1)\cdots A_{\text{in}}(x_n):\,,$$

then the reduction technique of Lehmann, Symanzik, and Zimmermann[13] shows that on the mass shell.

$$\omega(x_1\cdots x_n) = \langle 0|\Phi(A(x_1)\cdots A(x_n))|0\rangle\,, \qquad (23)$$

where the $\Phi$ product is also defined in Ref. 9. By Eqs. (22) and (23), an integral equation for the $S$ matrix off the mass shell can be derived in perturbation theory,

$$\omega^{(n)}(x_1\cdots x_m) - \int d^4y_1\cdots d^4y_m\,B(x_1\cdots x_m;\,y_1\cdots y_m) \times\omega^{(n)}(y_1\cdots y_m) = \lambda^{(n)}(x_1\cdots x_m)\,. \qquad (24)$$

Here, $\omega^{(n)}$ is the $n$th-order approximation to $\omega$, $B$ is an integral operator defined in Ref. 9, and $\lambda^{(n)}$ is an expression involving $S$-matrix elements of orders lower than $n$. The interaction is defined by postulating first-order values for certain of the $\omega$'s; for example, with a quartic coupling, the vertex function is

$$\omega^{(1)}(x_1,x_2,x_3,x_4) = -i\delta(x_1-x_2)\delta(x_3-x_4)\delta(x_1-x_3)\,.$$

With suitable boundary conditions, Eq. (24) admits only those vertex functions that correspond to renormalizable interactions. The problems of gauge invariance for Abelian and non-Abelian gauge fields has also been studied.[10,12,14]

## SPINOR ELECTRODYNAMICS IN TWO DIMENSIONS

In compliance with the formalism of asymptotic field theory, we introduce a complete set of spinor and vertex fields, $\Psi_{\text{in,out}}(x)$ and $A_{\mu\,\text{in,out}}(x)$, satisfying the equations of motion

$$\gamma\cdot\partial\Psi_{\text{in,out}}(x) = 0\,,$$
$$\Box A_{\mu\,\text{in,out}}(x) = 0 \qquad (25)$$

and the commutation relations

$$\{\Psi_{\text{in}}(x), \bar\Psi_{\text{in}}(y)\} = iS(x-y)\,,$$
$$\{\Psi_{\text{out}}(x), \bar\Psi_{\text{out}}(y)\} = iS(x-y)$$

and (26)

$$[A_{\mu\,\text{in}}(x), A_{\nu\,\text{in}}(y)] = -i\delta_{\mu\nu}D(x-y)\,,$$
$$[A_{\mu\,\text{out}}(x), A_{\nu\,\text{out}}(y)] = -i\delta_{\mu\nu}D(x-y)\,.$$

The $S$ matrix can be expanded in terms of the in-fields

$$S=\sum_{S,V}\frac{(-i)^V}{V!(S!)^2}\int d^2x_1\cdots d^2x_V d^2y_1\cdots d^2y_S\, d^2z_1\cdots d^2z_S \times \omega(x_1\cdots z_S):a_{\mu_1}(x_1)\cdots\bar{\psi}(z_S):,$$

where $\Psi_{\rm in}(x)$ and $A_{\mu\,\rm in}(x)$ have been denoted by $\psi(x)$ and $a_\mu(x)$.

Interpolating fields are defined by the equations

$$\Psi(x)=S^\dagger(\psi(x)S)_+$$

and (27)

$$A_\mu(x)=S^\dagger(a_\mu(x)S)_+\,.$$

By requiring that the interpolating fields undergo the infinitesimal transformation

$$A_\mu(x)\to A_\mu(x)+\partial_\mu\Lambda(x),$$
$$\Psi(x)\to[1-ie\gamma_5\Lambda(x)]\Psi(x)$$

when the in-fields undergo the transformation

$$a_\mu(x)\to a_\mu(x)+\partial_\mu\Lambda(x),$$
$$\psi(x)\to\psi(x),$$

commutation rules for the interpolating fields can be found, by the methods of Ref. 12. They are

$$\begin{aligned}&[\partial_\mu A_\mu(x),A_\nu(y)]=-i\partial_\nu D(x-y),\\&[\partial_\mu A_\mu(x),\Psi(y)]=eD(x-y)\gamma_5\Psi(y),\\&[\partial_\mu A_\mu(x),\bar{\Psi}(y)]=eD(x-y)\bar{\Psi}(y)\gamma_5\,.\end{aligned}\tag{28}$$

As in Ref. 14, the Ward identities resulting from these commutation rules are

$$\begin{aligned}\langle 0|(\partial_\mu A_\mu(\xi)A_{\mu_1}(x_1)\cdots A_{\mu_V}(x_V)\Psi(y_1)\cdots\bar{\Psi}(z_S))_+|0\rangle &\equiv(\partial\cdot\xi x_1\cdots x_V,y_1\cdots z_S)_+\\&=-i\sum_j\partial^\xi_{\mu_j}D_c(\xi-x_j)(x_1\cdots\Lambda_j\cdots x_V,y_1\cdots z_S)_+\\&\quad+e\sum_j D_c(\xi-y_j)\gamma_5^{(j)}(x_1\cdots z_S)_++e\sum_j D_c(\xi-z_j)(x_1\cdots z_S)_+\gamma_5^{(j)}\,.\end{aligned}\tag{29}$$

This can be shown to be equivalent to the Ward identity of Eq. (3), by the methods of Ref. 15.

A vertex function that is consistent with this Ward identity and unitarity is

$$\omega^{(1)}(x,y,z)=-ie\gamma_\mu\gamma_5\delta(x-y)\delta(y-z)\,.$$

By Eq. (27), the interpolating field to first order is

$$A^{(1)}_\mu(w)=-ie\int d^2x\,D_R(w-x):\bar{\psi}(x)\gamma_\mu\gamma_5\psi(x):\,.\tag{30}$$

Equation (24) gives for the second-order propagator

$$\omega^{(2)}(x_1,x_2)=\Box_1\Box_2\langle 0|(A^{(1)}_{\mu_1}(x_1)A^{(1)}_{\mu_2}(x_2))_+|0\rangle\,.\tag{31}$$

Explicit evaluation of this matrix element leads to the finite expression

$$\langle 0|(A^{(1)}_{\mu_1}(x_1)A^{(1)}_{\mu_2}(x_2))_+|0\rangle=\mathrm{Tr}\int d^2y_1d^2y_2D_R(x_1-y_1)D_R(x_2-y_2)\times[\theta(x_1-x_2)S_-(y_2-y_1)\gamma_{\mu_1}S_+(y_1-y_2)\gamma_{\mu_2}+\theta(x_2-x_1)S_-(y_1-y_2)\gamma_{\mu_2}S_+(y_2-y_1)\gamma_{\mu_1}]\,.$$

However, it is shown in Appendix B that

$$\mathrm{Tr}[S_-(-y)\gamma_\mu S_+(y)\gamma_\nu]=0\,,$$

in the two-dimensional case. Thus, the second-order propagator disappears completely, leaving only the free-field propagator. All integrals are well defined at the outset, no anomaly occurs, and the vector field remains massless.

Similar considerations, when applied to the massless Thirring model,[16,17] also show that it too is free of anomalies.

## SPINOR ELECTRODYNAMICS IN FOUR DIMENSIONS

The same considerations used in two dimensions can be applied in four dimensions, although the practical complications are considerable. The quantization procedure is identical, the Ward identities are the same, and the vertex function is unaltered. However, in four dimensions, it is the three-point function in third order that must be examined. Even in ordinary quantum electrodynamics, straightforward evaluation of the vertex function is very complicated when one is using asymptotic field theory.[18]

The proof that the dynamics of asymptotic field theory is consistent with the Ward identities of ordinary gauge invariance is given for the case of spinor electrodynamics by Pugh.[10] With a few modifications presented in this section, this proof can be used to establish the consistency of the dynamics of massless spinor electrodynamics with chiral Ward identities.

First of all, the Ward identities of Eq. (29) can be written as restrictions on the $\omega$ functions by converting time-ordered products to $\Phi$ products. The result is

$$\omega(\partial\cdot xx_1\cdots x_m)=ie\sum_{ij}P_{ij}K_iK_j\{\gamma_5 S_c(x_j-x_i)[\delta(x-x_i)-\delta(x-x_j)]\}\,\omega(x_1\cdots\Lambda_{ij}\cdots x_m)$$
$$+eK_1\cdots K_m\sum_i D_c(x-x_i)[\delta_{iF}\gamma_5^{(i)}\Phi(x_1\cdots x_m)+\delta_{i\bar F}\Phi(x_1\cdots x_m)\gamma_5^{(i)}].$$

The notation used is consistent with that of Ref. 10.

If this equation is assumed to be true for $l<n$, then

$$\partial_\mu A_\mu(x)=\partial_\mu a_\mu(x)+\sum_{p=n}^{\infty}\partial_\mu A_\mu^{(p)}(x)$$

and

$$[\partial_\mu a_\mu(x),\Psi^{(l)}(y)]=eD(x-y)\gamma_5\Psi^{(l-1)}(y),$$
$$[\partial_\mu a_\mu(x),\bar\Psi^{(l)}(y)]=eD(x-y)\bar\Psi^{(l-1)}(y)\gamma_5.$$

By the expression for $\partial_\mu A_\mu(x)$, we have

$$\{\langle 0|(\partial_\mu A_\mu(x)\phi_1\cdots\phi_m)_+|0\rangle\}^{(n)}=\sum_i\{\langle 0|(\phi_1\cdots[\partial_\mu a_\mu(x),\phi_i]_c\cdots\phi_m)_+|0\rangle\}^{(n)}+\langle 0|(\partial_\mu A_\mu^{(n)}(x)\phi_1^{\rm in}\cdots\phi_m^{\rm in})_+|0\rangle$$

Using the commutators of $\partial_\mu a_\mu(x)$ with $\Psi^{(l)}y$ and $\bar\Psi^{(l)}(y)$,

$$\langle 0|(\partial_\mu A_\mu(x)\phi_1\cdots\phi_m)_+|0\rangle^{(n)}$$
$$=e\sum_i D_c(x-x_i)[\delta_{iF}\gamma_5^{(i)}\langle 0|(\phi_1\cdots\phi_m)_+|0\rangle^{(n-1)}+\delta_{i\bar F}\langle 0|(\phi_1\cdots\phi_m)_+|0\rangle^{(n-1)}\gamma_5^{(i)}]$$
$$-i\sum_i\delta_{iB}\partial_{\mu_i}D_c(x-x_i)\langle 0|(\phi_1\cdots\Lambda_i\cdots\phi_m)_+|0\rangle^{(n)}$$
$$+\langle 0|(\partial_\mu A_\mu^{(n)}(x)\phi_1^{\rm in}\cdots\phi_m^{\rm in})_+|0\rangle$$
$$+\sum_i\langle 0|(\phi_1^{\rm in}\cdots\{[\partial_\mu a_\mu(x),\phi_i^{(n)}]_c-e\delta_{iF}D_c(x-x_i)\gamma_5\phi_i^{(n-1)}-e\delta_{i\bar F}D_c(x-x_i)\phi_i^{(n-1)}\gamma_5\}\cdots\phi_m^{\rm in})_+|0\rangle.$$

Converting this into an equation with $\Phi$ products, we obtain

$$\Phi^{(n)}(\partial\cdot xx_1\cdots x_n)=-ie\sum_{ij}P_{ij}\gamma_5 S_c(x_j-x_i)[D_c(x-x_i)-D_c(x-x_j)]\,\Phi^{(n-1)}(x_1\cdots\Lambda_{ij}\cdots x_m)$$
$$+e\sum_i D_c(x-x_i)[\delta_{iF}\gamma_5^{(i)}\Phi^{(n-1)}(x_1\cdots x_m)+\delta_{i\bar F}\Phi^{(n-1)}(x_1\cdots x_m)\gamma_5^{(i)}]$$
$$+\langle 0|\Phi(\partial_\mu A_\mu^{(n)}(x)\phi_1^{\rm in}\cdots\phi_m^{\rm in})|0\rangle$$
$$+\sum_i\langle 0|\Phi(\phi_1^{\rm in}\cdots\{[\partial_\mu a_\mu(x),\phi_i^{(n)}]-e\delta_{iF}D_c(x-x_i)\gamma_5\phi_i^{(n-1)}-e\delta_{i\bar F}D_c(x-x_i)\phi_i^{(n-1)}\gamma_5\}\cdots\phi_m^{\rm in})|0\rangle. \tag{32}$$

The commutation relation for $\Psi^{(n)}$ to $n$th order is now

$$[\partial_\mu a_\mu(x),\Psi^{(n)}(y)]_c=eD_c(x-y)\gamma_5\Psi^{(n-1)}(y)$$
$$+iS^\dagger\sum_{B,F}\frac{(-i)^B}{B!F!(F-1)!}$$
$$\times\int d^4\xi\,d^4x_1\cdots d^4x_m\gamma_5 S_R(y-x_m)D_{cy}(x-\xi)\Omega^{(n)}(\xi,x_1\cdots x_m):\bar\phi_1^{\rm in}\cdots\bar\phi_{m-1}^{\rm in}:,$$

and similar relations are found for $\bar{\Psi}^{(n)}$ and $A_\mu^{(n)}$. The function $\Omega$ is defined to be

$$\Omega(x, x_1 \cdots x_m) = \omega(\partial \cdot x x_1 \cdots x_m) - eK_1 \cdots K_m \sum_i D_c(x - x_i)[\delta_{iF}\gamma_5^{(i)}\Phi(x_1 \cdots x_m) + \delta_{i\bar{F}}\Phi(x_1 \cdots x_m)\gamma_5^{(i)}]$$

$$- ie\sum_{ij} P_{ij}K_iK_j\{\gamma_5 S_c(x_j - x_i)[\delta(x - x_i) - \delta(x - x_j)]\}\,\omega(x_1 \cdots \Lambda_{ij} \cdots x_m).$$

Upon substituting the expression for $[\partial_\mu a_\mu(x), \Psi(y)]_c$ into Eq. (32), it is found, as in Ref. 10, that

$$\Omega^{(n)}(x, x_1 \cdots x_m) = \int d^4\xi\, d^4\xi_1 \cdots d^4\xi_m B(x, x_1 \cdots x_m, \xi, \xi_1 \cdots \xi_m)\Omega^{(n)}(\xi, \xi_1 \cdots \xi_m),$$

where $B$ is the same operator as in Eq. (24). Pugh uses this equation[10] in order to show that $\Omega^{(n)}$ is zero. Consequently, if the Ward identity of Eq. (29) is satisfied to order $l$ for $l<n$, then it is satisfied to order $n$. This demonstrates that no axial-vector current anomaly occurs when one uses asymptotic field theory, and, in particular, the three-point function satisfies the chiral Ward identity in third order.

It is unfortunate that the equations for $\omega^{(3)}(x_1, x_2, x_3)$ do not lend themselves to explicit evaluation. However, even in Lagrangian field theory the triangle diagram has never been explicitly worked out, except by relating divergent quantities to convergent ones by imposing gauge invariance, which is a suspect procedure, when no attempt is made to preserve chiral symmetry.

## DISCUSSION

The disappearance of anomalies in chiral Ward identities removes many difficulties in constructing field theories with chiral gauge symmetry. The only place where anomalies have been involved in making a calculation for a physical process is the determination of the decay rate for the reaction $\pi^0 \to 2\gamma$ using current algebra.[1] However, quark-model calculations have accounted for the decay rates of pseudoscalar mesons without ever referring to anomalies.[19] In any case, PCAC (partial conservation of axial-vector current) has not been able to account for the decay $\eta \to 3\pi$. Any claim that anomalies are somehow "physical," and as such are not to be discarded, is somewhat suspect. There is simply no clear experimental necessity for the existence of anomalies.

It now becomes much easier to construct renormalizable field theories for the weak interactions. The constraint that anomalies have to cancel in order for a theory to be unitary and renormaliable has been thought to imply[4] certain restrictions on quark-lepton charge assignments, but this problem now disappears.

Many calculations involving the Bjorken limit[3] have been done on the basis of Eq. (6). The calculations must now be revised to remove any contributions from the anomalous term. For a discussion of the charge operator in asymptotic field theory, Ref. 20 is of particular interest.

Anomalies have been seen to play a role in spontaneous mass generation in two-dimensional electrodynamics. This mechanism has led to theorizing on the possibility of spontaneous mass generation in four dimensions, as, for example, in Ref. 21. The properties of two-dimensional quantum electrodynamics have also formed the basis of speculation for quark confinement in four dimensions.[22] With the disappearance of anomalies in two-dimensional electrodynamics, the physical basis of these models must be reconsidered.

The anomalies in the chiral Ward identities are usually related to anomalies in the Ward identities that are a result of scale invariance,[23] even though anomalous dimensions do not lead to problems with unitarity or renormalizability. It is difficult to draw such comparisons in the context of asymptotic field theory. The dynamic Eq. (24) is scale-invariant for the case $m=0$, but for $m\neq 0$ there is no operator corresponding to the "improved energy-momentum tensor" $\Theta_{\mu\nu}$. Explicit calculations show[14] that even in asymptotic field theory, infrared divergences can occur, thus resulting in the introduction of a mass in the form of an infrared cutoff. This, however, appears to be an inherent difficulty of the measuring process,[24] and not a weakness of asymptotic field theory. The Callan-Symanzik equations[25] are supposed to account for the anomalies in scale invariance, and these are usually considered to be equivalent to the Gell-Mann–Low equations.[26] As the Gell-Mann–Low equations occur in the formalism of asymptotic field theory,[14] it is safe to assume that anomalous corrections to scale-invariance Ward identities really do occur.

The power and utility of asymptotic field theory is demonstrated by its handling of axial-vector Ward identities. No divergent quantities appear in this formalism, allowing for freer manipulations of quantities that normally would be considered dangerous owing to their singular behavior.

Yet, with no appeal to renormalization theory, all the standard results of electrodynamics[10,12] and non-Abelian gauge theories[14] can be established.


## ACKNOWLEDGMENTS

The author has benefited greatly from the insights of Professor R. E. Pugh and Dr. Nathan Isgur.


## APPENDIX A

The Feynman integral corresponding to the graph of Fig. 2 is

$$A_{\mu\nu} = \mathrm{Tr}\left[\int d^2q \frac{\gamma\cdot q\gamma_\mu\gamma\cdot(q+p)\gamma_\nu}{q^2(q+p)^2}\right]. \qquad (A1)$$

To evaluate this integral, the condition $p_\mu A_{\mu\nu}=0$ must be imposed. However, if we look at

$$B_{\mu\nu} = \mathrm{Tr}\left[\int d^2q \frac{\gamma\cdot q\gamma_\mu\gamma_5\gamma\cdot(q+p)\gamma_\nu\gamma_5}{q^2(q+p)^2}\right], \qquad (A2)$$

then as $\gamma_\mu\gamma_5 = -i\epsilon_{\mu\lambda}\gamma_\lambda$, we must have

$$B_{\mu\nu} = -\epsilon_{\mu\lambda}\epsilon_{\nu\sigma}A_{\lambda\sigma}. \qquad (A3)$$

But if the matrices $\gamma_5$ in Eq. (A2) are allowed to contract on each other, we must have

$$B_{\mu\nu} = A_{\mu\nu}. \qquad (A4)$$

There is no nonzero transverse tensor $A_{\mu\nu}$ satisfying both Eqs. (A3) and (A4). This is a crude way of stating that an anomaly occurs in two-dimensional spinor electrodynamics. If the integral of Eq. (A1) is continued to $n$ dimensions,[7] we obtain

$$A_{\mu\nu} = n\int d^nq \frac{[q_\mu(q+p)_\nu + q_\nu(q+p)_\mu - \delta_{\mu\nu}q\cdot(q+p)]}{q^2(q+p)^2}. \qquad (A5)$$

After combining denominators by use of the integral

$$\left(\frac{1}{a+i\epsilon}\right)^\alpha = \frac{1}{i^\alpha\Gamma(\alpha)}\int_0^\infty dz\, z^{\alpha-1}e^{iz(a+i\epsilon)}$$

and inserting the identity

$$1 = \int_0^\infty \frac{d\lambda}{\lambda}\,\delta\left(1-\frac{z}{\lambda}\right),$$

Eq. (A5) becomes

$$A_{\mu\nu} = n\int d^nq\int_0^\infty d\lambda\int_0^1 dz\,\lambda[\delta_{\mu\nu}q^2 - 2q_\mu q_\nu - z(1-z)(\delta_{\mu\nu}p^2 - 2p_\mu p_\nu)] \times \exp\{i\lambda[q^2 + z(1-z)p^2]\}.$$

The integration over $q$ can be done using the relations

$$\int d^nq\, e^{i\lambda q^2} = i\left(\frac{i\pi}{\lambda}\right)^{n/2},$$

$$\int d^nq\, q_\mu q_\nu e^{i\lambda q^2} = -\frac{\delta_{\mu\nu}}{2\lambda}\left(\frac{i\pi}{\lambda}\right)^{n/2},$$

resulting in the expression

$$A_{\mu\nu} = n\int_0^\infty d\lambda\,\lambda\int_0^1 dz\left(\frac{i\pi}{\lambda}\right)^{n/2}\left[iz(1-z)(2p_\mu p_\nu - p^2\delta_{\mu\nu}) + \frac{1}{\lambda}\left(1-\frac{n}{2}\right)\delta_{\mu\nu}\right] \times \exp[i\lambda z(1-z)p^2].$$

If we naively pass to the limit $n=2$, the last term drops out and $A_{\mu\nu}$ satisfies Eqs. (A3) and (A4), but is not transverse. On the other hand, if the integration over $\lambda$ is first performed using the equation

$$\int_0^\infty d\lambda\,\lambda^x e^{i\lambda a} = \left(\frac{i}{a}\right)^{x+1}\Gamma(x+1)$$

and the relation $x\Gamma(x) = \Gamma(x+1)$, we obtain

$$A_{\mu\nu} = 2in\pi^{n/2}\Gamma\left(2-\frac{n}{2}\right) \times \int_0^1 dz\,\frac{p_\mu p_\nu - p^2\delta_{\mu\nu}}{(p^2)^{2-n/2}}\,[z(1-z)]^{n/2-1}.$$

This is finite at $n=2$, and is transverse, but does not satisfy Eqs. (A3) and (A4). This is an explicit demonstration of how a regularization procedure can fail to respect two symmetries that should be retained in the theory.

Furthermore, one could regularize the theory by defining the current[27] as

$$j_\mu = \bar\psi\left(x+\frac{\epsilon}{2}\right)\gamma_\mu\psi\left(x-\frac{\epsilon}{2}\right)\exp\left[ie\int_{x-\epsilon/2}^{x+\epsilon/2} dy\cdot A(y)\right] \qquad (A6)$$

and allowing $\epsilon$ to approach zero at the end of any calculation. In examining

$$M_{\mu\nu}(k) = \int d^2x\, e^{-ik\cdot x}\langle 0|T^*(j_\mu^5(x)A_\nu(0))|0\rangle,$$

we have to lowest order in $\epsilon$

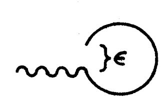


FIG. 4. Graph giving rise to the two-dimensional anomaly.

$$M_{\mu\nu}(k) \simeq \epsilon_{\mu\lambda} \int d^2x\, e^{-ik\cdot x} \langle 0| T^*(\bar{\psi}(x+\tfrac{1}{2}\epsilon)\gamma_\lambda \psi(x-\tfrac{1}{2}\epsilon)[1+e\epsilon\cdot A(x)]A_\nu(0))|0\rangle .$$

The anomalous contribution to $M_{\mu\nu}$ is given by the diagram of Fig. 4. This involves examining the integral

$$\Lambda_{\mu\nu} = \int d^2k \frac{\epsilon_\mu k_\nu e^{ik\cdot\epsilon}}{k^2 - i\epsilon} .$$

Performing the Wick rotation and integrating by parts yields

$$\Lambda_{\mu\nu} = -i \int d^2k\, \partial_\mu \left(\frac{k_\nu e^{ik\cdot\epsilon}}{k^2 - i\epsilon}\right) - e^{ik\cdot\epsilon}\partial_\mu\left(\frac{k_\nu}{k^2 - i\epsilon}\right). \tag{A7}$$

There are two types of contributions to the first term; those with $\mu=\nu$, and those with $\mu \neq \nu$. With $\mu=\nu=1$, we obtain

$$\Lambda_{11} = \int d^2k\, \partial_1\left(\frac{k_1}{k^2} e^{ik\epsilon\cos\phi}\right)$$

$$= \tfrac{1}{2}\int_0^{2\pi} d\phi \int_0^\infty dk\, k\left[\partial_i\left(\frac{k_i}{k^2} e^{ik\epsilon\cos\phi}\right)\right],$$

which upon applying Gauss's theorem becomes

$$\Lambda_{11} = \tfrac{1}{2}\lim_{K\to\infty}\int_0^{2\pi} d\phi\, K\left(\frac{K}{K^2} e^{iK\epsilon\cos\phi}\right).$$

Inserting the identity

$$e^{iK\epsilon\cos\phi} = \sum_{n=0}^{\infty} \{\epsilon_n (-1)^n J_{2n}(\epsilon K)\cos(2n\phi) + 2i(-1)^n J_{2n+1}(\epsilon K)\cos[(2n+1)\phi]\},$$

where $\epsilon_n = 2 - \delta_{n1}$, leaves us with

$$\Lambda_{11} = \pi \lim_{K\to\infty} J_0(\epsilon K) = 0 .$$

Similarly, the terms with $\mu\neq\nu$ can be shown to vanish.

The second term of Eq. (A7) can be written

$$\frac{i}{2}\int d^2k\, e^{ik\cdot\epsilon}\, \partial_\mu\partial_\nu \ln(k^2 - i\epsilon). \tag{A8}$$

As $\ln(k^2 - i\epsilon)$ is the Green function in two dimensions, satisfying

$$\Box \ln k = 2\pi\delta^2(k),$$

we see that the contribution of Eq. (A8) to Eq. (A7) is such that

$$\Lambda_{\mu\nu} = -\pi\delta_{\mu\nu} . \tag{A9}$$

This is exactly what is required to give the anomaly of Eq. (11).

We thus see that this method of regularization also gives anomalous contributions to the divergence of the axial-vector current. This is not unexpected, however, as the current defined by Eq. (A6) is not invariant under the chiral gauge transformations

$$\psi(x) \to \exp[-ie\gamma_5\Lambda(x)]\psi(x)$$

and

$$A_\mu(x) \to A_\mu(x) + \partial_\mu\Lambda(x).$$

We have dealt with the "point separation" method of regularization in two dimensions only. Hagen has discussed[28] the use of this method in handling chiral symmetries in four dimensions, and has found that several ambiguities arise.

## APPENDIX B

In this appendix, it will be proven that

$$A_{\mu\nu} \equiv \mathrm{Tr}[S_-(-y)\gamma_\mu S_+(y)\gamma_\nu] = 0 . \tag{B1}$$

First of all, substitution of

$$S_\pm(y) = \pm\frac{-i}{2\pi}\int d^2p\, e^{ip\cdot x}(i\gamma\cdot p)\delta(p^2)\theta(\pm p)$$

into this expression yields

$$A_{\mu\nu} = -\mathrm{Tr}\left\{\int \frac{d^2p_1}{2\pi}\frac{d^2p_2}{2\pi}\,\theta(p_1)\theta(-p_2)\delta(p_1^{\,2})\delta(p_2^{\,2})\gamma\cdot p_1\gamma_\mu\gamma\cdot p_2\gamma_\nu \exp[(-ip_1+ip_2)\cdot y]\right\}. \tag{B2}$$

Shifting variables of integration in this finite expression gives

$$A_{\mu\nu} = \frac{-1}{(2\pi)^2}\,\mathrm{Tr}\left\{\int d^2p\, d^2q\, e^{-ip\cdot y}\,\theta(p-q)\theta(q)\delta((p-q)^2)\delta(q^2)[\gamma\cdot(p-q)\gamma_\mu\gamma\cdot q\gamma_\nu]\right\}.$$

Evaluating the trace in this equation leads to

$$A_{\mu\nu} = \frac{4}{(2\pi)^2}\int d^2p\, e^{-ip\cdot y}\int d^2q\,\theta(p-q)\theta(q)\delta((p-q)^2)\delta(q^2)(p_\mu q_\nu + p_\nu q_\mu - 2q_\mu q_\nu - \tfrac{1}{2}\delta_{\mu\nu}p^2). \tag{B3}$$

If we now define the integrals

$$\begin{aligned} I &= \int d^2q\,\theta(p-q)\theta(q)\delta((p-q)^2)\delta(q^2),\\ I_\mu &= \int d^2q\,\theta(p-q)\theta(q)\theta((p-q)^2)\delta(q^2)q_\mu,\\ I_{\mu\nu} &= \int d^2q\,\theta(p-q)\theta(q)\delta((p-q)^2)\delta(q^2)q_\mu q_\nu, \end{aligned} \tag{B4}$$

then the requirements of Lorentz covariance imply that the forms of $I_\mu$ and $I_{\mu\nu}$ are given by

$$\begin{aligned} I_\mu &= Ap_\mu,\\ I_{\mu\nu} &= Bp_\mu p_\nu + Cp^2\delta_{\mu\nu}. \end{aligned} \tag{B5}$$

Employing the $\delta$ functions within the integrands yields

$$\begin{aligned} p_\mu I_\mu &= Ap^2\\ &= (p^2/2)I,\\ p_\mu I_{\mu\nu} &= Bp^2p_\nu + Cp^2p_\nu\\ &= (p^2/2)I_\nu,\\ I_{\mu\mu} &= Bp^2 + 2Cp^2\\ &= 0. \end{aligned}$$

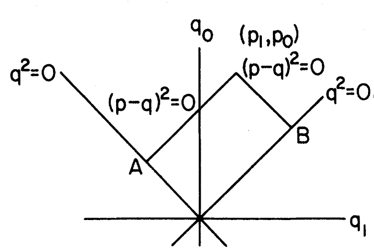


FIG. 5. Graph illustrating the integral of Eq. (B7).

As a result of the equations, we obtain

$$\begin{aligned} I_\mu &= \tfrac{1}{2}Ip_\mu,\\ I_{\mu\nu} &= \tfrac{1}{2}Ip_\mu p_\nu - \tfrac{1}{4}Ip^2\delta_{\mu\nu}. \end{aligned} \tag{B6}$$

These results can also be obtained by direct integration. Substitution of Eq. (B6) into Eq. (B3) gives us

$$\begin{aligned} A_{\mu\nu} = \frac{4}{(2\pi)^2}\int d^2p\, e^{-ip\cdot y}[&(\tfrac{1}{2}Ip_\mu p_\nu) + (\tfrac{1}{2}Ip_\mu p_\nu)\\ &- 2(\tfrac{1}{2}Ip_\mu p_\nu + \tfrac{1}{4}Ip^2\delta_{\mu\nu})\\ &- (\tfrac{1}{2}Ip^2\delta_{\mu\nu})]\\ =0. \end{aligned}$$

Consequently, we have proven Eq. (B1).

Actually, the integral $I$ in Eq. (B4) is not well defined at $p^2=0$. This is a manifestation of the familiar infrared problem, which also arises in Lagrangian field theory. To demonstrate the problems involved, an explicit evaluation of $I$ will now be made. The requirements that $q_0>0$ and $(p-q)_0>0$ immediately yield the restrictions $p_0>0$ and $p_0^{\,2}>p_1^{\,2}$, as can be seen from Fig. 5. Thus we have

$$I = \int_{-\infty}^{\infty} dq_0 \int_{-\infty}^{\infty} dq_1\,\theta(q)\left[\frac{\delta(q_0+q_1)+\delta(q_0-q_1)}{2|q_0|}\right] \times\delta(p^2-2p\cdot q)\theta(-p^2)\theta(p). \tag{B7}$$

Employing the $\delta$ functions to integrate first over $q_1$ and then over $q_0$ quickly yields

$$I = \frac{\theta(-p^2)\theta(p)}{2p^2}. \tag{B8}$$

If $p^2=0$, then the integrand in Eq. (B7), instead of being over the two points $A$ and $B$ in Fig. 5, now is over part of the forward light cone $|q^2=0$. This integral is not well defined; consequently, an infrared cutoff at $p^2=-\mu^2$ must be made in Eq. (B3).

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