PHYSICAL REVIEW D VOLUME 14, NUMBER 6 15 SEPTEMBER 1976

Unified approach to strings and vortices with soliton solutions

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(Received 19 May 1976)

A classical relativistic theory of one-dimensional extended objects interacting through a massless scalar field, of which they are in turn the source, is constructed. In the no-coupling limit, the string model is recovered. In another limit, the system that describes nonrelativistic vortex motion in a superfluid is obtained. The diverging self-interaction of these objects is shown to be regularizable through a renormalization of the slope of the Regge trajectories. Motion in an external field is studied in some detail and leads to a system of coupled nonlinear equations that generalizes the sine-Gordon system. Solitary wave solutions to these equations are obtained and a natural geometric interpretation to the associated linear equations of the inverse scattering method is given.

I. INTRODUCTION

In a previous paper by one of us in collaboration with Rasetti,[1] a canonical formalism was constructed that described the motion of vortices in an incompressible, inviscid fluid, of which superfluid He II is the best example. Among the problems that were opened by this work was the question of whether there exists a Lorentz-invariant theory that would generalize this nonrelativistic formalism in some sense. Also, in light of the work of Nielsen and Olesen[2] relating the string model with the vortex lines in a type-II superconductor, one would like to know what the relation is, if any, between the Nambu string[3] and vortices in a superfluid, apart from the mere fact that they are both one-dimensional extended objects.

We present here a classical Lagrangian that implements a unified theory of strings and (superfluid) vortices in the following sense: One-dimensional objects—that may indifferently be called strings or vortices—interact through a massless scalar field of which they are in turn the source. If the coupling constant vanishes, the action integral is just that of the Nambu string. On the other hand, if the slope of the Regge trajectory is infinite, that is, if the coefficient in front of the Nambu action vanishes, taking the special case of the one-dimensional object moving in a certain prescribed external field reduces the Lagrangian to the one of Ref. 1.

It turns out that our Lagrangian is the same as the one introduced some time ago by Kalb and Ramond[4] in their construction of a theory of interacting strings by analogy with the action-at-a-distance electrodynamics of Wheeler and Feynman.[5] Thus, another way of expressing our result is to say that the theory of Kalb and Ramond admits, as a particular case, the motion of a vortex in a superfluid.

The self-energy of an infinitely thin (classical) superfluid vortex is divergent. This is also true of our strings. However, the divergence is logarithmic and it can be taken care of by a renormalization of the slope of the Regge trajectories.

The equations of motion will be studied in some detail for the case of a uniform, static external field, and some pleasant surprises will emerge. In fact, the problem of the motion of the string is just the problem of finding the surface it describes in a Minkowski space-time. Taking the projection of this surface on a three-dimensional (Euclidean) hypersurface of constant time one recovers the ancient problem of embedding a surface in space, and our equation of motion is just a restriction on the admissible geometries for this surface. Under certain special initial conditions, the equations of Gauss and Codazzi, which say when a surface with a given geometry may be embedded in a three-dimensional Euclidean space, reduce to the sine-Gordon equation. This has been noticed before.[6] What apparently has not been realized is that the equations of Gauss and Weingarten—which actually build the surface—are first-order linear differential equations for the normal and tangent vectors that can be written in terms of a rotation operator, and that expressing this operator in the spin-$\frac{1}{2}$ representation, one recovers exactly the equations used by Ablowitz *et al.*[7] to solve the sine-Gordon equation. Thus we get a natural answer to the question of why a particular linear problem is of help in solving a certain nonlinear equation. For example, we might mention here that different eigenvalues of the linear problem correspond to projections of our surface along different three-dimensional spaces which are related to one another by a Lorentz transformation.

For general initial conditions, we are led to a set of two coupled nonlinear equations with an associated linear problem. We have not yet solved this problem in general. Static solutions can easily be found, however. These equations may be regarded as the dynamical equations of a theory of two coupled scalar fields in one space and one time dimensions. In this case a Lagrangian may be written, and some of the static solutions represent solitons in their rest frame.

Starting from a definite physical picture, namely, that of strings interacting through a scalar field in four-dimensional Minkowski space-time, we have been led by the mathematics of the problem to a quite different physical situation, namely, that of two coupled scalar fields in a two-dimensional Minkowski space-time. It would be very gratifying indeed if one could translate these formal analogies into a physical statement. We can only speculate at this stage, however. A serious obstacle to such a unified framework of quite different objects is that although the equations are the same, and thus locally a solution to one problem provides a solution to the other one, this is not so globally, as the boundary conditions are quite different. For example, closed strings are described by dynamical variables periodic in the space coordinate. This is not so in a field theory, the spatial variable being allowed to vary over the whole range of real numbers. Perhaps the way to look at this is to take infinite strings, but we have not yet done actual computations along this line. Another possibility is to interchange the role of the spatial and temporal coordinates, thus relating, one hopes, motions of a closed string with the so-called "breather" modes, which are periodic in time.

This work is organized as follows: In Sec. II we write the Lagrangian, the equations of motion, and show how the self-energy of a closed string may be regularized by a renormalization procedure. The canonical formalism for the theory is worked out in the Appendix. In Sec. III we study the motion of a closed string in a uniform external field and discuss the implications of the geometrical way of attacking the problem. Section IV contains some conclusions and summarizes the loose ends of this work. Quantum problems will not be discussed.

II. THE MODEL

In Ref. 1, a vortex was described by a closed curve Γ in R^3, parametrized by three functions $x^i(\sigma)$. This vortex moves in a fluid described by a divergenceless velocity field $\vec{v}(\vec{y},t)$. The dynamical ingredient is the statement that, at each point, the vortex velocity is given by the velocity of the fluid[8]

$$\frac{dx^i}{dt} = v^i\,|_{\vec{y}=\vec{x}}\,. \tag{2.1}$$

The dynamics of the system remains unchanged, however, if the parameter σ is changed in an arbitrary way, as it is just a coordinate along the vortex. In general, then, the parametrization is an arbitrary function of time, $\sigma=\sigma(t)$, and Eq. (2.1) acquires the more general form

$$\frac{\partial x^i}{\partial t} = v^i - \frac{d\sigma}{dt}\frac{\partial x^i}{\partial \sigma}\,, \tag{2.2}$$

where v^i is understood to be evaluated at $\vec{y}=\vec{x}$.

Since $d\sigma/dt$ is arbitrary, this equation says that the vectors $\partial\vec{x}/\partial\sigma$ and $\partial\vec{x}/\partial t - \vec{v}$ are parallel. In other words,

$$\epsilon^{ijk}\frac{\partial x^j}{\partial\sigma}\left(\frac{\partial x^k}{\partial t} - v^k\right) = 0\,. \tag{2.3}$$

To consider a vortex line in a fluid means that the vorticity $\vec{w}$ vanishes everywhere except along that line, to which it is tangent. That is,

$$\vec{w} \equiv \operatorname{curl}\vec{v} = k\frac{\partial\vec{x}}{\partial\sigma}\,\delta(\vec{y}-\vec{x}(\sigma))\,, \tag{2.4}$$

where k is a constant. Moreover, since $\operatorname{div}\vec{v}=0$, there exists a vector potential $\vec{A}$ such that

$$\vec{v} = \operatorname{curl}\vec{A}\,. \tag{2.5}$$

Substituting into (2.4) and using the fact that $\vec{A}$ is not completely determined so that $\operatorname{div}\vec{A}=0$ may be imposed, we get Poisson's equation for $\vec{A}$, which has the well-known solution

$$\vec{A}(\vec{y}) = \frac{k}{4\pi}\int_\Gamma \frac{1}{|\vec{y}-\vec{x}(\sigma)|}\frac{\partial\vec{x}}{\partial\sigma}\,d\sigma\,. \tag{2.6}$$

This expression, together with (2.5), gives the velocity $\vec{v}$ as a functional of the dynamical variables $\vec{x}(\sigma)$. The Lagrangian that, upon variation of the $\vec{x}(\sigma)$, gives the equation of motion (2.3) is[1,9]

$$\mathcal{L} = \frac{k\rho}{3}\int_\Gamma \epsilon^{ijk}x^i\frac{\partial x^j}{\partial\sigma}\frac{\partial x^k}{\partial t}\,d\sigma - \frac{\rho}{2}\int v^2 d^3y\,, \tag{2.7}$$

where ρ is the density of the fluid and $\vec{v}$ is understood to be expressed in terms of $\vec{x}(\sigma)$.

We now want to generalize the Lagrangian (2.7) in such a way that we will have a Lorentz-invariant theory. The only requirements are that the action integral be a Lorentz scalar and that it must have (2.7) as a special case. This special case is a theory in which the dynamical variables are the $x^i(\sigma)$. It is an obvious thing to do, then, to consider a theory with variables $x^\mu(\sigma)$ ($\mu=0,1,2,3$). The addition of one more variable reflects the fact that we shall have one more invariance, that which is under time reparametrizations.

In the nonrelativistic theory described by the Lagrangian (2.7) there is a field $\vec{v}(\vec{y})$ whose source is the vortex $\vec{x}(\sigma)$ that completely determines it through (2.5) and (2.6). The field $\vec{v}(\vec{y})$ does not have any life of its own, so to speak, just as the Newtonian gravitational field is completely determined by its source. This is, of course, just another way of saying that in these nonrelativistic theories interactions propagate instantaneously. We introduce, then, to replace v^i, a vector field F^μ with a Lagrangian density proportional to $F_\mu F^\mu$. How is this field coupled to $x^\mu(\sigma)$? The clue to answering this question lies in the fact that the first term in the right-hand side of the Lagrangian (2.7) is not translation invariant and must be replaced by a term that does have this invariance, i.e.,

$$A_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\nu}{\partial\tau}, \tag{2.8}$$

where $A_{\mu\nu}$ is an antisymmetric tensor. We have changed the notation for the time variable from t to τ to emphasize the fact that it is no longer the Newtonian time but an arbitrary parameter. It is natural now to take this term as the coupling between F^μ and x^μ, and we must find how F^μ can be related to $A_{\mu\nu}$. This is done as follows: In the nonrelativistic case, $x^0=\tau=t$ and the field $A_{\mu\nu}$ evaluated at the vortex must satisfy

$$A_{ij}\big|_{\vec{y}=\vec{x}}=\epsilon_{ijk}x^k\,. \tag{2.9}$$

This is most easily satisfied just by taking

$$A_{ij}(\vec{y})=\epsilon_{ijk}y^k\,. \tag{2.10}$$

On the other hand, still in this nonrelativistic case,

$$F^i=v^i=\epsilon^{ijk}\partial_jA_k\,. \tag{2.11}$$

The easiest way of generalizing this to four dimensions is to define

$$F^\mu\equiv\tfrac{1}{2}\epsilon^{\mu\nu\lambda\rho}\partial_\rho A_{\nu\lambda}\,. \tag{2.12}$$

We have been led then to the action

$$S'=f\int A_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\nu}{\partial\tau}\,d\sigma\,d\tau-\tfrac{1}{4}\int F^2d^4y\,, \tag{2.13}$$

with f a coupling constant and with F^μ given in terms of $A_{\mu\nu}=-A_{\nu\mu}$ by (2.12). The first integral is taken over the surface described by the vortex in space-time and the field $A_{\mu\nu}$ in the integrand is evaluated at this world surface. This action is a Lorentz scalar and is also invariant under coordinate transformations on the surface as well as under the gauge transformation

$$A_{\mu\nu}\to A_{\mu\nu}+\partial_\mu\Lambda_\nu-\partial_\nu\Lambda_\mu\,. \tag{2.14}$$

This is true owing to the fact that vortices are closed or, eventually, infinitely long.

We have then a relativistic system with variables $x^\mu(\sigma,\tau)$ and $A_{\mu\nu}(y)$. If the coupling constant vanishes, this is a free massless scalar field. The apparent tensor nature of the $A_{\mu\nu}$ does not mean that it carries spin. This is most directly seen using the canonical formalism. We do this in the Appendix. What we do not know is how the vortex moves when it is decoupled from the field. We need an extra term in the action. The obvious candidate is the Nambu action[3] for a relativistic string. In fact, whether we call our one-dimensional objects strings or vortices makes no difference, as they are abstractions having little resemblance to what we intuitively understand by either string or vortex. We shall use both names indistinctively. Also, as we shall see below, it is the Nambu action that must be added if we want to be able to regularize the self-interaction of a vortex. We have then

$$S=-N\int\sqrt{-g}\,d\sigma\,d\tau+f\int A_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\nu}{\partial\tau}\,d\sigma\,d\tau-\tfrac{1}{4}\int F^2d^4y\,, \tag{2.15}$$

with

$$\sqrt{-g}=\left[-\left(\frac{\partial x^\mu}{\partial\sigma}\frac{\partial x_\mu}{\partial\sigma}\right)\left(\frac{\partial x^\nu}{\partial\tau}\frac{\partial x_\nu}{\partial\tau}\right)+\left(\frac{\partial x^\mu}{\partial\sigma}\frac{\partial x_\mu}{\partial\tau}\right)^2\right]^{1/2} \tag{2.16}$$

It is clear that the action (2.15) has the same invariances as the preliminary action (2.13). The same expression has been obtained by Kalb and Ramond[4] through a different line of reasoning.

To summarize, then, we have an action that, for $f=0$, describes a system of noninteracting strings and fields. For $N=0$, that is, for Regge trajectories of infinite slope, we get the action (2.13). In the particular Lorentz frame $x^0=\tau$ and for

$$A_{i0}=A_i,\quad A_{ij}=\epsilon_{ijk}y^k \tag{2.17}$$

we recover the action (2.7) for nonrelativistic vortex motion in a superfluid, apart from an unimportant additive constant.

We now turn to the equations of motion. The variables $x^\mu(\sigma,\tau)$ describe a two-dimensional surface. Every such surface is conformally flat, and it is possible to choose coordinates in which this is apparent ("orthonormal gauge"). Since the metric of the surface is

$$ds^2 = \frac{\partial x^\mu}{\partial \sigma}\frac{\partial x_\mu}{\partial \sigma} d\sigma^2 + 2\frac{\partial x^\mu}{\partial \sigma}\frac{\partial x_\mu}{\partial \tau} d\sigma\, d\tau + \frac{\partial x^\mu}{\partial \tau}\frac{\partial x_\mu}{\partial \tau} d\tau^2 , \tag{2.18}$$

the conditions that must be satisfied are

$$\frac{\partial x^\mu}{\partial \sigma}\frac{\partial x_\mu}{\partial \sigma} + \frac{\partial x^\mu}{\partial \tau}\frac{\partial x_\mu}{\partial \tau} = 0 , \quad \frac{\partial x^\mu}{\partial \sigma}\frac{\partial x_\mu}{\partial \tau} = 0 . \tag{2.19}$$

Also, considering the gauge freedom (2.14) for the field $A_{\mu\nu}$, it is possible to impose the "Lorentz condition"

$$\partial_\mu A^{\mu\nu} = 0 . \tag{2.20}$$

These remarks are made more precise in the context of the canonical formalism in the Appendix. With the gauge choices (2.19) and (2.20), the equations of motion obtained by minimizing the action (2.15) are

$$\partial^\mu \partial_\mu A^{\alpha\beta} = -2f \int \left(\frac{\partial x^\alpha}{\partial \sigma}\frac{\partial x^\beta}{\partial \tau} - \frac{\partial x^\alpha}{\partial \tau}\frac{\partial x^\beta}{\partial \sigma}\right) \times \delta^{(4)}(x-y)\, d\sigma\, d\tau , \tag{2.21}$$

$$N\left(\frac{\partial^2 x^\mu}{\partial \tau^2} - \frac{\partial^2 x^\mu}{\partial \sigma^2}\right) = f\epsilon^{\mu\nu\lambda\rho} F_\nu \frac{\partial x_\rho}{\partial \tau}\frac{\partial x_\lambda}{\partial \sigma} . \tag{2.22}$$

We consider first Eq. (2.19), that gives the field produced by a moving string. It is just a wave equation with an external source and has the solution

$$A^{\mu\nu}(y) = f \int G_{\text{ret}}(y-x) \times \left(\frac{\partial x^\mu}{\partial \sigma}\frac{\partial x^\nu}{\partial \tau} - \frac{\partial x^\nu}{\partial \sigma}\frac{\partial x^\mu}{\partial \tau}\right) d\sigma\, d\tau , \tag{2.23}$$

where

$$G_{\text{ret}}(x) = \frac{1}{2\pi}\,\delta(x^\mu x_\mu)\,\theta(x_0) \tag{2.24}$$

is the retarded Green's function. From here one has

$$F^\mu(y) = -\frac{f}{\pi}\int \theta(y_0 - x_0)\,\delta'((y-x)^2) \times \epsilon^{\mu\nu\lambda\rho}(x-y)_\nu\, dx_\lambda \wedge dx_\rho , \tag{2.25}$$

with

$$dx_\mu \wedge dx_\nu = \left(\frac{\partial x_\mu}{\partial \sigma}\frac{\partial x_\nu}{\partial \tau} - \frac{\partial x_\mu}{\partial \tau}\frac{\partial x_\nu}{\partial \sigma}\right) d\sigma\, d\tau . \tag{2.26}$$

It is an illuminating exercise to show that F^μ is (locally) the gradient of a scalar field. As we said before, although we are working with an antisymmetric tensor $A_{\mu\nu}$, there is only one dynamical degree of freedom per space point. The rest is gauge. Explicitly,

$$F_\mu = \partial_\mu \phi , \tag{2.27}$$

with

$$\phi(y) = -\frac{f}{3\pi}\int \theta(y_0 - x_0)\,\delta'((y-x)^2) \times \epsilon^{\mu\nu\lambda\rho}(x-y)_\mu\, dx_\nu \wedge dx_\lambda \wedge dx_\rho . \tag{2.28}$$

If the vortex is static, ϕ is the velocity potential for the fluid or, if instead of a vortex we consider a static current loop, the scalar potential for the magnetic field. A short calculation shows that we recover in any case the classical result that ϕ is the solid angle subtended by the loop at the point of observation.

Consider now the interaction term of the action. Substitution into it of (2.23) gives the self-interaction of a vortex,

$$Y \equiv \frac{f^2}{4\pi}\int d\sigma\, d\sigma' d\tau\, d\tau'\, \theta(x_0(\sigma,\tau) - x_0(\sigma',\tau'))\,\delta([x(\sigma,\tau) - x(\sigma',\tau')]^2) \times \left(\frac{\partial x^\mu}{\partial \sigma}\frac{\partial x^\nu}{\partial \tau} - \frac{\partial x^\mu}{\partial \tau}\frac{\partial x^\nu}{\partial \sigma}\right)\left(\frac{\partial x_\mu}{\partial \sigma'}\frac{\partial x_\nu}{\partial \tau'} - \frac{\partial x_\mu}{\partial \tau'}\frac{\partial x_\nu}{\partial \sigma'}\right). \tag{2.29}$$

The integrand in this expression diverges when $\sigma \sim \sigma'$, $\tau \sim \tau'$. To study this divergence, consider the finite expression Y_ϵ obtained from Y by replacing $\delta([x(\sigma,\tau) - x(\sigma',\tau')]^2)$ by $\delta([x(\sigma,\tau) - x(\sigma',\tau')]^2 + \epsilon^2)$. Clearly, $\lim_{\epsilon \to 0} Y_\epsilon = Y$.

Introducing the new variables

$$\Delta\sigma \equiv \sigma - \sigma', \quad \Delta\tau \equiv \tau - \tau' , \tag{2.30}$$

and developing the integrand of (2.29) in a power series in $\Delta\sigma$ and $\Delta\tau$ we obtain

$$Y_\epsilon = \frac{f^2}{2\pi}\int d(\Delta\sigma)d(\Delta\tau)d\sigma\,d\tau\,\delta\left(\frac{\partial x^\mu}{\partial\sigma}\frac{\partial x_\mu}{\partial\sigma}(\Delta\sigma)^2 + 2\frac{\partial x^\mu}{\partial\sigma}\frac{\partial x_\mu}{\partial\tau}\Delta\sigma\Delta\tau + \frac{\partial x^\mu}{\partial\tau}\frac{\partial x_\mu}{\partial\tau}(\Delta\tau)^2 + \epsilon^2\right)$$
$$\times\left[\left(\frac{\partial x^\mu}{\partial\sigma}\frac{\partial x_\mu}{\partial\sigma}\right)\left(\frac{\partial x^\nu}{\partial\tau}\frac{\partial x_\nu}{\partial\tau}\right) - \left(\frac{\partial x^\mu}{\partial\sigma}\frac{\partial x_\mu}{\partial\tau}\right)^2\right]$$
$$+\text{higher-order terms}. \tag{2.31}$$

The integration over $\Delta\tau$ is readily carried out with the aid of the δ function, giving

$$Y_\epsilon = \frac{f^2}{4\pi}\int d(\Delta\sigma)d\sigma\,d\tau(g)\left[g(\Delta\sigma)^2 + \epsilon^2\frac{\partial x^\mu}{\partial\tau}\frac{\partial x_\mu}{\partial\tau}\right]^{-1/2} + \text{higher-order terms}, \tag{2.32}$$

where g is given by (2.16). For small ϵ we have

$$Y_\epsilon \underset{\epsilon\to 0}{\sim} -\frac{f^2}{2\pi}\ln\left(\frac{\epsilon_0}{\epsilon}\right)\int d\sigma\,d\tau\sqrt{-g}$$
$$+\text{finite terms}, \tag{2.33}$$

with ϵ_0 being a cutoff in $\Delta\sigma$.

We see that the divergent term has exactly the same form as the Nambu action, and we may absorb it with a renormalization of the constant N appearing in (2.15),

$$N_R = N + \frac{f^2}{2\pi}\ln\left(\frac{\epsilon_0}{\epsilon}\right). \tag{2.34}$$

III. MOTION OF A STRING IN A UNIFORM, STATIC, EXTERNAL FIELD

In this section we consider the equation of motion (2.22) for the $x^\mu(\sigma,\tau)$, subject to the constraints (2.19), when the field F^μ is a given static and uniform external quantity. We study the motion in a Lorentz frame in which

$$x^0 = \tau, \quad F^0 = \tilde{F}, \quad F^i = 0, \tag{3.1}$$

with $\tilde{F}$ a constant. In this case, the $\mu = 0$ of Eqs. (2.22) is identically satisfied. The others are

$$\frac{\partial^2\vec{x}}{\partial\tau^2} - \frac{\partial^2\vec{x}}{\partial\sigma^2} = -2c\left(\frac{\partial\vec{x}}{\partial\tau}\times\frac{\partial\vec{x}}{\partial\sigma}\right) \tag{3.2}$$

where

$$c = \frac{f\tilde{F}}{2N}, \tag{3.3}$$

and the constraints (2.19) become

$$\frac{\partial\vec{x}}{\partial\sigma}\cdot\frac{\partial\vec{x}}{\partial\tau} = 0, \tag{3.4}$$

$$\left(\frac{\partial\vec{x}}{\partial\tau}\right)^2 + \left(\frac{\partial\vec{x}}{\partial\sigma}\right)^2 = 1. \tag{3.5}$$

The metric of the world surface is given by (2.18). Its projection on three-dimensional space has the metric

$$ds^2 = \left(\frac{\partial\vec{x}}{\partial\sigma}\right)^2 d\sigma^2 + 2\left(\frac{\partial\vec{x}}{\partial\sigma}\cdot\frac{\partial\vec{x}}{\partial\tau}\right)d\sigma\,d\tau + \left(\frac{\partial\vec{x}}{\partial\tau}\right)^2 d\tau^2, \tag{3.6}$$

which may be written, according to (3.4) and (3.5), as

$$ds^2 = \cos^2\theta\,d\sigma^2 + \sin^2\theta\,d\tau^2. \tag{3.7}$$

Notice that as x^0 is fixed by the choice (3.1), all we need to know is $\vec{x}(\sigma,\tau)$, and is enough to consider the surface S obtained by projecting the world surface of the string on the $x^0 = \tau$ hyperplane.

In what follows, we shall use the notation[10]

$$\left(\frac{\partial\vec{x}}{\partial\sigma}\right)^2 = E, \quad \frac{\partial\vec{x}}{\partial\sigma}\cdot\frac{\partial\vec{x}}{\partial\tau} = F, \quad \left(\frac{\partial\vec{x}}{\partial\tau}\right)^2 = G \tag{3.8}$$

and we shall call D, D', and D'' the coefficients of the second fundamental form of the surface (also called the extrinsic curvature).

At a given point on S, $\partial\vec{x}/\partial\sigma$ and $\partial\vec{x}/\partial\tau$ are vectors tangent to the τ- and σ-coordinate lines, respectively. They span the plane tangent to S at that point and their vector product is the vector normal to S there. The angle θ in (3.7) is $\frac{1}{2}$ the angle between the two tangent vectors.[11] The length of the normal vector is $(EG - F^2)^{1/2}$.

One possible definition of the coefficients of the second fundamental form is

$$D = \frac{\partial^2\vec{x}}{\partial\sigma^2}\cdot\hat{X}_3, \tag{3.9}$$

$$D' = \frac{\partial^2\vec{x}}{\partial\sigma\partial\tau}\cdot\hat{X}_3, \tag{3.10}$$

$$D'' = \frac{\partial^2\vec{x}}{\partial\tau^2}\cdot\hat{X}_3, \tag{3.11}$$

where we have introduced the unit normal

$$\hat{X}_3 = (EG - F^2)^{-1/2}\frac{\partial\vec{x}}{\partial\sigma}\times\frac{\partial\vec{x}}{\partial\tau}. \tag{3.12}$$

Analogously, we shall use the unit tangents

$$\hat{X}_1 = \frac{1}{\sqrt{E}}\frac{\partial\vec{x}}{\partial\sigma}, \tag{3.13}$$

$$\hat{X}_2 = \frac{1}{\sqrt{G}}\frac{\partial\vec{x}}{\partial\tau}. \tag{3.14}$$

Taking the dot product of (3.2) with $\hat{X}_3$ we obtain

$$D''-D=2c(EG-F^2)^{1/2}. \tag{3.15}$$

It is a classical result of surface theory (see Ref. 10) that, given coordinates u_1, u_2, a metric tensor

$$g_{ij}=\begin{pmatrix} E & F \\ F & G \end{pmatrix}, \tag{3.16}$$

and a second fundamental form

$$L_{ij}=\begin{pmatrix} D & D' \\ D' & D'' \end{pmatrix}, \tag{3.17}$$

the equations (of Gauss and Weingarten)

$$\frac{\partial \hat{X}_i}{\partial u_k}=\Gamma^l{}_{ik}\hat{X}_l+L_{ik}\hat{X}_3, \tag{3.18a}$$

$$\frac{\partial \hat{X}_3}{\partial u_k}=-g^{ij}L_{kj}\hat{X}_i \tag{3.18b}$$

have a unique solution for $\hat{X}_1$, $\hat{X}_2$, and $\hat{X}_3$, provided the following compatibility conditions hold:

$$\frac{\partial L_{12}}{\partial u_1}-\frac{\partial L_{11}}{\partial u_2}+\Gamma^l{}_{12}L_{l1}-\Gamma^l{}_{11}L_{l2}=0, \tag{3.19a}$$

$$\frac{\partial L_{22}}{\partial u_1}-\frac{\partial L_{21}}{\partial u_2}+\Gamma^l{}_{22}L_{l1}-\Gamma^l{}_{21}L_{l2}=0, \tag{3.19b}$$

$$K=\frac{DD''-(D')^2}{EG-F^2}. \tag{3.20}$$

Here, K is the (intrinsic) scalar curvature of the surface and $\Gamma^l{}_{ik}$ are the Christoffel symbols. Equations (3.19) are called the Codazzi-Mainardi equations and (3.20) is the Gauss equation.

Moreover, these $\hat{X}_1, \hat{X}_2, \hat{X}_3$ so determined uniquely fix a surface $\vec{x}(\sigma,\tau)$, whose metric tensor is g_{ij}, whose second fundamental form is L_{ij}, and has $\hat{X}_1$ and $\hat{X}_2$ as unit tangents and $\hat{X}_3$ as unit normal.

We now go back to the equations of motion (3.2). It is easy to see that, as a consequence of Eq. (3.18a), and of the form (3.7) of the metric tensor, the three equations (3.2) are equivalent to the one equation (3.15). Our problem is then to find those surfaces with metric tensor of the form (3.7) and second fundamental form that satisfies (3.15).

Using (3.16) and (3.17), Eqs. (3.18) may be written as follows ($F=0$):

$$\frac{\partial}{\partial\sigma}\begin{pmatrix}\hat{X}_1\\ \hat{X}_2\\ \hat{X}_3\end{pmatrix}=\begin{pmatrix} 0 & -\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial\tau} & \frac{D}{\sqrt{E}} \\ \frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial\tau} & 0 & \frac{D'}{\sqrt{G}} \\ -\frac{D}{\sqrt{E}} & -\frac{D'}{\sqrt{G}} & 0\end{pmatrix}\begin{pmatrix}\hat{X}_1\\ \hat{X}_2\\ \hat{X}_3\end{pmatrix}, \tag{3.21}$$

$$\frac{\partial}{\partial\tau}\begin{pmatrix}\hat{X}_1\\ \hat{X}_2\\ \hat{X}_3\end{pmatrix}=\begin{pmatrix} 0 & \frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial\sigma} & \frac{D'}{\sqrt{E}} \\ -\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial\sigma} & 0 & \frac{D''}{\sqrt{G}} \\ -\frac{D'}{\sqrt{E}} & -\frac{D''}{\sqrt{G}} & 0\end{pmatrix}\begin{pmatrix}\hat{X}_1\\ \hat{X}_2\\ \hat{X}_3\end{pmatrix}, \tag{3.22}$$

and the integrability conditions (3.19) and (3.20), which are nothing more than the statement that crossed derivatives of (3.21) and (3.22) must coincide, become

$$\frac{\partial D}{\partial\tau}-\frac{\partial D'}{\partial\sigma}-\frac{D}{2E}\frac{\partial E}{\partial\tau}+\left(\frac{1}{2E}\frac{\partial E}{\partial\sigma}-\frac{1}{2G}\frac{\partial G}{\partial\sigma}\right)D'-\frac{1}{2G}\frac{\partial E}{\partial\tau}D''=0, \tag{3.23}$$

$$\frac{\partial D''}{\partial\sigma}-\frac{\partial D'}{\partial\tau}-\frac{D}{2E}\frac{\partial E}{\partial\sigma}+\left(\frac{1}{2G}\frac{\partial G}{\partial\tau}-\frac{1}{2E}\frac{\partial E}{\partial\tau}\right)D'-\frac{1}{2G}\frac{\partial G}{\partial\sigma}D''=0, \tag{3.24}$$

$$\frac{DD''-(D')^2}{EG}=-\frac{1}{\sqrt{EG}}\left[\frac{\partial}{\partial\sigma}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial\sigma}\right)+\frac{\partial}{\partial\tau}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial\tau}\right)\right]. \tag{3.25}$$

Substitution of (3.7) into (3.15) gives

$$D''-D=2c\sin\theta\cos\theta, \tag{3.26}$$

and defining

$$B\equiv\tfrac{1}{2}(D''+D), \tag{3.27}$$

the integrability conditions (3.23)–(3.25) become, using (3.7) and (3.26),

$$\frac{\partial B}{\partial\tau}-\frac{\partial D'}{\partial\sigma}+\frac{\partial\theta}{\partial\tau}\frac{B}{\sin\theta\cos\theta}-\frac{\partial\theta}{\partial\sigma}\frac{D'}{\sin\theta\cos\theta}=0, \tag{3.28}$$

$$\frac{\partial B}{\partial\sigma}-\frac{\partial D'}{\partial\tau}+\frac{\partial\theta}{\partial\tau}\frac{D'}{\sin\theta\cos\theta}-\frac{\partial\theta}{\partial\sigma}\frac{B}{\sin\theta\cos\theta}=0, \tag{3.29}$$

$$B^2-(D')^2=c^2\sin^2\theta\cos^2\theta+\sin\theta\cos\theta\left(\frac{\partial^2\theta}{\partial\tau^2}-\frac{\partial^2\theta}{\partial\sigma^2}\right). \tag{3.30}$$

The key remark now is, very simply, that (3.21) and (3.22) are three-dimensional rotations of $(\hat{X}_1, \hat{X}_2, \hat{X}_3)$ with rotation vectors

$$\vec{\omega}_\sigma=\left(\frac{D'}{\sqrt{G}},-\frac{D}{\sqrt{E}},-\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial\tau}\right)$$

and

$$\vec{\omega}_\tau=\left(\frac{D''}{\sqrt{G}},\frac{D'}{\sqrt{E}},\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial\sigma}\right),$$

respectively. We may then write these operations in the spin-$\frac{1}{2}$ representation:

$$\frac{\partial\psi}{\partial\sigma}=\left(\frac{i}{2}\vec{\omega}_\sigma\cdot\vec{\sigma}\right)\psi, \tag{3.31}$$

$$\frac{\partial\psi}{\partial\tau}=\left(\frac{i}{2}\vec{\omega}_\tau\cdot\vec{\sigma}\right)\psi, \tag{3.32}$$

where ψ is a two-component spinor and $\vec{\sigma}=(\sigma_1,\sigma_2,\sigma_3)$ are the Pauli matrices.

Let us consider first the special case $B=D'=0$. In this case, the Gaussian curvature is constant and Eq. (3.30) simplifies to

$$-\frac{\partial^2\theta}{\partial\tau^2}+\frac{\partial^2\theta}{\partial\sigma^2}=c^2\sin\theta\cos\theta, \tag{3.33}$$

the familiar sine-Gordon equation. Conditions (3.28) and (3.29) are, of course, trivially satisfied. Equations (3.31) and (3.32) take the form

$$2i\frac{\partial\psi}{\partial\sigma}=-\begin{pmatrix}\frac{\partial\theta}{\partial\tau} & -ic\sin\theta\\ ic\sin\theta & -\frac{\partial\theta}{\partial\tau}\end{pmatrix}\psi, \tag{3.34}$$

$$2i\frac{\partial\psi}{\partial\tau}=-\begin{pmatrix}\frac{\partial\theta}{\partial\sigma} & c\cos\theta\\ c\cos\theta & -\frac{\partial\theta}{\partial\sigma}\end{pmatrix}\psi. \tag{3.35}$$

In light-cone coordinates

$$\alpha\equiv\tfrac{1}{2}c(\tau+\sigma),\qquad \beta\equiv\tfrac{1}{2}c(-\tau+\sigma), \tag{3.36}$$

they are

$$2i\frac{\partial\psi}{\partial\alpha}=-\begin{pmatrix}\frac{\partial\theta}{\partial\alpha} & e^{-i\theta}\\ e^{i\theta} & -\frac{\partial\theta}{\partial\alpha}\end{pmatrix}\psi, \tag{3.37}$$

$$2i\frac{\partial\psi}{\partial\beta}=\begin{pmatrix}\frac{\partial\theta}{\partial\beta} & e^{i\theta}\\ e^{-i\theta} & -\frac{\partial\theta}{\partial\beta}\end{pmatrix}\psi. \tag{3.38}$$

The terms in $\partial\theta/\partial\beta$ may be eliminated from (3.37) by a rotation in spinor space. Defining

$$\overline{\psi}\equiv e^{i\sigma_3\theta/2}\psi, \tag{3.39}$$

one gets

$$2i\frac{\partial\overline{\psi}}{\partial\beta}=\begin{pmatrix}0 & e^{i\phi}\\ e^{-i\phi} & 0\end{pmatrix}\overline{\psi}, \tag{3.40}$$

$$2i\frac{\partial\overline{\psi}}{\partial\alpha}=-\begin{pmatrix}\frac{\partial\phi}{\partial\alpha} & 1\\ 1 & -\frac{\partial\phi}{\partial\alpha}\end{pmatrix}\overline{\psi} \tag{3.41}$$

where $\phi=2\theta$. The sine-Gordon equation in these coordinates is

$$\frac{\partial^2\phi}{\partial\alpha\partial\beta}=\sin\phi. \tag{3.42}$$

Leave now this line of reasoning and go back for a moment to the metric (2.18), which may be written, using the constraints (2.19), as

$$ds^2=\left(\frac{\partial x^\mu}{\partial\sigma}\frac{\partial x_\mu}{\partial\sigma}\right)(d\sigma^2-d\tau^2). \tag{3.43}$$

This expression is manifestly conformally flat, and it remains so under Lorentz transformations on the two-dimensional σ-τ parameter space; say

$$\sigma'=\cosh a\,\sigma+\sinh a\,\tau,\qquad \tau'=\sinh a\,\sigma+\cosh a\,\tau. \tag{3.44}$$

On the other hand, the equation of motion (2.22) is also invariant under these transformations. If, instead of taking the projection of the world surface on the hypersurface $x^0=\tau$, we take it on the hypersurface $x^0=\tau'$, it is still possible to choose $F^0=F$ and $F^i=0$. The $\mu=0$ of Eqs. (2.22) will still be identically satisfied, and Eq. (3.2) will keep its form. Writing out the constraints (2.19) with this new coordinate condition for x^0, we get

$$\frac{\partial\vec{x}}{\partial\tau}\cdot\frac{\partial\vec{x}}{\partial\sigma}=\sinh a\cosh a,\qquad \left(\frac{\partial\vec{x}}{\partial\tau}\right)^2+\left(\frac{\partial\vec{x}}{\partial\sigma}\right)^2=\cosh^2 a+\sinh^2 a, \tag{3.45}$$

and, in the primed coordinates,

$$\frac{\partial\vec{x}}{\partial\tau'}\cdot\frac{\partial\vec{x}}{\partial\sigma'}=0,\qquad \left(\frac{\partial\vec{x}}{\partial\tau'}\right)^2+\left(\frac{\partial\vec{x}}{\partial\sigma'}\right)^2=1. \tag{3.46}$$

In terms of these new coordinates then, the geometrical problem is unchanged. We are still looking for the geometry of a two-dimensional surface embedded in three-dimensional space, which is the projection of a world surface that is embedded in four-dimensional space-time. What *has* changed is the three-space where we are doing the projection.

The transformation (3.44) induces the following transformation on the light-cone coordinates:

$$\alpha'=e^a\alpha,\quad \beta'=e^{-a}\beta. \tag{3.47}$$

This leaves Eq. (3.42) unchanged, and (3.40) and

(3.41) take the form

$$2i\frac{\partial\bar{\psi}}{\partial\alpha'}=-\begin{pmatrix}\frac{\partial\phi}{\partial\alpha'} & k\\ k & -\frac{\partial\phi}{\partial\alpha'}\end{pmatrix}\bar{\psi}, \tag{3.48}$$

$$2i\frac{\partial\bar{\psi}}{\partial\beta'}=-\begin{pmatrix}0 & k^{-1}e^{i\phi}\\ k^{-1}e^{-i\phi} & 0\end{pmatrix}\bar{\psi}, \tag{3.49}$$

where $k=e^{-a}$. These are the equations employed by Ablowitz *et al.*[7] to solve the sine-Gordon equation (3.42) by the inverse scattering method. (The correspondence with that paper's notation is $k=2\xi$, $\bar{\psi}_1=v_1+iv_2$, and $\bar{\psi}_2=v_1-iv_2$.) They have also been used by Orfanidis[12] to study the relation between the Thirring and sine-Gordon models at the classical level.

It is interesting to see that there is a natural geometric interpretation for the linear eigenvalue problem which was used in Ref. 7 to solve the sine-Gordon equation: This equation expresses the "embeddability condition" that a surface with metric given by $E=\cos^2\theta$, $F=0$, $G=\sin^2\theta$, and extrinsic curvature $D''=-D=c\sin\theta\cos\theta$, $D'=0$, must satisfy in order to be embeddable in a three-dimensional Euclidean space. The linear equations (3.48) and (3.49) give the actual construction of the tangent and normal vectors to that surface, thus actually constructing it. Different values of the parameter k give different surfaces. These surfaces are related, however, all of them being projections of one and the same surface embedded in a four-dimensional space-time along different three-dimensional spaces. These spaces are, in turn, transformed into each other by Lorentz transformations.

The sine-Gordon equation has the interesting feature of admitting solitary waves among its solutions. It can also be considered as the evolution equation for a scalar field in one space and one time dimension with a highly nonlinear self-interaction. Our geometrical framework provides for an immediate generalization of this theory. Namely, one can ask what results if one considers surfaces whose extrinsic curvature in the three-dimensional space is not restricted by $B=D'=0$. In this case, the (nonlinear) embeddability conditions are (3.28)–(3.30) and the (linear) equations for the tangent and normal vectors are (3.31) and (3.32). Equations (3.28) and (3.29) may be written as follows:

$$\frac{\partial}{\partial\tau}(\tan\theta\, B)=\frac{\partial}{\partial\sigma}(\tan\theta\, D'), \tag{3.50}$$

$$\frac{\partial}{\partial\sigma}(\cot\theta\, B)=\frac{\partial}{\partial\tau}(\cot\theta\, D'). \tag{3.51}$$

From (3.50) it follows that there exist λ such that

$$B=\cot\theta\frac{\partial\lambda}{\partial\sigma},\quad D'=\cot\theta\frac{\partial\lambda}{\partial\tau}. \tag{3.52}$$

Substitution into (3.51) yields an equation for λ:

$$\frac{\partial}{\partial\sigma}\left(\cot^2\theta\frac{\partial\lambda}{\partial\sigma}\right)=\frac{\partial}{\partial\tau}\left(\cot^2\theta\frac{\partial\lambda}{\partial\tau}\right). \tag{3.53}$$

The other embeddability condition, Eq. (3.30) is, according to (3.52),

$$\frac{\partial^2\theta}{\partial\tau^2}-\frac{\partial^2\theta}{\partial\sigma^2}+c^2\sin\theta\cos\theta+\frac{\cos\theta}{\sin^3\theta}\left[\left(\frac{\partial\lambda}{\partial\tau}\right)^2-\left(\frac{\partial\lambda}{\partial\sigma}\right)^2\right]=0. \tag{3.54}$$

The last two equations can now be interpreted as the dynamical equations for two coupled scalar fields. They can be derived from the Lagrangian

$$\mathcal{L}=\frac{1}{2}\left[\left(\frac{\partial\theta}{\partial\sigma}\right)^2-\left(\frac{\partial\theta}{\partial\tau}\right)^2\right]+\frac{c^2}{2}\sin^2\theta+\tfrac{1}{2}\cot^2\theta\left[\left(\frac{\partial\lambda}{\partial\sigma}\right)^2-\left(\frac{\partial\lambda}{\partial\tau}\right)^2\right], \tag{3.55}$$

which is a Lorentz scalar in σ-τ space-time. The linear equations (3.31) and (3.32) are now

$$2i\frac{\partial\psi}{\partial\sigma}=-\begin{pmatrix}\frac{\partial\theta}{\partial\tau} & \frac{\cos\theta}{\sin^2\theta}\frac{\partial\lambda}{\partial\tau}+\frac{i}{\sin\theta}\frac{\partial\lambda}{\partial\sigma}-ic\sin\theta\\ \frac{\cos\theta}{\sin^2\theta}\frac{\partial\lambda}{\partial\tau}-\frac{i}{\sin\theta}\frac{\partial\lambda}{\partial\sigma}+ic\sin\theta & -\frac{\partial\theta}{\partial\tau}\end{pmatrix}\psi, \tag{3.56}$$

$$2i\frac{\partial\psi}{\partial\tau}=-\begin{pmatrix}\frac{\partial\theta}{\partial\sigma} & \frac{\cos\theta}{\sin^2\theta}\frac{\partial\lambda}{\partial\sigma}+c\cos\theta-\frac{i}{\sin\theta}\frac{\partial\lambda}{\partial\tau}\\ \frac{\cos\theta}{\sin^2\theta}\frac{\partial\lambda}{\partial\sigma}+c\cos\theta+\frac{i}{\sin\theta}\frac{\partial\lambda}{\partial\tau} & -\frac{\partial\theta}{\partial\sigma}\end{pmatrix}\psi. \tag{3.57}$$

It is easy to check directly by cross-differentiation that (3.53) and (3.54) are indeed the integrability conditions for (3.56) and (3.57).

We now look for static (τ-independent) solutions to the field equations. They are, in this case,

$$\frac{\partial}{\partial\sigma}\left(\cot^2\theta\frac{\partial\lambda}{\partial\sigma}\right)=0\,, \tag{3.58}$$

$$\frac{\partial^2\theta}{\partial\sigma^2}-c^2\sin\theta\cos\theta+\frac{\cos\theta}{\sin^3\theta}\left(\frac{\partial\lambda}{\partial\sigma}\right)^2=0\,. \tag{3.59}$$

It is easy to solve these equations using elliptic functions to obtain periodic waves. A particular solution that vanishes at infinity is given by

$$\theta=\arcsin\left((1-A^2)^{1/2}\sin\{2\tan^{-1}\exp[\sigma(1-A^2)^{1/2}]\}\right)\,, \tag{3.60}$$

$$\frac{\partial\lambda}{\partial\sigma}=A\tan^2\theta\,, \tag{3.61}$$

where A is a constant. The form of θ is sketched in Fig. 1 and that of λ in Fig. 2. Performing a Lorentz transformation we get a localized wave traveling at a constant velocity and remaining localized for all time. When $A\to 0$ the well-known sine-Gordon soliton is recovered.

IV. CONCLUDING REMARKS

We have constructed a classical theory of one-dimensional extended objects interacting through a scalar field. Special cases are the Nambu string and vortices in a superfluid. The divergence of the self-energy of a vortex may be regularized by a renormalization of the slope of the Regge trajectories. The action integral describing our system is the same as the one found previously by Kalb and Ramond[4] in a somewhat different context. At this stage, our contributions are then the relation of this action with the one describing nonrelativistic vortex motion in a superfluid and the regularization of the self-energy.

The study of the motion of a closed string in an external field led us to a geometrical problem throwing some light on the sine-Gordon equation: We found a natural geometric interpretation to the linear eigenvalue problem used to solve this equation by the inverse scattering method. This system was obtained under certain simplifying assumptions on the motion of the string. In the general case we were led to a set of two coupled nonlinear equations. This relativistic system, as does the sine-Gordon equations, admits solitary wave solutions, whose explicit form we have determined. The Lagrangian for this system was also found.

At this point, several problems are left open. Firstly, the system of nonlinear coupled equations (3.53) and (3.54) can be studied as a purely mathematical problem. Can the inverse scattering problem for the associated linear system (3.56) and (3.57) be solved? It is unfortunate that, at least at first sight, this set of equations does not seem to fit among the general class considered by Ablowitz *et al.*[13] in their systematic search for solutions to nonlinear equations. One should be able at least to determine whether a Bäcklund transformation exists and whether there is an infinite set of conservation laws[14] or not. The fact that we have a linear problem naturally associated with the nonlinear equations should be of help in resolving these points.

Secondly, one should find out whether the Lagrangian (3.55) describes a system with any relation to the physics of elementary particles. The first question to be settled in this context is whether the theory described by this Lagrangian

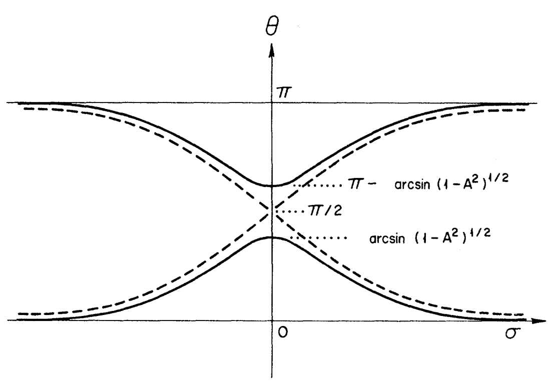

FIG. 1. The solid line is the solitary wave given by (3.60). In the limit $A\to 0$ it goes over to the sine-Gordon soliton (dashed line). Notice that if θ is a solution, so is $\pi-\theta$. Both cases are drawn here.

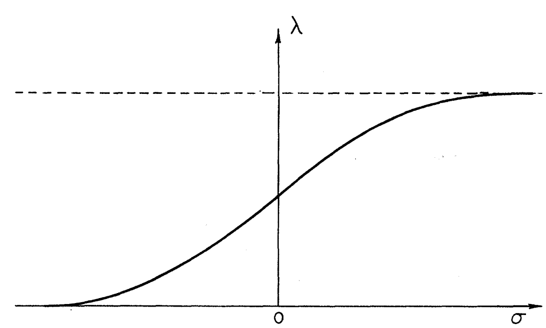

FIG. 2. Shape of field λ given by (3.61) when θ is a solitary wave.

is renormalizable.

Thirdly, as we pointed out in the Introduction, we have started from one physical system, namely strings in four dimensions, and through the mathematics of the problem we have arrived at quite a different one, coupled fields in two dimensions. It would be very interesting to find whether there is more than just formalism in this relation.

Finally, we have not tried to quantize either the system described by the Lagrangian (2.15) or the one described by the Lagrangian (3.55).

Note added in proof. We have been informed by B. Julia that the Lagrangian (3.55) has also been obtained by K. Pohlmeyer, Commun. Math. Phys. 46, 207 (1976).

ACKNOWLEDGMENTS

Stimulating discussions with B. Julia are gratefully acknowledged. Thanks are also due to T. Damour, R. Dashen, M. Kruskal, A. Mueller, H. Szu, and M. Virasoro for informative conversations.

APPENDIX

Here we shall consider the canonical formalism for the system described by the action (in units such that $N=1$)

$$S=-\int\sqrt{-g}\,d\sigma d\tau+f\int A_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\nu}{\partial\tau}d\sigma d\tau -\tfrac{1}{4}\int F_\mu F^\mu d^4y\,. \quad (A1)$$

There are two gauge freedoms: one is the invariance under reparametrizations of the surface $x^\mu(\sigma,\tau)$ and the other is the invariance under

$$A_{\mu\nu}\to A_{\mu\nu}+\partial_\mu\Lambda_\nu-\partial_\nu\Lambda_\mu\,. \quad (A2)$$

This is reflected by the presence of constraints among the canonical variables. We shall follow the general treatment of Dirac[15] for such systems. First we consider the variables $x^\mu(\sigma)$. The canonical momenta are (the overdot means τ derivative)

$$\mathcal{P}_\mu(\sigma)\equiv\frac{\delta S}{\delta\dot{x}^\mu(\sigma)}, \quad (A3)$$

and it is simple to see that they satisfy the (primary) first-class constraints

$$\psi_1\equiv\left(\mathcal{P}_\mu-fA_{\nu\mu}\frac{\partial x^\nu}{\partial\sigma}\right)\left(\mathcal{P}^\mu-fA^{\lambda\mu}\frac{\partial x_\lambda}{\partial\sigma}\right)+\frac{\partial x^\mu}{\partial\sigma}\frac{\partial x_\mu}{\partial\sigma}=0\,, \quad (A4)$$

$$\psi_2\equiv\left(\mathcal{P}_\mu-fA_{\nu\mu}\frac{\partial x^\nu}{\partial\sigma}\right)\frac{\partial x^\mu}{\partial\sigma}=0\,. \quad (A5)$$

This shows that this system is obtained from the free string by the "minimal coupling" $\mathcal{P}_\mu\to\mathcal{P}_\mu-fA_{\nu\mu}\partial x^\nu/\partial\sigma$. In fact, with this subsitution the constraints (A4) and (A5) are those of the free string. It is also simple to see that there are no more constraints: ψ_1 and ψ_2 form a closed algebra under Poisson-bracket multiplication. Since the Lagrangian is homogeneous of the first degree in the velocity $\dot{x}^\mu$, the quantity $(\mathcal{L}-\mathcal{P}_\mu\dot{x}^\mu)$ vanishes and the Hamiltonian is given by

$$H=\int d\sigma(\lambda_1\psi_1+\lambda_2\psi_2)\,, \quad (A6)$$

where λ_1 and λ_2 are arbitrary functions, reflecting the reparametrization invariance of the system. The equations of motion are then

$$\dot{x}^\mu=\frac{\delta H}{\delta\mathcal{P}_\mu}=2\lambda_1\left(\mathcal{P}^\mu-fA^{\lambda\mu}\frac{\partial x_\lambda}{\partial\sigma}\right)+\lambda_2\frac{\partial x^\mu}{\partial\sigma}, \quad (A7)$$

$$\dot{\mathcal{P}}_\mu=-\frac{\delta H}{\delta x^\mu}=2\frac{\partial}{\partial\sigma}\left(\lambda_1\frac{\partial x_\mu}{\partial\sigma}\right)-2f\frac{\partial}{\partial\sigma}\left[\lambda_1A_{\mu\nu}\left(\mathcal{P}^\nu-fA^{\lambda\nu}\frac{\partial x_\lambda}{\partial\sigma}\right)\right]+\frac{\partial}{\partial\sigma}\left[\lambda_2\left(\mathcal{P}_\mu-fA_{\nu\mu}\frac{\partial x^\nu}{\partial\sigma}\right)\right]-f\frac{\partial}{\partial\sigma}\left(\lambda_2A_{\mu\nu}\frac{\partial\lambda^\nu}{\partial\sigma}\right). \quad (A8)$$

The orthonormal gauge is obtained with $\lambda_1=\frac{1}{2}$, $\lambda_2=0$. In fact, in this case

$$\dot{x}^\mu=\mathcal{P}^\mu-fA^{\nu\mu}\frac{\partial x_\nu}{\partial\sigma} \quad (A9)$$

and (A8) reduces to (2.22) and (A4) and (A5) to (2.19).

Now we take the field $A_{\mu\nu}=-A_{\nu\mu}$. The canonical momenta are

$$\Pi^{\alpha\beta} \equiv \frac{\delta S}{\delta A_{\alpha\beta,0}} = -\tfrac{1}{8}(\partial^0 A^{\alpha\beta} + \partial^\beta A^{0\alpha} + \partial^\alpha A^{\beta 0}) = -\Pi^{\beta\alpha}. \tag{A10}$$

We immediately have three primary constraints,

$$\Pi^{0\beta} = 0. \tag{A11}$$

The Hamiltonian

$$H = \int (\tfrac{1}{2}\Pi^{\alpha\beta} A_{\alpha\beta,0} - \mathcal{L}) d^3y$$

is given by (Latin indices take the values 1, 2, 3)

$$H = \int d^3y [\Pi_{ij}\Pi^{ij} + \tfrac{1}{4}(F_0)^2 + \Pi^{0i} A_{0i,0} - A_{0i}\Pi^{ij}{}_{,j}]. \tag{A12}$$

This Hamiltonian is not uniquely defined, as an arbitrary linear combination of the constraints may be added to it to obtain

$$H' = H + \int u_j \Pi^{0j} d^3y. \tag{A13}$$

The constraints (A11) must be preserved in time. Thus, their time derivative, given by the Poisson bracket with H', must vanish. This leads to the additional (secondary) constraints

$$\Pi^{ij}{}_{,j} = 0. \tag{A14}$$

Since $\Pi^{ij}{}_{,ij} \equiv 0$, these are two independent constraints only. The preservation in time of these constraints leads to no further ones. It is also easily seen that both (A11) and (A14) are first class. That is, they generate gauge transformations. It is easy to see now that, as asserted in the text, $A_{\mu\nu}$ corresponds to a scalar field. In fact, there are twelve canonical variables: six coordinates and six conjugate momenta. These variables must satisfy five constraints and five gauge-fixing conditions, leaving two independent variables, one coordinate and its conjugate momentum. That is, one degree of freedom per space point.

The extended Hamiltonian H_E, in the notation of Dirac,[15] is

$$H_E = \int d^3y (\Pi_{ij}\Pi^{ij} + \tfrac{1}{4}F_0{}^2) + \int d^3y (v_j \Pi^{0j} + w_i \Pi^{ij}{}_{,j}), \tag{A15}$$

where v_j are three arbitrary functions and w_i are three functions subject to one condition, for example $w_{i,i} = 0$. The gauge-independent part of (A15) is positive-definite, as it should be.

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*Research sponsored in part by the National Science Foundation under Grant No. GP-40768X.

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[9]It is perhaps not out of place to sketch how the variation of the second term in the right-hand side of Eq. (2.7) leads to the second term in the equation of motion (2.3):

$$\frac{1}{2}\frac{\delta}{\delta x^i(\sigma)}\int v^2 d^3y = \int d^3y\, v^k(y) \frac{\delta v^k}{\delta x^i(\sigma)} = \frac{k}{4\pi}\int d^3y\, v^k(y)\epsilon^{klr}\frac{\partial}{\partial y^l} \times \left[\frac{\partial}{\partial x^i}\left(\frac{1}{|\vec{y}-\vec{x}|}\right)\frac{\partial x^r}{\partial\sigma} - \delta_{ir}\frac{\partial}{\partial\sigma}\left(\frac{1}{|\vec{y}-\vec{x}|}\right)\right].$$

Since

$$\frac{\partial}{\partial x^i}\left(\frac{1}{|\vec{y}-\vec{x}|}\right) = -\frac{\partial}{\partial y^i}\left(\frac{1}{|\vec{y}-\vec{x}|}\right),$$

one gets

$$\frac{1}{2}\frac{\delta}{\delta x^i(\sigma)}\int v^2 d^3y = \frac{k}{4\pi}\frac{\partial x^p}{\partial\sigma}\left(\epsilon^{kli}\frac{\partial}{\partial x^p} - \epsilon^{klp}\frac{\partial}{\partial x^i}\right) \times \frac{\partial}{\partial x^l}\int d^3y \frac{v^k(\vec{y})}{|\vec{y}-\vec{x}(\sigma)|}.$$

Using now the identity

$$\epsilon^{kli}\frac{\partial}{\partial x^p} - \epsilon^{klp}\frac{\partial}{\partial x^i} = \epsilon^{pik}\frac{\partial}{\partial x^l} - \epsilon^{pil}\frac{\partial}{\partial x^k}$$

and integrating by parts, one obtains finally

$$\frac{1}{2}\frac{\delta}{\delta x^i(\sigma)}\int v^2 d^3y = \epsilon^{ijk}\frac{\partial x^j}{\partial\sigma}v^k .$$

[10]This notation was first introduced by Gauss in his *Disquisitiones Circa Superficies Curvas,* and is used by geometers to this day.

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