Anisotropic parametrized post-Newtonian gravitational metric field*

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The anisotropic generalization of the parametrized post-Newtonian (PPN) gravitational metric field is made for the case of theories with energy and momentum conservation laws. Such an anisotropic metric field will generally result in two-tensor or bimetric theories of gravity in an anisotropic universe. New anisotropic 3×3 spatial PPN matrices are introduced into the general metric expansion. Earth gravimeter measurements strongly restrict some anisotropies, while anisotropic inertial and gravitational mass for celestial bodies result from other combinations of the PPN matrices.

I. INTRODUCTION

During the era of the classical tests of general relativity (light deflection, perihelion precession of Mercury, clock gravitational frequency shift) Eddington[1] and Robertson[2] used a general parametrized expansion for the exterior gravitational metric field $g_{\mu\nu}$ of a spherically symmetric mass source;

$$g_{00} = 1 - 2\alpha(GM/c^2r) + 2\beta(GM/c^2r)^2 + \cdots, \quad (1a)$$

$$g_{ab} = -\left[1 + 2\gamma(GM/c^2r)\right]\delta_{ab} + \cdots, \quad (1b)$$

$$g_{0a} = 0 \quad (1c)$$

(isotropic spatial coordinates are used; $a, b = x, y, z$). Such a metric-field expansion facilitated determination of the dependence of each experimental prediction on the various aspects of the metric field, and comparison of different gravitational theories by means of experiment could elegantly be stated in terms of the metric-field parameters $\alpha, \beta, \gamma, \ldots$. $\alpha = 1$ was required for recovery of Newtonian gravity as a first approximation of relativistic gravity. Clock frequency-shift results were then uniquely predicted for all metric theories (equivalence principle). Light-deflection experiments measured γ, while perihelion precession of the inner planets was sensitive to both γ and β. ($\gamma = \beta = 1$ in general relativity.) Later Schiff[3] used this metric-field expansion to analyze the spin-axis precession of orbiting gyroscopes.

Nordtvedt[4] generalized the parametrized metric-field expansion beyond the static spherically symmetric case to include the possibilities of multiple sources and moving sources. His immediate goal was the study of gravitational self-energy effects on the equation of motion of massive bodies and experimental consequences. A number of new parameters were added to γ and β in Nordtvedt's general metric-field expansion. Soon thereafter Will[5] formulated the metric-field expansion for the case of continuous hydrodynamical sources of energy density, pressure, and momentum density. The metric field became known as the parametrized post-Newtonian (PPN) metric. Will and Nordtvedt[6] then unified their metric-field expansions into a single formulation and studied the properties of the PPN metric field under Lorentz transformations. Specific PPN parameters were identified as "preferred frame" parameters which when nonzero indicated that gravitational physics was dependent on motion relative to a particular inertial frame. Experiments to search for preferred-frame gravitational effects were suggested by Nordtvedt and Will.[7]

More recent work[8] has added a new PPN parameter — the Whitehead parameter ξ—associated with noncentral gravitational fields produced by multiple sources. Other studies[9] have examined the conditions for energy-momentum and angular momentum conservation laws in the gravitational physics of PPN systems.

The metric field of Will and Nordtvedt, though generally having a preferred inertial frame, did have the property that in this preferred frame gravitational physics was isotropic with no reference to any global directionally dependent boundary conditions. In the special inertial frame equations (1a)–(1c) were considered as giving the most general metric field of a static spherically symmetric mass source. However, consideration of "bimetric" or two-tensor theories of gravity, where either the second tensor augmenting $g_{\mu\nu}$ is a nondynamical, flat background tensor $\eta_{\mu\nu}$, or the second tensor is a dynamical field $k_{\mu\nu}$, have brought to attention the possibilities of intrinsic anisotropies in the PPN expansion for the gravitational metric field. In such theories a second tensor cannot be expected to have global isotropic diagonal form in the coordinate system in which the metric field $g_{\mu\nu}$ is asymptotically Minkowskian. A residual anisotropy in a second tensor in the theory would be a consequence of the cosmological anisotropy which we have no right to assume is precisely zero. Then depending on the field equations of the specific two-tensor theory of gravity, the anisotropy in the global value of the second tensor will induce anisotropic post-Newtonian gravitational potentials in $g_{\mu\nu}$.

The purpose of this paper is to generate a general anisotropic PPN metric field, and then discuss experiments which can put stringent limits on the anisotropies or detect them if they are present. As an example of a theory which leads to an anisotropic PPN metric field, we calculate in an appendix the PPN metric for Rosen's bimetric theory of gravity,[10] which has previously been shown to be consistent with the PPN metric of general relativity under certain conditions including isotropy.[11]

II. CONSERVATIVE PPN METRIC FIELD FROM A POST-NEWTONIAN LAGRANGIAN

The PPN metric field developed by Will and Nordtvedt[6] has inertial frame-dependent potentials appearing in the various components of $g_{\mu\nu}$. However, this metric field has a "preferred frame" often identified with a mean universe rest frame in which the PPN metric field is isotropic. This metric might therefore be called the "isotropic universe PPN."

Generalizing the PPN metric field for use in a possibly anisotropic universe is the purpose of this paper. Gravitational effects associated with possible universe anisotropies can then be calculated and experiments proposed. Our anisotropic generalization will be made under a self-imposed restriction: Only PPN metric fields which have energy and momentum conservation laws are considered. This seems a reasonable restriction, as dynamical systems without such conservation laws usually possess runaway configurations, in disagreement with experience. Also, most reasonable field theories of gravity lead to energy and momentum conservation via Noether's theorem plus the absence of explicit space-time dependence of the theory's action integral.

A useful algorithm exists for simply generating a conservative PPN metric field. First, a many-body post-Newtonian gravitational Lagrangian L is produced which generates the single-particle equations of motion;

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{v}_i}\right) - \frac{\partial L}{\partial \vec{r}_i} = 0 \ . \tag{2}$$

Because L is independent of explicit space or time dependence (L is solely a function of particle velocities $\vec{v}_i$ and interparticle distances $\vec{r}_i - \vec{r}_j$), there is conserved energy and momentum;

$$\frac{dH}{dt} = \frac{d\vec{P}}{dt} = 0 \ , \tag{3}$$

for

$$H = \sum \vec{v}_i \cdot \frac{\partial L}{\partial \vec{v}_i} - L \tag{4a}$$

and

$$\vec{P} = \sum \frac{\partial L}{\partial \vec{v}_i} \ . \tag{4b}$$

On the other hand an assumption of metric theories of gravity is that a particle's equation of motion is derivable from a metric field $g_{\mu\nu}(\vec{r}, t)$ which is used to form one-particle Lagrangians;

$$L_i = \left(g_{\mu\nu}(\vec{r}_i, t)\, \frac{dx_i^\mu}{dt}\, \frac{dx_i^\nu}{dt} \right)^{1/2} \ , \tag{5}$$

and the resulting equation of motion for each particle

$$\frac{d}{dt}\left(\frac{\partial L_i}{\partial \vec{v}_i}\right) - \frac{\partial L_i}{\partial \vec{r}_i} = 0 \ . \tag{6}$$

A post-Newtonian Lagrangian yields a conservative PPN metric field by equating the equations of motion obtained by the two methods, Eq. (2) and Eq. (6).

The fact that the equations of motion given by Eq. (2) lead to conserved energy and momentum by Eqs. (3), (4a), and (4b) guarantees that a metric whose equation of motion given by Eq. (6) matches Eq. (2) leads also to conservation laws for energy and momentum. A Lagrangian is arbitrary up to the addition of a total time derivative term, while the metric field is arbitrary up to addition of terms generated by the general coordinate transformation. These freedoms are used to put L into the simplest general form and to put the resulting $g_{\mu\nu}(\vec{r}, t)$ into a "standard" gauge form.

III. THE ISOTROPIC CASE

The procedure outlined in the previous section is first employed to review the case of the isotropic PPN metric. In such PPN metric fields there exists an inertial frame in which the metric-field potentials have no anisotropic potentials. In other inertial frames anisotropic potentials might result from the Lorentz transformations,[6] but they are then called "preferred-frame" potentials and would be proportional to $\vec{w}/c$ or $(\vec{w}/c)^2$ with $\vec{w}$ being the velocity of the inertial frame relative to the preferred frame. The most general isotropic PPN Lagrangian is

$$L=\sum \tfrac{1}{2} m_i v_i^2+\sum \tfrac{1}{8} m_i v_i^4+\sum \tfrac{1}{2} m_i m_j/r_{ij}+\tfrac{1}{2}(2\gamma+\zeta+1)\sum m_i m_j v_i^2/r_{ij}$$
$$-\tfrac{1}{4}(4\gamma+3+\alpha_1-\alpha_2+2\zeta)\sum m_i m_j \vec{v}_i\cdot\vec{v}_j/r_{ij}-\tfrac{1}{2}\zeta\sum m_i m_j(\vec{r}_{ij}\cdot\vec{v}_i)^2/r_{ij}^3$$
$$+\tfrac{1}{4}(2\zeta-1-\alpha_2)\sum m_i m_j \vec{r}_{ij}\cdot\vec{v}_i\,\vec{r}_{ij}\cdot\vec{v}_j/r_{ij}^3+(\tfrac{1}{2}-\beta)\sum m_i m_j m_k/r_{ij}r_{ik}\,. \tag{7}$$

Particle labels i,j,k are to be summed over; units in which the speed of light c and Newton's gravitational constant G are 1 are used. $\vec{r}_{ij}$ is the interparticle vector $\vec{r}_i-\vec{r}_j$. For $\alpha_1=\alpha_2=0$ and γ, β, and ζ arbitrary this Lagrangian is form invariant (to the necessary approximation) under Lorentz transformations and hence produces no preferred-frame effects; the equations of motion are the same in all inertial frames. For $\beta=1$, $\zeta=\alpha_1=\alpha_2=0$, and γ arbitrary we have the Lagrangian for the post-Newtonian approximation to the Brans-Dicke scalar-tensor theory. The general form of the PPN metric field developed by Will and Nordtvedt[6] for the case of conserved energy and momentum results from a Lagrangian with $\zeta=0$ and the other four parameters arbitrary. Later it was shown that conservative metric fields could have a nonzero "Whitehead" parameter ζ,[12] and Will[8] has studied theories of gravity which lead to these types of PPN metrics.

The Lagrangian of Eq. (7) yields the PPN metric field:

$$g_{00}=1-2U+2\beta U^2+(4\beta-2-\zeta)\sum m_i m_j/r_i r_{ij}-(2\gamma+1+\zeta)\sum m_i v_i^2/r_i$$
$$+\zeta\sum m_i(\vec{r}_i\cdot\vec{v}_i)^2/r_i^3+\zeta\sum m_i m_j\,\vec{r}_i\cdot(\vec{r}_j/r_j-\vec{r}_{ij}/r_{ij})/r_i^3\;, \tag{8a}$$

$$g_{0a}=\tfrac{1}{2}(4\gamma+3+\alpha_1-\alpha_2+\zeta)\sum m_i(v_i)^a/r_i+\tfrac{1}{2}(1+\alpha_2-\zeta)\sum m_i\vec{v}_i\cdot\vec{r}_i(r_i)^a\;, \tag{8b}$$

and

$$g_{ab}=-(1+2\gamma U)\delta_{ab}\;. \tag{8c}$$

$\vec{r}_i$ is the vector $\vec{r}-\vec{r}_i$. A coordinate transformation has been used to put the spatial metric components into a "standard" diagonal isotropic form. These metric fields given in Eqs. (8a)–(8c) will generate new potential terms proportional to $\vec{w}$ when Lorentz-transformed if the parameters α_1 and/or α_2 are nonzero.[6]

IV. THE ANISOTROPIC CASE

Using Eq. (7) as a guide we add anisotropic Lagrangian terms in order to generate the anisotropic but conservative PPN metric field. Several 3×3 anisotropic but traceless matrices are introduced as new PPN parameters; the trace of any such matrices are incorporated into the existing scalar PPN parameters γ, β, ζ, α_1, and α_2. The anisotropic Lagrangian terms are

$$L_A=\sum m_i m_j\Omega_{ab}(r_{ij})^a(r_{ij})^b/r_{ij}^3+\sum m_i m_j m_k\Theta_{ab}(r_{ij})^a(r_{ij})^b/r_{ik}r_{ij}^3+\sum m_i m_j v_i^2\Lambda_{ab}(r_{ij})^a(r_{ij})^b/r_{ij}^3$$
$$+\sum m_i m_j\bar{\Lambda}_{ab}(v_i)^a(v_j)^b/r_{ij}+\sum m_i m_j\Psi_{ab}(r_{ij})^a(r_{ij})^b\vec{v}_i\cdot\vec{v}_j/r_{ij}^3+\sum m_i m_j\Psi_{ab}(v_i)^a(v_j)^b/r_{ij}$$
$$+\sum m_i m_j\Gamma_{ab}(r_{ij})^a(v_i)^b\,\vec{r}_{ij}\cdot\vec{v}_i/r_{ij}^3+\sum m_i m_j\,\Xi_{ab}(r_{ij})^a(v_i)^b\,\vec{r}_{ij}\cdot\vec{v}_j/r_{ij}^3$$
$$+\sum m_i m_j\vec{\Phi}\cdot\vec{v}_i/r_{ij}+\sum m_i m_j\vec{\Upsilon}\cdot\vec{r}_{ij}\,\vec{r}_{ij}\cdot\vec{v}_i/r_{ij}^3\,. \tag{9}$$

The indices a and b are to be summed over x, y, and z according to the Einstein convention. The matrices Ω, Θ, Λ, $\bar{\Lambda}$, Ψ, and $\bar{\Psi}$ are symmetric while the matrices Γ and Ξ may have antisymmetric parts. $\vec{\Phi}$ and $\vec{\Upsilon}$ are vectors. Following the method of the previous section, L_A yields anisotropic metric-field components;

$$\delta g_{00}(A) = -4\sum m_i\Omega_{ab}(r_i)^a(r_i)^b/r_i^3 + 4\sum m_i m_j\Omega_{ab}(r_i)^a(r_i)^b/r_j r_i^3$$
$$-2\sum m_i m_j\Theta_{ab}[(r_i)^a(r_i)^b/r_i^3 r_j + (r_{ij})^a(r_{ij})^b/r_{ij}^3 r_j + (r_i)^a(r_j)^b/r_i^3 r_{ij}]$$
$$-2\sum m_i(\Lambda+\Omega)_{ab}(r_i)^a(r_i)^b v_i^2/r_i^3 - 2\sum m_i\bar{\Lambda}_{ab}(v_i)^a(v_i)^b/r_i - 2\sum m_i\Gamma_{ab}(r_i)^a(v_i)^b\vec{r}_i\cdot\vec{v}_i/r_i^3$$
$$+2\sum m_i m_j\Gamma_{ab}[(r_j)^a/r_j - (r_{ij})^a/r_{ij}](r_i)^b/r_i^3\,, \tag{10a}$$

$$\delta g_{0a}(A) = \sum m_i\Gamma_{ba}(v_i)^b/r_i - \sum m_i\Gamma_{ba}(r_i)^b\vec{r}_i\cdot\vec{v}_i/r_i^3 - 2\sum m_i\bar{\Psi}_{ab}(v_i)^b/r_i - 2\sum m_i\Psi_{bc}(r_i)^b(r_i)^c(v_i)^a$$
$$-\sum m_i\,\Xi_{ba}(r_i)^b\vec{r}_i\cdot\vec{v}_i/r_i^3 - \sum m_i\,\Xi_{bc}(r_i)^b(v_i)^c(r_i)^a/r_i^3 + \sum m_i\Phi^a/r_i + \sum m_i\vec{\Upsilon}\cdot\vec{r}_i(r_i)^a/r_i^3\,, \tag{10b}$$

and

$$\delta g_{ab}(A) = -2\sum m_i[\Gamma_{ab}^{(s)} + \bar{\Lambda}_{ab}]/r_i - 2\sum m_i\,\Lambda_{cd}(r_i)^c(r_i)^d/r_i^3\delta_{ab}\,. \tag{10c}$$

(s) stands for the symmetric part of the matrix. A coordinate transformation has been used to make the spatial metric field g_{ab} as isotropic as possible, although complete isotropy cannot be produced. This anisotropic metric-field expansion is valid in one preferred inertial frame. The method of Will and Nordtvedt[6] must be used to generate the new frame-dependent terms which will in general be present.

V. THE ANISOTROPIC NEWTONIAN POTENTIAL

Very stringent limits can be placed on the PPN matrix Ω. Will and Nordtvedt[7] have shown that earth tidal gravimeter data will place limits on anisotropic Newtonian potentials resulting from preferred-frame ($\alpha_2 \neq 0$) effects resulting from motion of the solar system in the galaxy. The same analysis can be used to place an upper limit of

$$|\Omega| < 10^{-9}$$

by using the most recent results of gravimeter experiments.[13]

VI. ANISOTROPIC INERTIAL AND GRAVITATIONAL MASS

Past work has shown that a massive body's gravitational to inertial mass ratio (M_G/M_I) will differ from 1 in the case of the general PPN metric field.[4] The difference $M_G/M_I - 1$ is proportional to a combination of PPN metric coefficients multiplying the ratio of the body's gravitational self-energy to total mass-energy. This has led to observations in the lunar laser ranging experiment which give stringent limits on the scalar PPN parameters γ, β, ζ, α_1, and α_2.[14,15]

For the case of a nonrotating, spherically symmetric body the M_G/M_I ratio is a scalar number in the isotropic PPN metric field, as one would expect. However, a rotating body was found to develop an M_G/M_I anisotropic spatial tensor part which was proportional to the rotational kinetic energy of the body divided by total mass-energy.[16]

Here we calculate contributions to a massive body's gravitational and inertial masses resulting from the anisotropic PPN matrices and obtain an anisotropic mass tensor for nonrotating spherically symmetric bodies. If Eq. (5) is written in the form

$$L_i = [1 - 2U + h_{00}^{(2)} + 2\vec{h}\cdot\vec{v}_i - v_i^2 + h_{ab}(v_i)^a(v_i)^b]^{1/2} \tag{12a}$$
$$\simeq 1 - U - \tfrac{1}{2}v_i^2 - \tfrac{1}{8}v_i^4 + \tfrac{1}{2}(h_{00}^{(2)} - U^2) + \vec{h}\cdot\vec{v}_i + \tfrac{1}{2}(h_{ab} - U\delta_{ab})(v_i)^a(v_i)^b\,, \tag{12b}$$

with U being the Newtonian potential

$$U = \sum m_i/r_i \tag{13}$$

and $h_{00}^{(2)}$ being the second-order correction to g_{00}, the equation of motion of a particle becomes

$$\frac{d}{dt}[\vec{v} + \tfrac{1}{2}v^2\vec{v} - \vec{h} + (U - h^{(3)})\vec{v}] = \vec{\nabla}[U + \tfrac{1}{2}(U^2 - h_{00}^{(2)}) - \vec{h}\cdot\vec{v} + \tfrac{1}{2}(U - h^{(3)})v^2]\,. \tag{14}$$

Here $h^{(3)}$ stands for the spatial metric h_{ab}.

VII. INERTIAL MASS

Contributions to inertial mass are considered first; these effects come from terms in the equation of motion proportional to the body's actual acceleration. Examination of Eq. (14) shows that anisotropic contributions of this type result from the terms

$$\frac{-d}{dt}\,(\vec{\mathrm{h}}+h^{(3)}\vec{\mathrm{v}})\ . \tag{15}$$

Consider a nonrotating spherically symmetric massive assembly of gravitationally interacting particles. After using Eqs. (10b) and (10c) and after weighting each particle's equation of motion by m_i/total mass, Eq. (15) produces inertial-mass contributions for the body:

$$\delta M_I=\sum\frac{m_i m_j}{r_{ij}}\,[(2\bar{\Lambda}+\tfrac{2}{3}\Gamma^{(s)}+2\bar{\Psi}+\tfrac{2}{3}\Xi^{(s)})_{ab}+\tfrac{2}{3}\Gamma_{ab}]\ . \tag{16}$$

(s) stands for symmetric part of the matrices. The inertial mass of the body is now a spatial 3×3 anisotropic matrix.

VIII. ACTIVE GRAVITATIONAL MASS

Considering the same massive body now as the source of a gravitational field acting on another body, we concentrate on contributions to Eq. (14) where the Newtonian potential M/R is augmented by anisotropic potential terms proportional to internal gravitational energies and dependent on distance as R^{-1}. (R is the distance from the massive body to the field point.) The additional potential terms of this type are

$$\delta U_A=\sum\frac{m_i m_j}{r_{ij}}\,(\Theta_{ab}R^aR^b/R^3)+\sum m_i v_i^{\,2}(\Lambda+\tfrac{1}{3}\Gamma^{(s)})_{ab}R^aR^b/R^3\ . \tag{17}$$

But by using the virial relation for the particle in a body in equilibrium

$$\sum m_i v_i^{\,2}=\tfrac{1}{2}\sum m_i m_j/r_{ij}\ , \tag{18}$$

we arrive at

$$\delta U_A=\sum\frac{m_i m_j}{r_{ij}}(\Theta+\tfrac{1}{2}\Lambda+\tfrac{1}{6}\Gamma^{(s)})_{ab}R^aR^b/R^3\ . \tag{19}$$

IX. PASSIVE GRAVITATIONAL MASS

Terms in Eq. (14), where the Newtonian potential of the external body which accelerates a second body is multiplied by either the gravitational potentials of the second body or squared velocities of elements of the second body, produce what can be called passive gravitational mass corrections. Such terms come from

$$-\tfrac{1}{2}\vec{\nabla}h^{(3)}v^2\ , \tag{20}$$

where the part of $h^{(3)}$ produced by the external body multiplies v^2 of elements of the second body; also from part of the anisotropic "Whitehead" potential in $\vec{\nabla}h_{00}^{(2)}$, and finally from the nonlinear potentials in $\vec{\nabla}h_{00}^{(2)}$ proportional to Θ. Altogether these terms produce a potential acting on a second body of

$$M_{\rm ex}\sum\frac{m_i m_j}{r_{ij}}(\Theta+\tfrac{1}{2}\Lambda+\tfrac{1}{6}\Gamma^{(s)})_{ab}R^aR^b/R^3\ , \tag{21}$$

plus an additional acceleration

$$\tfrac{2}{3}\sum\frac{m_i m_j}{r_{ij}}\,\Gamma_{ab}g^{\,b}_{\rm ex}\ , \tag{22}$$

which combines with the nonsymmetric term in the inertial mass given by Eq. (15) to symmetrize it. The identity of the form of Eq. (19) and Eq. (21) (except for the inversion of the roles of the bodies) shows the equality of active and passive gravitational mass in this conservative theory.

X. TWO-BODY EQUATION OF MOTION

The combined result of these anisotropic effects is to have the two-body gravitational equation of motion take the form[17]

$$(M_1\delta_{bc}+U_1C_{bc})a_1^c=M_1M_2\nabla_b(1/R)+(M_1U_2+M_2U_1)\nabla_b(B_{cd}R^cR^d/R^3)\ , \tag{23}$$

with another equation for the acceleration of the second body obtained by interchanging labels 1 and 2. $\vec{a}$ is acceleration, spatial indices b, c, and d are summed according to the Einstein convention. B and C are anisotropic matrices given by

$$C_{bc}=-[4(\bar{\Lambda}+\bar{\Psi})+\tfrac{4}{3}(\Gamma^{(s)}+\Xi^{(s)})]_{bc} \tag{24}$$

and

$$B_{bc}=-[2\Theta+\Lambda+\tfrac{1}{3}\Gamma^{(s)}]_{bc} \tag{25}$$

A few years observations of the frequency shifts due to the orbital motion of the binary pulsar PSR 1913+16 can put a stringent limit on the B and C matrices[17];

$$|B|\simeq|C|<10^{-4}\text{ or }10^{-5}\ .$$

APPENDIX: PPN METRIC FOR ROSEN'S BIMETRIC THEORY OF GRAVITY

Rosen's field equations for the metric field $g_{\mu\nu}$ take the form[10,11]

$$\tfrac{1}{2}\eta^{\mu\nu}g_{\lambda\rho|\mu\nu}-\tfrac{1}{2}\eta^{\mu\nu}g^{\gamma\sigma}g_{\gamma\lambda|\mu}g_{\rho\sigma|\nu}=-8\pi(-g)^{1/2}G(T_{\lambda\rho}-\tfrac{1}{2}g_{\lambda\rho}T)\ , \tag{A1}$$

in a coordinate system where $g_{\mu\nu}$ is asymptotically Minkowskian;

$$g_{\mu\nu} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad \text{as } r \to \infty , \tag{A2}$$

In an anisotropic universe the "flat" background metric $\eta^{\mu\nu}$ must have the general form

$$\eta^{\mu\nu} = \begin{bmatrix} \chi & 0 & 0 & 0 \\ 0 & -1+\epsilon_x & 0 & 0 \\ 0 & 0 & -1+\epsilon_y & 0 \\ 0 & 0 & 0 & -1+\epsilon_z \end{bmatrix} , \tag{A3}$$

with

$$\epsilon_x + \epsilon_y + \epsilon_z = 0 . \tag{A4}$$

The form of Eq. (A3) indicates that spatial rotations were made to find the principal axes of the anisotropic 3×3 spatial matrix ϵ^{ab} contained within $\eta^{\mu\nu}$, and also that a preferred inertial frame in which the space-time components of $\eta^{\mu\nu}$ are zero is used.

To obtain the PPN metric field the above equations can be approximated by

$$\nabla_*^2 g_{00} = 16\pi(-g)^{1/2} G(T_{00} - \tfrac{1}{2} g_{00} T) + \chi \ddot{g}_{00} + \vec{\nabla}_* g_{00} \cdot \vec{\nabla}_* g_{00} , \tag{A5}$$

$$\nabla_*^2 g_{0a} = 16\pi G T_{0a} , \tag{A6}$$

$$\nabla_*^2 g_{aa} = 8\pi G T , \tag{A7}$$

where $a = x, y, z$. The $\vec{\nabla}_*$ operator is an anisotropic gradient operator

$$\vec{\nabla}_* = (\partial/\partial x_*, \partial/\partial y_*, \partial/\partial z_*) , \tag{A8}$$

with

$$x_* = (-\eta_{xx})^{1/2} x, \text{ etc} . \tag{A9}$$

Equations (A5)–(A7) have the following solutions:

$$g_{00} = 1 - 2U_* + 2U_*^2 + \sum m_i m_j / r_{*i} r_{*ij} - 3 \sum m_i v_i^2 / r_{*i} , \tag{A10}$$

$$g_{0a} = 4 \sum m_i (v_i)^a / r_{*i} - \tfrac{1}{2}\chi \sum m_i (v_{*i})^a / r_{*i} + \tfrac{1}{2}\chi \sum m_i \vec{r}_{*i} \cdot \vec{v}_{*i} (r_{*i})^a / r_{*i}^3 , \tag{A11}$$

and

$$g_{ab} = -(1 + 2U_*)\delta_{ab} , \tag{A12}$$

with

$$U = \sum m_i / r_{*i} ,$$

$$r_{*i} = [-\eta_{ab}(r_i)^a (r_i)^b]^{1/2} \simeq r_i + \tfrac{1}{2} \epsilon_{ab}(r_i)^a (r_i)^b / r_i^3 ,$$

$$\vec{r}_{*i} \cdot \vec{v}_{*i} = -\eta_{ab}(r_i)^a (v_i)^b \simeq \vec{r}_i \cdot \vec{v}_i + \epsilon_{ab}(r_i)^a (v_i)^b ,$$

and

$$(v_{*i})^a = -\eta_{ab}(v_i)^b \simeq (v_i)^a + \epsilon_{ab}(v_i)^b .$$

For the case of no anisotropy in the matrix η_{ab} and the parameter value $\chi = 1$, Rosen's theory has the same conservative PPN metric as general relativity. The anisotropic PPN potentials are obtained by expanding the expressions in Eqs. (A10)–(A12) to lowest order in ϵ_{ab};

$$\delta g_{00}(A) = \sum m_i \epsilon_{ab}(r_i)^a (r_i)^b / r_i^3 + \tfrac{3}{2} \sum m_i v_i^2 \epsilon_{ab}(r_i)^a (r_i)^b / r_i^3 - 2 \sum m_i m_j \epsilon_{ab}(r_j)^a (r_j)^b / r_j^3 r_i - \sum m_i m_j \epsilon_{ab}(r_i)^a (r_i)^b / r_i^3 r_{ij} - \sum m_i m_j \epsilon_{ab}(r_{ij})^a (r_{ij})^b / r_{ij}^3 r_i , \tag{A13}$$

$$\delta g_{0a}(A) = -\tfrac{7}{4} \sum m_i \epsilon_{bc}(r_i)^b (r_i)^c (v_i)^a / r_i^3 - \sum m_i \epsilon_{ab}(v_i)^b / r_i + \tfrac{1}{2} \sum m_i \epsilon_{bc}(r_i)^b (v_i)^c (r_i)^a / r_i^3 + \sum m_i \vec{r}_i \cdot \vec{v}_i \epsilon_{ab}(r_i)^b / r_i^3 - \tfrac{3}{4} \sum m_i \vec{r}_i \cdot \vec{v}_i \epsilon_{bc}(r_i)^b (r_i)^c (r_i)^a / r_i^5 , \tag{A14}$$

and

$$\delta g_{ab}(A) = \sum m_i \epsilon_{cd}(r_i)^c (r_i)^d / r_i^3 \delta_{ab} . \tag{A15}$$

Except for a term in Eq. (A14) proportional to $1/r_i^5$, this metric derived from Rosen's theory is of the form given by Eqs. (10a)–(10c). An additional term added to the Lagrangian can generate this $1/r_i^5$ term, and therefore Rosen's anisotropic metric field leads to post-Newtonian energy and momentum conservation laws.

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