PHYSICAL REVIEW D VOLUME 14, NUMBER 6 15 SEPTEMBER 1976

Dirac equation around a charged, rotating black hole*

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(Received 21 June 1976)

The Dirac equation for an electron around a Kerr-Newman black hole is separated into decoupled ordinary differential equations.

Chandrasekhar[1] has recently separated the Dirac equation in the Kerr geometry. This allows an analysis of the wave behavior of electrons and muons around a rotating black hole with no external matter fields. However, one would also like to analyze the situation when the black hole is electrically or otherwise charged so that it has an external electromagnetic or other massless vector field. For example, a small hole emitting electrons and positrons stochastically by Hawking's process[2] would have random charge fluctuations that could affect the average emission rates through the electromagnetic coupling.[3] Carter[4] has also suggested the existence of another long-range vector field coupled to the baryon or lepton number of the hole which could affect the geometry and thus the behavior of Dirac particles whether or not they couple directly to the vector field.

In this paper, Chandrasekhar's analysis is extended to a Dirac particle in the Kerr-Newman field. With a given energy and axial angular momentum, the wave function can be written in terms of four components which satisfy a set of four coupled partial-differential equations in the radius and polar angle of Boyer-Lindquist[5] coordinates. These components can be expressed as products of radial and angular functions that satisfy decoupled ordinary differential equations, which will be given below.

The analysis starts with the Dirac equation coupled to a general gravitational and electromagnetic or other vector field. In two-component spinor notation,[6,7] the equation is a simple generalization of Eqs. (C1) and (C2) (where C denotes Chandrasekhar's paper[1]):

$$\sqrt{2}\,(\nabla_{BB'} + ieA_{BB'})P^B + i\mu_e Q^*_{B'} = 0\,, \tag{1}$$

$$\sqrt{2}\,(\nabla_{BB'} - ieA_{BB'})Q^B + i\mu_e P^*_{B'} = 0\,. \tag{2}$$

Here $\nabla_{BB'}$ is the symbol for covariant differentiation in the pseudo-Riemannian geometry representing the gravitational field, $A_{BB'}$ is the electromagnetic or other vector field potential, e is the charge or coupling constant of the Dirac particle to the vector field, μ_e is the particle mass ($\sqrt{2}$ times the μ_e used by Chandrasekhar), and P^B and Q^B are the two-component spinors representing the wave function. The asterisk replaces Chandrasekhar's general use of overbars to denote complex conjugation; $Q^*_{B'}$ is the complex conjugate of Q_B. The vector potential must enter with the opposite signs in Eqs. (1) and (2) to preserve gauge invariance, since the spinors in the two equations are related by complex conjugation. Planck units are used, so $\hbar = c = G = 1$.

In the Newman-Penrose formalism[8] with a null tetrad $(\vec{l}, \vec{n}, \vec{m}, \vec{m}^*)$ and spinor components P^a and Q^a, the equations become [cf. Eqs. (C7) and (C8)]

$$(D + \epsilon - \rho + ie\vec{A}\cdot\vec{l})P^0 + (\delta^* + \pi - \alpha + ie\vec{A}\cdot\vec{m}^*)P^1 = 2^{-1/2} i\mu_e Q^{*1'}\,, \tag{3}$$

$$(\Delta + \mu - \gamma + ie\vec{A}\cdot\vec{n})P^1 + (\delta + \beta - \tau + ie\vec{A}\cdot\vec{m})P^0 = -2^{-1/2} i\mu_e Q^{*0'}\,, \tag{4}$$

$$(D + \epsilon^* - \rho^* + ie\vec{A}\cdot\vec{l})\,Q^{*0'} + (\delta + \pi^* - \alpha^* + ie\vec{A}\cdot\vec{m})Q^{*1'} = -2^{-1/2} i\mu_e P^1\,, \tag{5}$$

$$(\Delta + \mu^* - \gamma^* + ie\vec{A}\cdot\vec{n})Q^{*1'} + (\delta^* + \beta^* - \tau^* + ie\vec{A}\cdot\vec{m})Q^{*0'} = 2^{-1/2} i\mu_e P^0\,. \tag{6}$$

Here the electromagnetic or other vector field is expressed as a four-vector $\vec{A}$.

These equations may be written out explicitly for the Kerr-Newman[9] field in Boyer-Lindquist[5] coordinates by using the Kinnersley[10] tetrad. The equations simplify if one sets [cf. Eqs. (C9) and (C25)]

$$P^0 = (r - ia\cos\theta)^{-1} e^{i(\sigma t + m\varphi)} f_1(r, \theta)\,, \tag{7}$$

$$P^1 = e^{i(\sigma t + m\varphi)} f_2(r, \theta)\,, \tag{8}$$

$$Q^{*1'} = e^{i(\sigma t + m\varphi)} g_1(r, \theta)\,, \tag{9}$$

$$Q^{*0'} = -(r + ia\cos\theta)^{-1} e^{i(\sigma t + m\varphi)} g_2(r, \theta)\,. \tag{10}$$

The (t, φ) dependence is that of a wave function with energy σ and axial angular momentum m. The tetrad components and spin coefficients have the same form in the Kerr-Newman geometry as in the Kerr geometry, except that the quantity Δ [not to be confused with Δ, the directional derivative along $\vec{n}$ in Eqs. (4) and (6)] has an extra term from the charge Q of the hole[11]:

$$\Delta \equiv r^2 - 2Mr + a^2 + Q^2 . \tag{11}$$

The Kerr-Newman vector field[12] has the tetrad components

$$\vec{A}\cdot\vec{l} = -\frac{Qr}{\Delta}\,,\quad \vec{A}\cdot\vec{n} = -\frac{Qr}{2\Sigma}\,,\quad \vec{A}\cdot\vec{m} = \vec{A}\cdot\vec{m}^* = 0\,, \tag{12}$$

where

$$\Sigma \equiv r^2 + a^2\cos^2\theta \tag{13}$$

is what Chandrasekhar calls ρ^2.

Now if Chandrasekhar's quantity K is replaced with

$$K \equiv (r^2 + a^2)\sigma - eQr + am\,, \tag{14}$$

then Eqs. (3)–(6) separate into ordinary differential equations by precisely the same procedure Chandrasekhar[1] uses for the uncharged case. That is, we have

$$f_1(r,\theta) = R_{-1/2}(r)S_{-1/2}(\theta) \equiv R(r)S(\theta)\,, \tag{15}$$

$$f_2(r,\theta) = R_{+1/2}(r)S_{+1/2}(\theta)\,, \tag{16}$$

$$g_1(r,\theta) = R_{+1/2}(r)S_{-1/2}(\theta)\,, \tag{17}$$

$$g_2(r,\theta) = R_{-1/2}(r)S_{+1/2}(\theta)\,. \tag{18}$$

$S(\theta)$ and $R(r)$ must satisfy Eqs. (C44) and (C45), which, with Chandrasekhar's $\sqrt{2}\lambda$ replaced by my λ, may be written out as

$$\frac{1}{\sin\theta}\frac{1}{d\theta}\left(\sin\theta\frac{dS}{d\theta}\right) + \frac{a\mu_e\sin\theta}{\lambda + a\mu_e\cos\theta}\frac{dS}{d\theta} + \left[(\tfrac{1}{2} - a\sigma\cos\theta)^2 - \left(\frac{m - \frac{1}{2}\cos\theta}{\sin\theta}\right)^2 - \tfrac{3}{4} - 2a\sigma m - a^2\sigma^2 + \frac{a\mu_e(\frac{1}{2}\cos\theta - a\sigma\sin^2\theta - m)}{\lambda + a\mu_e\cos\theta} - a^2\mu_e{}^2\cos^2\theta + \lambda^2\right]S = 0\,, \tag{19}$$

$$\sqrt{\Delta}\frac{d}{dr}\left(\sqrt{\Delta}\frac{dR}{dr}\right) - \frac{i\mu_e\Delta}{\lambda + i\mu_e r}\frac{dR}{dr} + \left[\frac{K^2 - i(r-M)K}{\Delta} + 2i\sigma r - ieQ + \frac{\mu_e K}{\lambda + i\mu_e r} - \mu_e{}^2 r^2 - \lambda^2\right]R = 0\,. \tag{20}$$

Then $S_{+1/2}(\theta)$ and $R_{+1/2}(r)$ can be obtained from Eqs. (C41) and (C40). The condition that S be regular over $0 \le \theta \le \pi$ makes Eq. (19) an eigenvalue equation for the separation constant λ, which is then used in the radial Eq. (20). Equation (19) is independent of the charge, so λ is a function only of $a\sigma$ and $a\mu_e$ for each half-integral m. When $a\mu_e = 0$, Eq. (19) reduces Teukolsky's[13] Eq. (4.9) with $s = -\frac{1}{2}$, $\omega = -\sigma$, and $A = \lambda^2 - 2am\sigma - a^2\sigma^2$. Equation (20) similarly reduces to Teukolsky's Eq. (4.10) for neutrinos when both $\mu_e = 0$ and $Q = 0$, since his K is then the negative of mine.

Note added in proof. S. Chandrasekhar has informed me that Toop has independently separated the Dirac equation in the Kerr-Newman field and has also done the cases when there is a cosmological constant and when the Dirac particle is a magnetic monopole.

I wish to thank D. M. Chitre for informing me of the reference[11] on the spin coefficients in the Kerr-Newman geometry and K. S. Thorne, J. B. Hartle, and J. L. Friedman for suggesting how to check the validity of Eqs. (1) and (2). K. S. Thorne also gave helpful comments on the manuscript.

*Work supported in part by the National Science Foundation under Grant No. AST75-01398 A01.

[1]S. Chandrasekhar, Proc. R. Soc. London A349, 571 (1976).

[2]S. W. Hawking, Nature 248, 30 (1974); Commun. Math. Phys. 43, 199 (1975).

[3]D. N. Page, Phys. Rev. D 13, 198 (1976).

[4]B. Carter, Phys. Rev. Lett. 33, 558 (1974).

[5]R. H. Boyer and R. W. Lindquist, J. Math. Phys. 8, 265 (1967).

[6]F. A. E. Pirani, in *Lectures on General Relativity*, Brandeis 1964 Summer Institute on Theoretical Physics, edited by S. Deser and K. W. Ford (Prentice-Hall, Englewood Cliffs, N. J., 1965), Vol. I.

[7]R. Penrose, in *Battelle Rencontres: 1967 Lectures in Mathematics and Physics*, edited by C. M. DeWitt and J. A. Wheeler (W. A. Benjamin, New York, 1968).

[8]E. Newman and R. Penrose, J. Math. Phys. 3, 566 (1962).

[9]E. T. Newman, E.Couch, K. Chinnapared, A. Exton, A. Prakash, and R. Torrence, J. Math. Phys. 6, 918 (1965).

[10]W. Kinnersley, J. Math. Phys. 10, 1195 (1969).

[11]S. K. Bose, J. Math. Phys. 16, 772 (1975).

[12]C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W. H. Freeman, San Francisco, 1973), p. 898.

[13]S. A. Teukolsky, Astrophys. J. 185, 635 (1973).