The mechanism of blackbody radiation from an incipient black hole*

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On the basis of the phenomenon of zero-point energy an account is given of the mechanism of the emission of blackbody radiation from an incipient (about-to-be-formed) black hole, which results from the gravitational collapse of a star. The account is given in terms of three related points of view: (1) the emitted blackbody radiation results from a Fourier spectrum analysis of the zero-point fluctuations on the surface of the collapsing star; (2) the radiation results from a parametric amplification by a time-dependent potential of waves emerging from the collapsing surface of the star; (3) the radiation results from the star passing continually through states of resonance mutual to the natural modes internal and those external to the star. These three points of view are related by virtue of the underlying principle that explains the blackbody radiation mechanism: the nonadiabatic red-shift process operating on the randomly correlated zero-point fluctuation modes. The picture that emerges from these analyses is that all zero-point oscillation modes emerging from the star give rise to statistically identical blackbody radiation packets. Their only difference lies in their time of emission. The packets are emitted in a time sequential order and each is created during a limited time interval at the surface of the star: those packets caused by low-frequency zero-point modes first, those caused by high-frequency modes later. The blackbody radiation continuously drains away the irreducible mass of the black hole at an ever increasing rate. The lifetime of the incipient black hole is therefore finite. Consequently, the total number of zero-point fluctuation modes taking part in the emission of blackbody radiation packets is finite. The logarithm of this number (multiplied by Boltzmann's constant) equals the entropy of a black hole. This suggests that the internal microscopic states (in the statistical-mechnical sense) of a macroscopic black hole, i.e., the "hairs" lost by the black hole, are to be identified with those degrees of freedom that are capable of and will be causing the emission of blackbody radiation. The statistical fluctuation spectrum of the emitted energy is exhibited and found to be identical to that associated with a blackbody, showing thereby that radiation emitted from a black hole is thermal radiation in the precise sense of the term. The relevance of these statistical fluctuations to the formation of a black hole is discussed briefly. Brief mention is made of the sense in which a radiating incipient black hole lends support to Sakharov's idea that gravitation is a manifestation of the alteration of the zero-point fluctuations of space. The formulation of the radiation mechanism in terms of successively amplified zero-point radiation modes allows us to conclude that a star can never pass through its instantaneous $r = 2M$ surface. In view of the fact that the evaporation and the final demise of an incipient black hole are visible to a distant observer, it is necessary to reformulate the classical version of the issue of the final state of gravitational collapse. A qualitative account of the evolution of a classical incipient black hole is given. The issue of the final state of stellar gravitational collapse is restated in the form of a question: *What is the ground state of an incipient black hole?*

I. INTRODUCTION

During the late stages of gravitational collapse of a star, all radiatable multipole perturbation in its gravitational field are either radiated away or reflected by the centrifugal potential barrier and thus disapperar from the view of a distant observer.[1] Consequently, the configuration tends towards a stationary state that is characterized by only three descriptors: mass, angular momentum, and charge.[2] This state of affairs is captured by the phrase "a black hole has no hair." The process of an incipient black hole (a star during its late stages of collapse) ridding itself of all those multipole moments ("hairs") that are not forced on it by the existence of dynamically conserved descriptors is condensed into the phrase "anything which can be radiated gets radiated away completely."[3] It is natural therefore to inquire to what extent this principle is also applicable to those asymmetries on the surface of a star that are caused by the zero-point fluctuations of the electromagnetic (or any other) field.

The first result of such an inquiry is that during the late stages of stellar collapse the incipient black hole gives rise to an outgoing power spectrum whose spectral flux is

$$\frac{\text{(total energy)}}{\text{(unit spatial volume)(unit momentum volume)}} = \frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{8\pi\omega M}-1}. \quad (1.1)$$

Here $\hbar$, ω, and M are Planck's constant, the frequency of the radiation at a distance observer, and the mass (in geometrical units $M = M_{\text{conv}} G/c^2$ cm^{-1}) of the black hole, respectively. The first term refers to the zero-point fluctuations. The second can be identified with thermal radiation actually emitted by a black hole at a temperature

$$T=\frac{\hbar}{k}\frac{1}{8\pi M}\,. \tag{1.2}$$

The second result is that the total net power loss due to the thermal radiation is

$$\frac{dM}{dt}=0.708\times 10^{-4}\,\frac{L_W^2}{M^2} \tag{1.3}$$

$[L_W^2=(\hbar G/c^2)^{1/2}$, the squared Wheeler length, t is in cm] at any given moment of time and that therefore the incipient black hole has a finite lifetime given by

$$\tau=4700\,\frac{M^3}{L_W^2}\,. \tag{1.4}$$

Expressions in Eqs. (1.1) and (1.2) are derived in Sec. VII. Those in Eqs. (1.3) and (1.4) are derived in Sec. IX.

Hawking arrived at similar results by applying the formalism of quantized fields to the spacetime setting of a black hole formed by a collapsed star.[4,5] This article developes a more detailed treatment of the mechanism of blackbody radiation. By applying classical field theory to zero-point fluctuations, which permeate all space including the interior of a collapsing star, we endeavor to find the evolution of these fluctuations. The original motivation came from trying to apply Price's theorem ("anything which can be radiated gets radiated away completely") to the vacuum fluctuation modes emerging from the surface of a star during its late stages of collapse. As each mode gets redshifted nonadiabatically and finally switched off from the view of a distant observer, it gets amplified.

The third result is that a collapsing star continually passes through states of *simultaneous resonance* during each of which a vacuum fluctuation mode gets amplified.

The collapsing star is found in each of such states by virtue of the fact that a given red-shifted vacuum fluctuation mode upon crossing the star's surface agrees in frequency and is thus identified with an outward-traveling (Schwarzschild) mode. Amplification occurs only as long as the two frequencies agree. The (Schwarzschild) time interval during which the collapsing star brings about this amplification (and hence is in a well-determined state of resonance) is

$$\Delta t=4\pi M;$$

it is a function of the incipient black-hole descriptor M only. This interval surrounds a well-determined time and hence a well-determined radius, the "resonance radius."

The fourth result is that in the mean the amplification of a vacuum fluctuation mode gives rise to a corresponding finite packet of radiative energy, which has a Planck blackbody spectrum. Furthermore, the packets, which in the mean are identical, are emitted in a time-sequential order.

The fifth result is that the fluctuations in the emitted energy—fluctuations due to the variations in the amount by which each vacuum fluctuation mode gets amplified by the collapsing star—have a spectrum corresponding to the sum of the mean squared fluctuations of a mixture of statistically independent Boltzmann gases, composed respectively fo single quanta, pairs, trios, etc. In other words, the radiation emitted by an incipient black hole is indistinguishable from blackbody radiation. The sixth result is that

$$\begin{aligned}(\text{black-hole entropy}) &= k\ln(\text{total number of vacuum fluctuation modes responsible for the emission of blackbody radiation})\\ &= k\ln(\text{number of packets of blackbody radiation emitted by an incipient black hole})\,.\end{aligned}$$

The seventh result is that the production of blackbody radiation by a collapsing spherically symmetric shell star prevents its collapse through its instantaneous $r=2M$ surface. A black hole, i.e., an event horizon, is never formed; instead the collapsing star settles into a quasistatic evolutionary patter, an "incipient" black hole.

Section II defines the general setting of the two systems under consideration, a star and the set of standing wave modes, each one undergoing its zero-point oscillations.

Section III defines the exterior and interior geometry associated with the collapsing star.

In Sec. IV the red-shifting of the zero-point radiation emerging from the star's surface is discussed and the Fourier spectrum of that radiation is exhibited. It will be discussed in Sec. VI.

In Sec. V the parametric amplification of the vacuum fluctuation modes is discussed heuristically in terms of (a) the nonapplicability of the WKB (geometrical optics) approximation and (b) a time-dependent blue- (and red-) shift associated with a time-dependent potential. In this section the "radius of *simultaneous resonance*" is introduced. This radius plays a central role in the amplification mechanism.

In Sec. VI the resonance nature of the amplification process is exhibited. With the help of the orthonormal wave-packet representation of the outgoing radiation modes the qualitative results of Sec. V are made precise. Thus (a) the "time (or radius) of *simultaneous resonance*" which arises quite naturally is defined, and (b) the finite and frequency independent resonance width and hence the local nature of the amplification process are exhibited. Furthermore, Hawking's assumptions[5] are contrasted with ours.

In Sec. VII the radiation spectrum is identified and its most important properties are listed. Among them are the facts that the emitted radiation packets are statistically identical and are emitted in a time-sequential order.

In Sec. VIII the black-hole entropy is identified with the logarithm of the total number of radiation packets, i.e., with the total number of vacuum fluctuation waves (inside the star) that cause the emission of the radiation.

In Sec. IX the net rate of energy emitted by an incipient black hole is exhibited and its accuracy is discussed.

In Sec. X the subtraction, necessary to determine the change in energies of the zero-point fluctuations, is discussed and justified.

In Sec. XI the energy spectrum as well as the spectrum of the energy fluctuations of blackbody radiation are compared and found indentical to the same properties of radiation from a black hole. The origin of the violation of the energy condition, which is central to standard black-hole formulation, is identified. Furthermore, our classical amplification mechanism, is related to the probability of creating photons and thus to Hawking's approach.

In Sec. XII the relationship between (1) the zero-point fluctuation formulation of a radiating incipient black hole and the Casimir effect and (2) Sakharov's ideas about the basis of gravitation is discussed.

In Sec. XIII the effect of the generation of blackbody radiation on the outcome of gravitational collapse is discussed. Here we conclude that a star will not pass through its instantaneous $r=2M$ surface, the incipient event horizon.

In Sec. XIV the classical version of the issue of the final state of gravitational collapse is reformulated to take into account the finite lifetime of an incipient black hole.

In Appendix A the Fourier spectrum due to zero-point fluctuations at the star's surface is evaluated for late stages of collapse. The result is used in Sec. VI.

In Appendix B the instantaneous frequency of a wave train emerging from the star is determined as a function of time. The length of time that the wave train is visible to a distant observer is determined. The results of this appendix are used in Sec. IX.

In Appendix C the rate at which such wave trains are amplified and switched off from the view of a distant observer is given. It is used in obtaining the rate of mass loss expression in Sec. IX.

II. ZERO-POINT FLUCTUATIONS BEFORE COLLAPSE

Consider a cold noncollapsed star enclosed in a large conducting spherical cavity of radius a. The reflective walls of this cavity allow the existence of standing electromagnetic waves, which also permeate the whole star. Even in their state of lowest energy these standing-wave modes undergo oscillations (zero-point fluctuations).[6] The scalar wave function associated with the electromagnetic field oscillating at frequency ω_n is

$$\Phi_{nlm}=\left(\frac{2\hbar}{\omega_n a}\right)^{1/2}\sin(\omega_n t+\beta_{nlm}) \times\frac{\sin(\omega_n r-\frac{1}{2}l\pi)}{r}Y_l^m(\theta,\phi)\,. \tag{2.1}$$

It satisfies the usual scalar wave equation $\Box\Phi=0$ in flat space throughout the conducting spherical cavity. In the short-wavelength (WKB) approximation[7] this scalar function yields, for example, the electromagnetic potential $\psi_n^\mu=e^\mu\psi_n$ with the help of the polarization vector e^μ.

The amplitude of the above mode is determined by the fact that it must have the zero-point energy

$$\int\int\int_{\text{inside the sphere}}|\Phi_{nlm}|^2\omega_n^2\,d(\text{volume})=\tfrac{1}{2}\hbar\omega_n \tag{2.2}$$

of a quantum-mechanical oscillator vibrating at the frequency

$$\omega_n=\frac{\pi}{a}\left(n+\frac{l}{2}\right) \tag{2.3}$$

(n = radial quantum number, l = total angular momentum quantum number) determined by a, the radius of the "large" spherical cavity. The radiation emitted by an incipient black hole is however, as will be seen in Eq. (9.8), independent of the size of the spherical cavity.

The presence of the phase angle β_{nlm} for each mode, Eq. (2.1), expresses the fact that the oscillation of different modes are uncorrelated.[8] In other words, the phase angle $\beta_{nlm}=\beta(n,l,m)$ is a random function of the integers n,l,m.

The fact that $\hbar\neq0$ together with the fact that β_{nlm} is a random phase angle consitute the only quantum-mechanical input into our theory of the mechanism of blackbody radiation. The rest is merely classical field theory.

The analysis of the evolution of the standing wave modes is facilitated by recognizing that they are a superposition of outgoing and ingoing traveling waves,

$$\Phi_{nlm}^{\text{outgoing}} + \Phi_{nlm}^{\text{ingoing}} = \left(\frac{2\hbar}{a\omega_n}\right)^{1/2} \frac{1}{2r}\{\cos[\omega_n(t-r)+\beta_{nlm}] - \cos[\omega_n(t+r)+\beta_{nlm}]\}\, Y_l^m(\theta,\phi)\,. \tag{2.4}$$

It is evident therefore, that the existence of the standing-wave pattern may be viewed as a continuous scattering process in which the second term is an ingoing wave converging onto the star and the first term is the outgoing wave, which has been scattered by the uncollapsed star.

III. GEOMETRY INSIDE AND OUTSIDE A COLLAPSING STAR

Now, at $t=0$, let the star undergo gravitational collapse, and inquire as to the effect of the evolving geometry on the "scattering process," i.e., on the outgoing waves.

During the initial stages of collapse, space inside and outside the star is flat, or nearly so, and the wave field has the space-time dependence exhibited by Φ_{nlm} in Eq. (2.4). As the collapse evolves, in particular during the intermediate and late stages of collapse, the geometry outside the star is the one given by the Schwarzschild metric,

$$(ds^2)_{\text{outside}} = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-2M/r} + r^2(d\theta^2+\sin^2\theta\, d\phi^2)\,, \tag{3.1}$$

while the geometry inside the star is given by the Friedmann geometry, if the star is a homogeneous sphere of pressureless dust, or by a flat geometry,

$$(ds^2)_{\text{inside}} = -dt_{\text{in}}{}^2 + dr^2 + r^2(d\theta^2+\sin^2\theta\, d\phi^2)\,, \tag{3.2}$$

if the star is a hollow shell star with all the mass concentrated in the thin shell, or by another geometry of a spherically symmetric mass undergoing gravitational collapse.[9]

During the late stages of collapse the history of any star assumes a feature which, as seen by a distant observer, is universal in nature[1]: The history of a collapsing star made of only radially traveling photons, i.e., the history of the surface tends (as viewed in terms of Schwarzschild time t and tortoise[10] radial coordinate r^*, which exhibit radial null histories as straight lines; see Fig. 1) to be along the inwardly pointing light cone whose apex is located somewhere on the world line of the origin $r=0$. Consequently, one may, and we shall, consider the structurally simplest type of star, a shell star, composed of photons with zero angular momentum in whose geometry the evolution of the vacuum fluctuations is to be determined. The geometry inside such a star is flat and is given by Eq. (3.2); outside it is given by Eq. (3.1). The history of the star's surface, which separates spacetime inside from that outside, is

$$t_{\text{in}} + r = r_0 \tag{3.3a}$$

in terms of the interior coordinates, and

$$t + r + 2M\ln\left(\frac{r}{2M}-1\right) = t + r^* = v_0 \tag{3.3b}$$

in terms of the exterior Schwarzschild coordinates. Here r_0 and v_0 are the respective values of the initial radius r and tortoise coordinate r^* of the shell star composed of radially traveling photons.

IV. EVOLUTION OF A STANDING WAVE IN THE GEOMETRY OF A COLLAPSING STAR

What effects does a collapsing shell star have on a standing wave Φ_{nlm}, i.e., on the scattering process? During the initial stages of collapse Φ_{nlm} is totally unaffected: Space inside and outside the star is flat to an excellent degree of approximation. Thus the wave field both inside and outside is that of the zero-point fluctuations as given by Eq. (2.4). A distant observer measuring radiation coming from the star's surface will observe outgoing zero-point radiation modes which are matched by ingoing zero-point radiation[11] in Eq. (2.4). In other words, he measures no net flux of radiation during the initial stages of collapse.

During the intermediate and late stages of collapse the picture is different. Although the geometry is flat inside and given by Eq. (3.2), outside the geometry is that of Schwarzschild. Consequently, when an outgoing zero-point radiation wave packet of mean frequency ω_n travels from the flat inner region of the star to the Schwarzschild geometry outside, its frequency as seen by a distant observer is [see Eq. (A20)]

$$\omega = \left(1-\frac{2M}{r}\right)\omega_n\,. \tag{4.1}$$

A wave undergoes a continuous changing red-shift. (This red-shift is exhibited pictorially in Fig. 1. The "interior wave" characterized by ω_n gives rise along the "history of collapsing surface" to outgoing radiation which is of lower frequency the more negative r^* is, i.e., the closer r is to $2M$.) As a matter of fact, as the radius of the star approaches $2M$, the radiation emerging from the star will ultimately have such a long wavelength that it will not be able to surmount the well-known radial potential barrier[12]

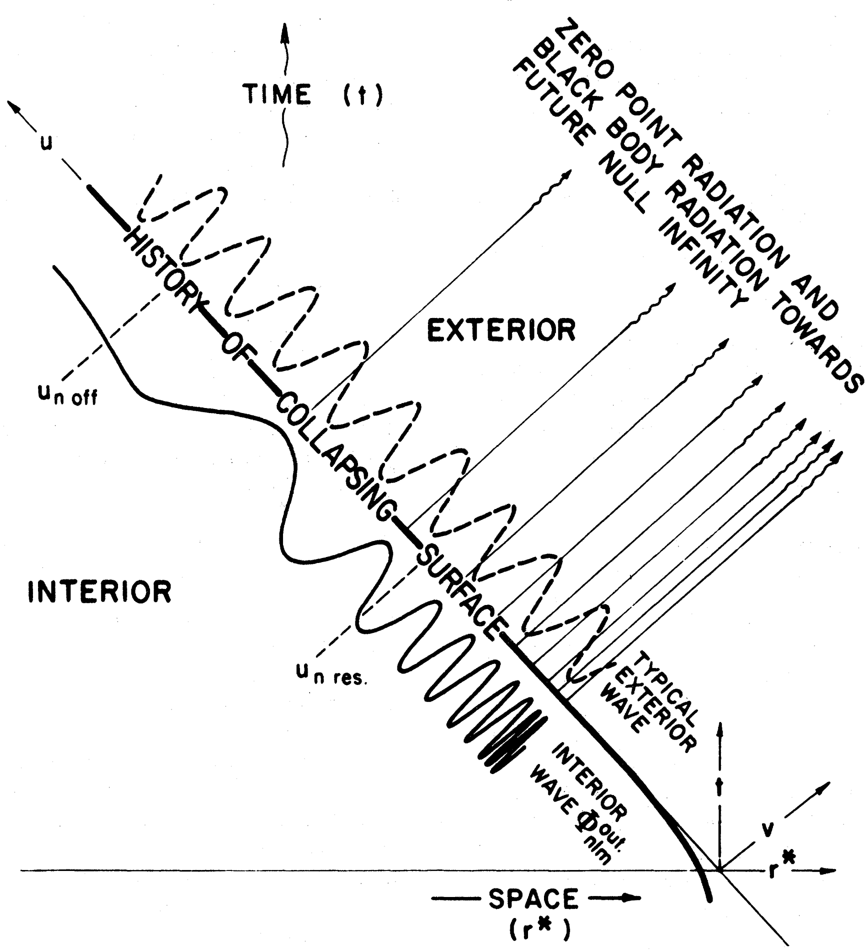

FIG. 1. Spacetime history of waves emerging from the surface of a collapsing star. The horizontal coordinate is the tortoise coordinate $r^*=r+2M\ln(r/2M-1)$. During late retarded time (large u) the history of the collapsing star approaches asymtotically the ingoing light cone as viewed in the Schwarzschild time t and r^* coordinate system. In this coordinate system the amplitude of the outgoing part $\Phi_{nlm}^{\text{outgoing}}$ of an interior standing wave at the surface of the collapsing star is indicated by "interior wave". As the star collapses, the oscillation frequency tends to zero as seen by a distant observed, i.e., the histories (null rays) of the wave crests emerging from the star are spaced successively farther apart. As the mode Φ_{nlm} crosses the star's surface it will cause the emission of photons into the exterior. Photons identified with a "typical exterior wave" will be created only as long as $\Phi_{nlm}^{\text{outgoing}}$ is in resonance with, i.e., has the same instantaneous frequency as, the "typical exterior wave." This happens in that interval $\Delta u=8\pi M$ along the history of the star's surface which is centered around the resonance (retarded) time $u_{n\,\text{res}}$. It is defined both in Secs. V and VI. Thus, owing to the nonadiabatic nature of the red-shift, $\Phi_{nlm}^{\text{outgoing}}$ gets amplified and thereby gives rise to a whole spectrum of photons, high-frequency photons first, low-frequency photons later. At the switch-off time, u_{off} (see Secs. V or VI), the instantaneous frequency of $\Phi_{nlm}^{\text{outgoing}}$ is zero and its behavior changes from an oscillatory to an exponential behavior. The production of photons is therefore exponentially negligible beyond the switch-off time. Even before $\Phi_{nlm}^{\text{outgoing}}$ is finished producing photons, the next zero-point fluctuation mode $\Phi_{nlm}^{\text{outgoing}}$ (not exhibited in this picture) is getting amplified and produces its spectrum of photons. The corresponding modes identified with the "typical exterior wave" are produced also in an interval $\Delta u=8\pi M$, which is centered around the resonance time $u_{n+1\,\text{res}}$ ($>u_{n\,\text{res}}$), however. The mean spectrum associated with the various zero-point fluctuation modes $\Phi_{nlm}^{\text{outgoing}}$, $\Phi_{n+1lm}^{\text{outgoing}}$, $\Phi_{n+2lm}^{\text{outgoing}}$, etc. is a blackbody spectrum.

$$V=\frac{l(l+1)}{r^2}\left(1-\frac{2M}{r}\right) \tag{4.2}$$

which separates the incipient event horizon from the distant observer.

It follows that an emerging wave train, which consists of a succession of wave packets, is not monochromatic. Rather, the frequency of the wave train (i.e., of successive wave packets) is decreasing. When the wave train (characterized by the angular momentum quantum numbers l and m, conserved during the scattering process) emerging from the star's surface has frequency lower than

$$\left[\frac{l(l+1)}{27}\right]^{1/2}\frac{1}{M}=\omega_{\min}, \tag{4.3}$$

the train will be reflected by the potential barrier V, Eq. (4.2), and will not reach a distant observer (see Appendix B). The time necessary for a wave train to decrease its frequency from ω_n to $\omega_{\min}$, Eq. (4.3), is (see Appendix B)

$$t=(v_0-2M)+2M\ln(4M\omega_n)+M\ln\frac{16l(l+1)}{27}. \tag{4.4}$$

It is the length of Schwarzschild time the outgoing part of the standing wave inside the star,

$$\Phi_{nlm}=\left(\frac{2\hbar}{a\omega_n}\right)^{1/2}\frac{1}{2r}\{\cos[\omega_n(t_{\rm in}-r)+\beta_{nlm}] - \cos[\omega_n(t_{\rm in}+r)+\beta_{nlm}]\}, \tag{4.5}$$

contributes at the star's surface to the outgoing radiation, which travels to a distant observer. The time coordinate $t_{\rm in}$ designates the proper time inside the shell star.

The observer, who resides in the asymptotically flat region of the Schwarzschild geometry, uses the unique time coordinate which reflects the static nature of the exterior of the incipient black hole. However, it is with respect to this unique Schwarzschild time that the radiation coming from the star's surface is *not* monochromatic, but is of changing frequency and of finite duration as given by Eq. (4.4). The radiation is a Fourier superposition of the outgoing waves associated with the static Schwarzschild geometry

$$\psi_{lm}(t,r,\theta,\phi)=\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\psi_\omega\frac{e^{i\omega(t-r^*)}}{r}d\omega\, Y_l^m(\theta,\phi). \tag{4.6}$$

The wave field is continuous across the history of the surface of the collapsing shell star, Eqs. (3.3),

$$\Phi_{nlm}\big|_{t_{\rm in}+r=r_0}=\psi_{lm}\big|_{t+r^*=v_0}.$$

Thus it can be said that the outgoing radiation field is literally caused by the standing-wave mode Φ_{nlm}. However, it is only the outgoing part of Φ_{nlm}, Eq. (4.5), which excites the outgoing Schwarzschild traveling modes exhibited in the integrand of the Fourier integral, Eq. (4.6). As shown in Appendix A, the Fourier amplitude ψ_ω is

$$\psi_\omega(\omega_n,\beta_{nlm})=\left(\frac{2\hbar}{a\omega_n}\right)^{1/2}\frac{M}{(2\pi)^{1/2}}\Gamma(-4i\omega M)\times(e^{-2\pi\omega M}e^{i\beta_{nlm}}+e^{+2\pi\omega M}e^{-i\beta_{nlm}})\times e^{i\omega(u_{n\,\rm off})}. \tag{4.7}$$

Here

$$u_{n\,\rm off}=v_0-4M+4M\ln4M\omega_n$$

is the (retarded) "switch off" time which is introduced in the next section.

The squared magnitude is

$$|\psi_\omega(\omega_n,\beta_{nlm})|^2=\frac{2\hbar}{a\omega_n}\frac{M^2}{2\pi}\frac{\pi}{\omega M}\times\left(\frac{1}{2}+\frac{1}{e^{8\pi\omega M}-1}+\frac{\cos2\beta_{nlm}}{2\sinh4\pi\omega M}\right). \tag{4.8}$$

The first term refers to the zero-point energy associated with $\Phi_{nlm}^{\rm outgoing}$. The second term refers to the blackbody energy spectrum. The third term refers to the statistical fluctuations of the energy; it goes to zero when averaged over several modes (n,l,m). A discussion of the physical significance of each factor of the Fourier amplitude ψ_ω is given in the paragraphs following Eq. (7.1). But first we give a qualitative and quantitative identification of the mechanism of blackbody radiation itself.

V. PARAMETRIC EXCITATION OF VACUUM FLUCTUATIONS

The production of blackbody radiation by a collapsing star can be accounted for in terms of a single idea governing the evolution of vacuum fluctuation modes: parametric excitation, i.e., the nonapplicability of the WKB approximation.

(1) A first qualitative picture of the evolution of waves is invariably obtained by determining to what extent the WKB (quasiclassical, adiabatic, geometric optics) approximation is applicable. The relevant criterion for waves emerging from the star's surface is found in their emitted instantaneous frequency [see Eq. (B7) and set $v_0=2M$]

$$\omega(t)=\omega_n e^{-t/2M}. \tag{5.1}$$

(A pictorial representation is given in Fig. 1.) The applicability of the WKB approximation (in Schwarzschild coordinates) near the surface of the star is gauged by the smallness of the quantity

$$\frac{1}{\omega^2}\frac{d\omega}{dt}=-\frac{1}{2M\omega_n}e^{t/2M}. \tag{5.2}$$

As $t\to\infty$, the star's surface approaches the incipient event horizon and the WKB approximation becomes inapplicable. Is that inapplicability merely a coordinate effect, or does it signal something physically significant?

That the latter is the case follows from an examination of the nature of the wave equation

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(g^{\mu\nu}\sqrt{-g}\,\frac{\partial\psi}{\partial x^\nu}\right)=0, \tag{5.3}$$

which governs the evolution of the wave field.

Evaluated inside the shell star, this equation is

$$\left[-\frac{\partial^2(r\Phi)}{\partial t_{\rm in}{}^2}+\frac{\partial^2(r\Phi)}{\partial r^2}-\frac{l(l+1)}{r^2}\right](r\Phi)=0\,. \tag{5.4}$$

Outside it is

$$\left[-\frac{\partial^2(r\psi)}{\partial t^2}+\frac{\partial^2(r\psi)}{\partial r^{*2}}-\frac{l(l+1)}{r^2}+\frac{2M}{r^2}\right]\left(1-\frac{2M}{r}\right)(r\psi)=0\,. \tag{5.5}$$

Here

$$r^*=r+2M\ln\left(\frac{r}{2M}-1\right) \tag{5.6}$$

is the "tortoise" coordinate,[10] which together with t straightens out the radial null geodesics and thereby pushes the $r=2M$ surface to $r^*=-\infty$.

First, observe that the wave equation has constant coefficients with respect to the $(t_{\rm in}, r)$ and the (t, r^*) coordinates inside and outside the star, respectively. Consequently, any disturbance is propagated without amplification or attenuation (i.e., without parametric excitation in the wider sense of the term). The disturbance is merely scattered by a time-independent potential barrier, which distorts the disturbance quantitatively, but not qualitatively.

Second, observe that the (t, r^*) coordinates go over into the usual Minkowskian (t, r) coordinates for a distant observer, who can thus use these coordinates to define (i.e., identify) radiative energy flowing out of a black hole.

These two properties, the first local, the second global, uniquely single out the (t, r^*) coordinate system as one of special physical significance. As a consequence, they carry with them two implications: (a) The applicability of the WKB approximation as gauged by the smallness of Eq. (5.2) is not merely a coordinate peculiarly, but instead refers to some physical phenomenon of nature. (b) In those regions of spacetime where the wave-equation coefficients are constant, zero-point or any other type of radiation is propagated without any (parametric) amplification; in other words, only the time-dependent region of the star, the history of its thin shell, assigns abruptly changing coefficients to the wave equation and thereby causes a parametric amplification of waves passing through that shell. The amplification is elaborated upon in paragraph (2) below and in Sec. VI.

This breakdown of the WKB approximation at the surface of the collapsing star is propagated to a distant observer by means of the following null-initial-value problem:

(i) Along the history of the star's surface

$$t+r^*=v_0\,, \tag{5.7}$$

where the retarded (Finkelstein) time

$$u=t-r^*$$

is related to the coordinates r^* and t by

$$u=-2r^*+v_0\,, \tag{5.8a}$$

$$u=2t-v_0\,, \tag{5.8b}$$

a typical outgoing vacuum fluctuation mode $\Phi_{nlm}^{\rm outgoing}$, Eq. (2.4), which satisfies Eq. (5.4), given by

$$\Phi_{nlm}^{\rm outgoing}=\left(\frac{2\hbar}{a\omega_n}\right)^{1/2}\frac{1}{2r}\cos\left[\exp\left(\frac{u_{n\,\rm off}-u}{4M}\right)+\beta_{nlm}\right]. \tag{5.9}$$

Here

$$u_{n\,\rm off}=v_0-4M+4M\ln 4M\,\omega_n \tag{5.10}$$

is, as we shall see in paragraph (2), the (retarded) "switch-off time" which, for the mode $\Phi_{nlm}^{\rm outgoing}$ of frequency ω_n, designates when it changes from an oscillatory to an exponential behavior.

(ii) Across the history of the star's surface the outgoing radiation mode is continuous,

$$\Phi_{nlm}^{\rm outgoing}=\psi_{lm}\,.$$

(iii) This field is propagated outside the star by the wave Eq. (5.5) to a distant observer. In other words, given a zero-point standing-wave mode Φ_{nlm}, the (null) initial-value problem is characterized by Eqs. (5.5), (5.7), (5.8), and (5.9). The law of the emission of blackbody radiation is governed by the solutions to such initial-value problems.

The rest of this section gives a qualitative account of the amplification process. First [i.e., in paragraph (2)], focus on the salient features of the wave field on the star's surface. These features are propagated to a distant observer. Second [in paragraph (3)], discuss the amplification process in terms of a time-dependent blue- (and red-) shift associated with a time-dependent potential.

(2) Although the field is continuous across the surface of the collapsing star, the coefficients of radial Helmholtz equation for the modes inside ($\sim e^{i\omega_n t_{\rm in}}$) and outside ($\sim e^{i\omega t}$) the star change discontinuously. Indeed, for these modes Eqs. (5.4) and (5.5) are respectively

$$\left[\frac{d^2}{dr^{*2}}-\frac{2M}{r^2}\frac{d}{dr^*}+\left(\omega_n{}^2-\frac{l(l+1)}{r^2}\right)\left(1-\frac{2M}{r}\right)^2\right]r\Phi_{nlm}=0 \tag{5.11}$$

and

$$\left[\frac{d^2}{dr^{*2}}+\omega^2-\left(\frac{l(l+1)}{r^2}+\frac{2M}{r^3}\right)\left(1-\frac{2M}{r}\right)\right]r\psi_{lm}^{\omega}=0\,. \tag{5.12}$$

The behavior of a typical vacuum fluctuation mode $\Phi_{nlm}^{\text{outgoing}}$ is most directly exhibited by means of the differential equation, Eq. (5.11), it satisfies along the history of the star's surface:

$$\left[\frac{d^2}{dr^{*2}} + \omega^2(r^*)\right]\left(1 - \frac{2M}{r}\right)^{1/2} r\Phi_{nlm}^{\text{outgoing}} = 0\,. \quad (5.13)$$

Here both the variable r and the squared instantaneous frequency

$$\omega^2(r^*) = \left(\omega_n{}^2 - \frac{l(l+1)}{r^2}\right)\left(1 - \frac{2M}{r}\right)^2 - \frac{2M}{r^3}\left(1 - \frac{2M}{r}\right) - \frac{M^2}{r^4}\Bigg|_{\text{star's surface}} \quad (5.14)$$

are understood by means of Eq. (5.6) to be functions of the tortoise coordinate r^*, which, via Eq. (5.8a), coordinatizes an ingoing null cone (= the history of the star's surface during late collapse) (see Fig. 1). Evidently the field undergoes the well-known time-dependent gravitational red-shift. Indeed, along the history of the star's surface a high-frequency vacuum fluctuation mode during late collapse ($1 - 2M/r \ll 1 \ll M\omega_n$) has an asymptotic frequency given by

$$\omega^2(u) = \left(\frac{1}{4M}\right)^2 \exp\left(\frac{u_{n\text{off}} - u}{2M}\right) - \left(\frac{1}{4M}\right)^2. \quad (5.15)$$

Thus, at the well-determined instance of (retarded) time, determined by

$$\omega^2(u_{n\text{off}}) = 0$$

and, thus, given by

$$u_{n\text{off}} = v_0 - 4M + 4M \ln 4M\omega_n\,, \quad (5.10)$$

the field $\Phi_{nlm}^{\text{outgoing}}$ on the star's surface changes its oscillatory behavior to an exponential behavior. This evidently happens when the tortoise coordinate and the (Schwarzschild) time have reached the respective values

$$r^*_{n\,\text{off}} = 2M - 2M \ln 4M\omega_n\,, \quad (5.16\text{a})$$

$$t_{n\,\text{off}} = v_0 - 2M + 2M \ln 4M\omega_n\,. \quad (5.16\text{b})$$

As expected, the higher the frequency ω_n of $\Phi_{nlm}^{\text{outgoing}}$ inside the star, the longer it takes the collapsing star to switch off [i.e., red-shift to zero frequency, $\omega^2(u) = 0$] that mode from view outside the star. If one takes into consideration the centrifugal barrier, Eq. (5.16b) must be replaced by Eq. (4.4). Similarly, Eqs. (5.10) and (5.16a) are to be replaced by analogous expressions.

If each mode $\Phi_{nlm}^{\text{outgoing}}$ is characterized by its switch-off time $u_{n\,\text{off}}$, how does the switch-off process itself, i.e., the exponential behavior of $\Phi_{nlm}^{\text{outgoing}}$, depend upon n, l, and m? A central proposition, again proved in Sec. VII, is that *the process is time translation invariant*. Indeed, the two quantities relevant to the switch-off process,

$$\lim_{r\to 2M} \omega^2(u) = -(4M)^{-2} \quad (5.17)$$

and

$$\frac{d\omega^2}{dr^*}(u_{n\,\text{off}}) = -(2M)^{-3}\,, \quad (5.18)$$

depend only on the parameter M of the incipient black hole. Thus the asymptotic behavior of $\Phi_{nlm}^{\text{outgoing}}$ at the star's surface is independent of the integers n, l and m that characterize the vacuum fluctuation mode $\Phi_{nlm}^{\text{outgoing}}$.

(3) If the unifying concept dealing with the behavior of a vacuum fluctuation mode $\Phi_{nlm}^{\text{outgoing}}$ on the star's surface is its *switch-off* (retarded) time,

$$u_{n\,\text{off}} = v_0 - 4M + 4M \ln 4M\omega_n\,, \quad (5.10)$$

then the unifying concept dealing with the parametric amplification of zero-point fluctuations is the *simultaneous resonance* time

$$u_{n\,\text{res}}(\omega) = v_0 - 4M + 4M \ln 4M\omega_n - 4M \ln 4M\omega. \quad (5.19)$$

It constitutes that instance of time in whose neighborhood the collapsing star produces photons of frequency ω by amplifying the vacuum fluctuation mode $\Phi_{nlm}^{\text{outgoing}}$, whose frequency is ω. In other words, it is that instance of time when the vacuum fluctuation mode Φ_{nlm} is in resonance with, i.e., has the same frequency by virtue of the gravitational red-shift as, the mode ψ_{lm}^{ω} at the surface of the collapsing star.

A heuristic account of this amplification process, based on the notion of a time-dependent potential, is obtained by considering the asymptotic radial Helmholtz equation inside and outside the collapsing star during late collapse:

$$\left[\frac{d^2}{dr^{*2}} + \left(\frac{1}{4M}\right)^2 \exp\left(\frac{r^* - r^*_{n\text{off}}}{M}\right) - \left(\frac{1}{4M}\right)^2\right] \times r\left(1 - \frac{2M}{r}\right)^{-1/2} \Phi_{nlm} = 0, \quad (5.20\text{a})$$

$$\left(\frac{d^2}{dr^{*2}} + \omega^2\right) r\psi_{lm}^{\omega} = 0\,. \quad (5.20\text{b})$$

The domain of definition of these two equations is separated by a moving boundary, the collapsing surface of the star. The vacuum fluctuation mode Φ_{nlm} inside the star will give rise to the mode ψ_{lm}^{ω} outside the star when the two modes are in resonance. This happens when the "instantaneous frequency" at the star's surface agrees with the frequency ω of the exterior mode,

$$\left(\frac{1}{4M}\right)^2 \exp\left(\frac{r^*_{n\,\mathrm{res}} - r^*_{n\,\mathrm{off}}}{M}\right) - \left(\frac{1}{4M}\right)^2 = \omega^2 , \qquad (5.21)$$

i.e., when

$$r^*_{n\,\mathrm{res}} = r^*_{n\,\mathrm{off}} + 2M\ln 4M\omega . \qquad (5.22)$$

The concept "frequency" in this context is only applicable to the extent that the second (negative) term on the left-hand side of Eq. (5.21) is negligible. This approximation is incorporated into the expression for the "resonance radius" $r^*_{n\,\mathrm{res}}$, Eq. (5.22).

To exhibit the time-dependent potential associated with Eqs. (5.20), expand the "instantaneous frequency" of $\Phi^{\mathrm{outgoing}}_{nlm}$ on the star's surface in the neighborhood of the instance when $\Phi^{\mathrm{outgoing}}_{nlm}$ is in "resonance" with ψ^{ω}_{lm},

$$\omega^2(r^*) = \omega^2(r_{n\,\mathrm{res}}) + \left.\frac{d\omega^2}{dr^*}\right|_{r^*_{n\,\mathrm{res}}} (r^* - r^*_{n\,\mathrm{res}}) + \cdots . \qquad (5.23)$$

Here $\omega^2(r_{n\,\mathrm{res}})$ is given by Eq. (5.21) and

$$\left.\frac{d\omega^2}{dr^*}\right|_{r^*_{n\,\mathrm{res}}} = \frac{\omega^2(r^*_{n\,\mathrm{res}}) + (4M)^{-2}}{M} . \qquad (5.24)$$

It is evident therefore that in the vicinity of the "time of resonance" the two asymptotic radial Helmholtz equations are

$$\left(\frac{d^2}{dr^{*2}} + [\omega^2 - W(r^*)]\right) r\left(1 - \frac{2M}{r}\right)^{-1/2} \Phi_{nlm} = 0 \quad \text{inside}, \qquad (5.25)$$

$$\left(\frac{d^2}{dr^{*2}} + \omega^2\right) r\psi^{\omega}_{lm} = 0 \quad \text{outside}. \qquad (5.26)$$

Here ω^2 is the squared Schwarzschild frequency given by Eq. (5.21). The time-dependent effective potential for Eqs. (5.25) and (5.26) is

$$W(r^*, t) = \begin{cases} \dfrac{\omega^2 + (4M)^{-2}}{M} (r^* - r^*_{n\,\mathrm{res}}) & \text{inside} \\ 0 & \text{outside} \end{cases}$$

$$= \theta(v_0 - t - r^*_{n\,\mathrm{res}}) \frac{\omega^2 + (4M)^{-2}}{M} (r^* - r^*_{n\,\mathrm{res}}) . \qquad (5.27)$$

Here θ is the unit step function. Figure 2 exhibits the diagram of this potential at the instant the collapsing surface crosses the resonance radius $r^*_{n\,\mathrm{res}}$ pertaining to the modes ψ^{ω}_{lm} and $\Phi^{\mathrm{outgoing}}_{nlm}$. This moving potential determines the shape and evolution of only those waves which have the specific (squared) frequency ω^2 given by Eq. (5.21). Evidently, for different Schwarzschild frequencies (which jointly make up the spectrum seen by a distant observer) different choices of resonance radii, Eq. (5.22), and potentials, Eq. (5.27), are necessary.

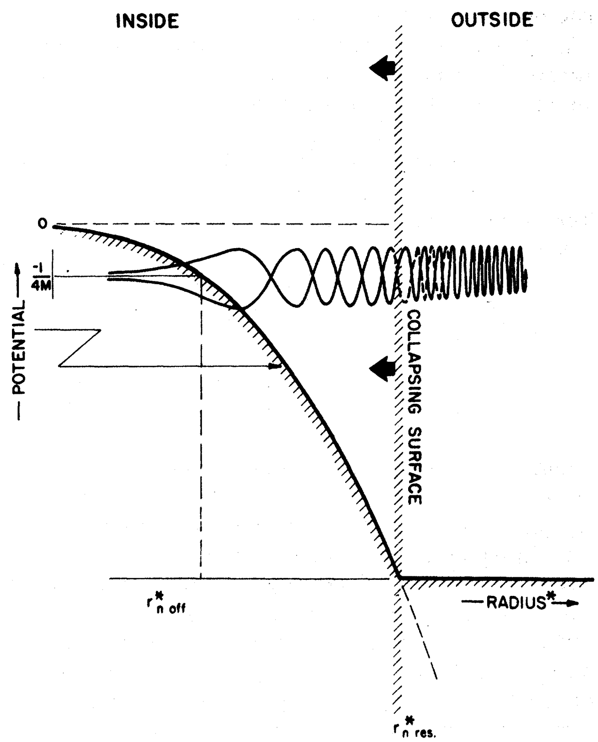

FIG. 2. Effective potential at $t_{n\,\mathrm{res}} = v_0 - r^*_{n\,\mathrm{res}}$ for waves traveling inside and outside a collapsing star. The horizontal coordinate is the radial "tortoise" coordinate $r^* = r + 2M\ln(r/2M - 1) + \mathrm{const}$. "Inside," to the left of $r^*_{n\,\mathrm{res}}$, the vertical coordinate designates the "effective potential," $-\omega_n{}^2 e^{r^*/M}$ (striped curve), of Eq. (5.20a). This curve is joined continuously, across the instantaneous position of the star's surface, to the outside effective potential of Eq. (5.20). During the late stages of collapse that potential is zero in the vicinity of the surface. The wave pattern indicates a standing wave inside and an outgoing wave outside. Inside these waves (Φ_{nlm}) have an exponential decay to the left of the switch-off radius, $r^*_{n\,\mathrm{off}}$; this is the point at which the instantaneous oscillation (squared) frequency, $\omega_n{}^2 e^{r^*/M} - (4M)^{-2}$, the coefficient of the interior radial Helmholtz equation (5.20a), turns negative. The collapsing surface moves uniformly ($r^* = -t + \mathrm{const}$) with respect to Schwarzschild time t through the standing-wave pattern. The wavelength of these waves is evidently such that their instantaneous oscillation frequency at the surface is exactly that observed by an (e.g., distant) observer using Schwarzschild time t.

It follows from the WKB approximation that, to the extent that it is applicable, photon number is conserved across the discontinuity of the potential, Eq. (5.27). This happens when the squared oscillation frequency, Eq. (5.23), of the wave train emerging from the star changes sufficiently slowly over one oscillation period, i.e.,

$$\frac{1}{\omega^2(r^*_{n\,\mathrm{res}})}\frac{d\omega}{dr^*}\bigg|_{r^*_{n\,\mathrm{res}}} = \frac{\omega^2(r^*_{n\,\mathrm{res}})+(4M)^{-2}}{2M\omega^3(r^*_{n\,\mathrm{res}})} \approx \frac{1}{2M\omega(r^*_{n\,\mathrm{res}})} \ll 1\,.$$

It is evident from Eq. (5.21) that the WKB criterion is violated especially for low frequencies. This happens when the collapsing surface of the star approaches the switch-off radius $r_{n\,\mathrm{off}}$, Eq. (5.16a). The corresponding Schwarzschild and retarded times are given by Eqs. (5.10) and (5.16).

The importance of the mechanism of parametric resonance excitation increases to the extent that the size of the second term relative to the first term in Eq. (5.23), i.e., the applicability of the WKB approximation, decreases. For a given outgoing zero-point fluctuation mode $\Phi^{\mathrm{outgoing}}_{nlm}$, notice that the time-dependent jump discontinuity in the effective potential entails a time-dependent "force," which is proportional to a moving δ function located at the star's surface. That "force" changes signs as the star's surface crosses the simultaneous resonance radius $r^*_{n\,\mathrm{res}}$. It is evident that if an interior standing wave has the correct frequency and correct phase at $r^*_{n\,\mathrm{res}}$, i.e., has a node there, then the time-dependent "force" will increase the amplitude of the outgoing part of the wave as it crosses the surface. [Indeed, phase synchronization (or the lack thereof) is important enough to enhance or diminish the amplified wave field; this fact is expressed by the "fluctuation" terms of Eq. (4.8).] In other words, "photons" of frequency ω are created, (see Fig. 3). However, as this force continues acting on this wave, its effect over one cycle will average more closely to zero the farther the force is located from the resonance radius $r^*_{n\,\mathrm{res}}$. As a consequence, *one expects that the ω spectral component of the outgoing wave $\Phi^{\mathrm{outgoing}}_{nlm}$ is only amplified at the star's surface in a very narrow neighborhood surrounding this simultaneous resonance radius $r^*_{n\mathrm{res}}$.* Although the formulation in this section constitutes only a qualitative heuristic summary of the relevant properties of the partial-differential equation, Eqs. (5.4) and (5.5), the expectations of this summary are formulated and proved more precisely with the help of the complete set of orthonormal wave packets also used by Hawking in his analysis. This is done in the next section.

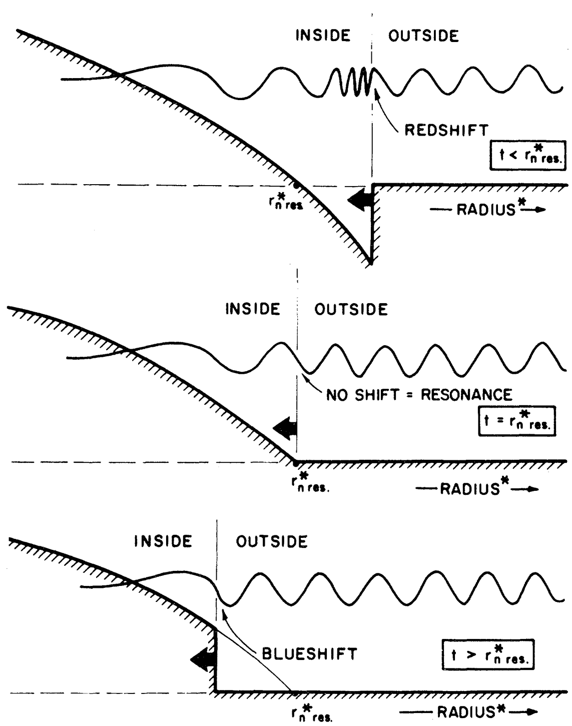

FIG. 3. Three snapshots of the time-varying step potential which governs the evolution of zero-point fluctuation mode $\Phi^{\mathrm{outgoing}}_{nlm}$ ("inside") and a specific Schwarzschild mode $\Psi_{\omega lm}$ ("outside"). As discussed at the end of Sec. V and as shown in Sec. VI this potential amplifies Φ_{nlm} and gives rise to photons of frequency ω only during the limited time interval $\Delta t = 4\pi M$ surrounding the resonance time $t^{(\omega)}_{n\,\mathrm{res}}$ $[=v_0 - r^*_{n\,\mathrm{res}}(\omega)]$. At this instant the red-shifted frequency of Φ_{nlm} at the star's surface equals the frequency (ω) of the excited mode $\Psi_{\omega lm}$.

VI. RESONANCE: THE NATURE OF A SINGLE AMPLIFIED ZERO-POINT FLUCTUATION MODE

The Fourier spectrum due to the single outgoing vacuum fluctuation mode $\Phi^{\mathrm{outgoing}}_{nlm}$ is given by Eq. (4.7). The squared magnitude of this spectrum is given by Eq. (4.8). These quantities are produced by a wave field whose properties are condensed into and referred to by means of real solution of the (Klein-Gordon) wave equation. Alternatively, one may, and in this section we shall, consider this real-valued field as the sum of positive- and negative-frequency complex fields. Such a decomposition, we recall, constitutes a more refined classification of the wave field—a classification which brings into sharp focus the particle-antiparticle properties of the wave field phenomenon.[13] Although the neutral particles, considered here, are their own antiparticle, the distinction allows us to discuss the amplification of a wave very efficiently as the mere production of particles (say, photons) and antiparticles (again photons) from, say, the positive-frequency wave field of a vacuum fluctuation mode, Eq. (2.4),

$$\Phi^{\mathrm{outgoing}}_{nlm} = \left(\frac{2\hbar}{a\omega_n}\right)^{1/2}\frac{1}{4r}\,e^{-i\omega_n(t_{\mathrm{in}}-r)-i\beta_{nlm}}\,Y^m_l(\theta,\varphi)\,.$$

The spectral amplitudes [analogous to those defined by Eq. (4.7)] of the resultant positive- and negative-frequency wave fields are, as one can see by inspecting the derivation in Appendix A, respectively

$$\psi_{+\omega}(\omega_n, \beta_{nlm}) = \left(\frac{2\hbar}{a\omega_n}\right)^{1/2} \frac{M}{(2\pi)^{1/2}} e^{-i\beta_{nlm}} \Gamma(-4i\omega M) \times e^{2\pi\omega M} e^{i\omega u_{n\,\mathrm{off}}} \qquad (6.1a)$$

and

$$\psi_{-\omega}(\omega_n, \beta_{nlm}) = \left(\frac{2\hbar}{a\omega_n}\right)^{1/2} \frac{M}{(2\pi)^{1/2}} e^{-i\beta_{nlm}} \Gamma(4i\omega M) \times e^{-2\pi\omega M} e^{-i\omega u_{n\,\mathrm{off}}}. \qquad (6.1b)$$

Here

$$u_{n\,\mathrm{off}} = v_0 - 4M + 4M \ln 4M\,\omega_n$$

is the switch-off (retarded) time which separates the oscillatory from the exponential behavior of Φ_{nlm} on the star's surface (see Fig. 1). Their absolute squares are respectively

$$|\psi_{+\omega}|^2 = \frac{2\hbar}{a\omega_n} \frac{M^2}{2\pi} \frac{\pi}{M\omega} \left(\frac{1}{2} + \frac{\frac{1}{2}}{e^{8\pi\omega M} - 1}\right) \qquad (6.2)$$

and

$$|\psi_{-\omega}|^2 = \frac{2\hbar}{a\omega_n} \frac{M^2}{2\pi} \frac{\pi}{M\omega} \left(\frac{\frac{1}{2}}{e^{8\pi\omega M} - 1}\right). \qquad (6.3)$$

In these expressions it is understood that $0 < \omega$. The amplification phenomenon is now reflected very distinctly in the expressions of Eqs. (6.2) and (6.3). The first term in Eq. (6.2) refers to the unamplified zero-point radiation spectrum. The second term refers to the spectrum of the amplified part only. Equation (6.3) gives the corresponding "antiparticle" intensity of the amplified portion. In other words, half the amount by which the mode $\Phi_{nlm}^{\mathrm{outgoing}}$ got amplified is given directly by the negative-frequency intensity, Eq. (6.3). The sum of Eqs. (6.2) and (6.3) gives, modulo the intensity due to fluctuations, the total spectral intensity already exhibited by Eq. (4.8).

The amplification mechanism cannot be understood within the WKB approximation. Indeed, computing the spectral amplitude in Appendix A within this approximation, i.e., by using the method of steepest descent, yields the unamplified zero-point spectrum only,

$$|\psi_{+\omega}|^2 = \frac{2\hbar}{a\omega_n} \frac{M^2}{2\pi} \frac{\pi}{M\omega} \left(\frac{1}{2}\right). \qquad (6.4)$$

All properties pertaining to the nature of an amplified vacuum fluctuation mode are contained in Eqs. (6.1). Two questions must therefore be asked and answered: (1) Given the outgoing part of a vacuum fluctuation mode, $\Phi_{nlm}^{\mathrm{outgoing}}$, where along the history of the surface of the collapsing shell star does the amplification take place? (2) What is the half-width of the retarded time interval along the history of the star's surface during which this amplification takes place? Both questions can be answered by expressing the amplified portion of the wave field in terms of orthonormal wave packets which constitute a complete set, already used by Hawking.[5] The basis functions of this set are wave packets,

$$P_{jk}(u) = \frac{1}{\sqrt{\epsilon}} \int_{j\epsilon}^{(j+1)\epsilon} e^{2\pi ik\omega/\epsilon} \frac{1}{(2\pi)^{1/2}} e^{-i\omega u} d\omega \qquad (6.5)$$

of mean frequency $(j+\frac{1}{2})\epsilon$. The predominant contribution to each packet is localized in an interval whose half-width is

$$\Delta u = \frac{2\pi}{\epsilon} > 0,$$

and which is centered around

$$u = \frac{2\pi k}{\epsilon}.$$

The completeness relation is

$$\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} P_{jk}(u) \overline{P}_{jk}(u') = \delta(u-u'). \qquad (6.6)$$

A function $\psi(u)$ can thus be decomposed into a superposition of wave packets,

$$\psi(u) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \alpha_{jk} P_{jk}(u). \qquad (6.7)$$

Here the expansion coefficients

$$\alpha_{jk} = \frac{1}{\sqrt{\epsilon}} \int_{j\epsilon}^{(j+1)\epsilon} e^{-2\pi ik\omega/\epsilon} \psi_\omega d\omega, \qquad (6.8)$$

$$j, k = 0, \pm 1, \pm 2, \ldots$$

are expressed in terms of the usual Fourier transform

$$\psi_\omega = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \psi(u') e^{i\omega u'} du'$$

of the given function $\psi(u)$.

The Fourier transform of the amplified wave field, Eqs. (6.1), gives rise to the corresponding wave-packet coefficients,

$$\alpha_{jk} = \left(\frac{2\hbar}{a\omega_n}\right)^{1/2} \frac{M}{(2\pi)^{1/2}} e^{-i\beta_{nlm}} \frac{1}{\sqrt{\epsilon}} \times \int_{4j\epsilon M}^{4(j+1)\epsilon M} \Gamma(-iw) e^{\pi w/2} \times \exp\left[iw\left(\frac{u_{n\,\mathrm{off}}}{4M} - \frac{2\pi k}{4M\epsilon}\right)\right] \frac{dw}{4M}. \qquad (6.9)$$

Here $w = 4M\omega$. Positive (negative) integers j refer to positive- (negative-) frequency wave packets composed of Fourier components, Eq. (6.1a) [Eq. (6.1b)], which are made up of amplified (amplified part of the) zero-point radiation. The integer k locates via Eq. (6.6) the position of the wave packets along the history of the star's surface, which is coordinatized by the retarded time u.

To evaluate the integral, let $1 \ll |j|$. This condition does not exclude low-frequency wave packets provided ϵ is chosen small enough. The condition facilitates the evaluation by allowing a first-order expansion of the exponent of the integrand. Consider negative-frequency wave packets only. Let

$$w = -w_0 - \Delta w,$$

where $w_0 = 4|j|M\epsilon > 0$. Equation (6.9) thereby becomes

$$\alpha_{jk} = \left(\frac{2\hbar}{a\omega_n}\right)^{1/2} \frac{-1}{4(2\pi\epsilon)^{1/2}} e^{i\beta_{nlm}} \Gamma(iw_0) \times e^{-\pi w_0/2} e^{iw_0\left(\frac{u_{n\,\mathrm{off}}}{4M} - \frac{2\pi k}{4M\epsilon}\right)} \times \int_0^{4M\epsilon} e^{\Delta w}(\gamma + i\delta)\, d(\Delta w)\,, \tag{6.10}$$

where

$$\gamma + i\delta = \frac{\Gamma'(iw_0)}{\Gamma(iw_0)} - \frac{\pi}{2} - i\left(\frac{u_{n\,\mathrm{off}}}{4M} - \frac{2\pi k}{4M\epsilon}\right). \tag{6.11}$$

The intensity of a wave packet is therefore

$$|\alpha_{jk}|^2 = \frac{2\hbar}{a\omega_n} \frac{1}{32\pi\epsilon} |\Gamma(iw_0)|^2 e^{-\pi w_0} \times 4e^{\gamma} \frac{\sin^2 2M\epsilon\delta + \sinh^2 2M\epsilon\gamma}{\delta^2 + \gamma^2}. \tag{6.12}$$

This intensity exhibits by virtue of the "resonance difference" δ and the "half width" γ a typical resonance behavior along the history of the collapsing star in the following sense: Selectively focus attention on the set of all those wave packets that have fixed mean frequency $j\epsilon$. These wave packets are distinguished from each other by the integer k, i.e., by the (retarded) time $u = 2\pi k/\epsilon$ at which they emerge from the surface of the star. The intensity of these wave packets, as seen below, is large for that integer k for which

$$\frac{2\pi k}{\epsilon} = u_{n\,\mathrm{res}}\,, \tag{6.13}$$

where $u_{n\,\mathrm{res}}$ is given by Eq. (5.23). The intensity is small for other values of k. Indeed, as seen below, the full width of the retarded time interval over which wave packets have "appreciable" intensity is only

$$\Delta\left(\frac{2\pi k}{\epsilon}\right) = \Delta u = 8\pi M\,. \tag{6.14}$$

The location of the resonance, and its width, are governed respectively by the imaginary and the real part of $\gamma + i\delta$, Eq. (6.11). At high frequencies, $1 \ll w_0 = 4|j|M\epsilon$, for example,

$$\frac{\Gamma'(iw_0)}{\Gamma(iw_0)} \to i\ln w_0 - \frac{\pi}{2} - \frac{1}{2w_0}\,, \tag{6.15}$$

$$\gamma = -\pi\,, \tag{6.16a}$$

$$\delta = \frac{1}{4M}\left(\frac{2\pi k}{\epsilon} - u_{n\,\mathrm{res}}\right), \tag{6.16b}$$

where

$$u_{n\,\mathrm{res}} = u_{n\,\mathrm{off}} - 4M\ln w_0 \tag{6.17}$$

is the (retarded) moment of resonance already introduced by Eq. (5.23) in the context of parametric resonance. Introduce Eqs. (6.16) into Eq. (6.12), use the well-known expression for $|\Gamma(iw_0)|^2$ near the end of Appendix A, reintroduce $4M\omega\epsilon = w_0$, drop the subscript zero, and thus obtain explicitly the intensity of the packets under consideration (namely, $j\epsilon$ fixed),

$$|\alpha_{jk}|^2 = \frac{2\hbar}{a\omega_n} \frac{1}{32\pi\epsilon} \frac{\pi}{2M\omega} \frac{1}{e^{8\pi M\omega} - 1} 4e^{-\pi} \times 16M^2 \frac{\sin^2[\frac{1}{2}\epsilon(2\pi k/\epsilon - u_{n\,\mathrm{res}})] + \sinh^2 2\pi M\epsilon}{(2\pi k/\epsilon - u_{n\,\mathrm{res}})^2 + (4\pi M)^2}. \tag{6.18}$$

The relative intensity of the wave packets under consideration ($j\epsilon$ fixed) is governed by the very last factor of Eq. (6.12):

$$|\alpha_{jk}|^2 \propto 16M^2 \frac{\sin^2[\frac{1}{2}\epsilon(2\pi k/\epsilon - u_{n\,\mathrm{res}})] + \sinh^2 2\pi M\epsilon}{(2\pi k/\epsilon - u_{n\,\mathrm{res}})^2 + (4\pi M)^2}.$$

It is now clear that, considered as a function of

$$u = \frac{2\pi k}{\epsilon},$$

$|\alpha_{jk}|^2$ expresses the afore-mentioned resonance behavior of the wave packets along the history of the surface of the collapsing star. It is this behavior that gives $u_{n\,\mathrm{res}}$, Eq. (6.17), its special significance: It locates *where* the zero-point fluctuation mode is getting amplified. Equation (6.18) indicates that the spectral component $\omega = j\epsilon$ is produced predominantly in a total interval

$$\Delta u = 8\pi M \tag{6.19}$$

centered around this simultaneous resonance time given by Eq. (6.17).

The damped resonance behavior is also present for low frequencies ($w = |j|\epsilon \sim 1/4M$). Indeed, for such frequencies, $w_0 = 4|j|\epsilon M \sim 1$, the real and imaginary part of $(\ln\Gamma)'$ in Eq. (6.11) are of order unity. Although the half-width $(=\gamma)$ and the resonance difference $(=\delta)$ are altered slightly from those of Eqs. (6.16), we expect that the qualitative behavior of Eq. (6.18), and the concomitant results, Eqs. (6.13) and (6.14), still hold.

The results of the wave-packet analysis of this section can be summarized by rewriting Eq. (6.18), the intensity of a wave packet, in the form

$$|\alpha_{\omega_u}|^2 = \frac{2\hbar}{a\omega_n}\frac{e^{-\pi}}{M\omega}\frac{M^2}{\epsilon}\frac{1}{e^{8\pi M\omega}-1} \times \frac{\sin^2\{\frac{1}{2}\epsilon[u-u_{n\,\mathrm{res}}(\omega)]\}+\sinh^2 2\pi M\epsilon}{[u-u_{n\,\mathrm{res}}(\omega)]^2+(4\pi M)^2}, \quad (6.20)$$

where

$$u_{n\,\mathrm{res}}(\omega) = v_0 - 4M + 4M\ln 4M\omega_n - 4M\ln 4Mw \quad (6.21)$$

is the resonance time, Eq. (6.17), with its dependence on ω_n and ω emphasized and exhibited. The spectral component ω is produced in a neighborhood of size $\Delta u = 8\pi M$ surrounding the resonance time $u_{n\,\mathrm{res}}(\omega)$.

The various spectral components are produced only over a limited part of the history of the collapsing star. Indeed, the high-frequency $(M^{-1} \ll \omega = |j|\epsilon)$ intensity contribution [see Eq. (6.20)] produced at early retarded time [for how early, see Eq. (6.21)], is exponentially small because of the Planck factor. The very-low-frequency $(|j|\epsilon = \omega \ll (4M)^{-1})$ intensity contributions are produced at late retarded times. Although the corresponding integrand, Eq. (6.20), diverges $(\sim\omega^{-2})$ as $\omega = |j|\epsilon$ approaches zero, the associated spectral photon flux $(\propto|\alpha_{\omega_u}|^2\omega^3)$ or "energy flux" $(\propto|\alpha_{\omega_u}|^2\omega^4$, see Sec. IX) does tend towards zero during late retarded time. Indeed, without even referring to the above detailed wave-packet analysis, one may observe [see Eq. (5.9)] that the vacuum fluctuation mode $\Phi_{nlm}^{\mathrm{outgoing}}$ is being switched off by the star after $u = u_{n\,\mathrm{off}}$; it approaches a contant in an exponential way and thereby will not give rise to any radiation appreciably after $u = u_{n\,\mathrm{off}}$.

Within the context of gravitational collapse, the primary significance of the above result about the resonance aspect of the amplification process lies in its local nature. Within the framework of classical field theory the above analysis shows that a wave emerging from the surface of the star will be amplified precisely at the collapsing surface and no where else.

The amplification process is a linear one. Consequently, it applies to any wave field satisfying the Klein-Gordon equation, and the wave fields of vacuum fluctuation modes, designated by Φ_{nlm} throughout this article, are no exception. Averaged over several such modes, the spectrum of the amplified portion of these modes is that of a blackbody.

It is therefore very difficult to associate the production of blackbody radiation with anything but the surface of the collapsing star.

It is interesting to contrast Hawking's basic assumptions with ours. His picture and computations rest very heavily on the assumption that a collapsing star will actually pass through its event horizon, leaving behind a black hole in the standard sense of the word. The origin of the blackbody radiation, he argues, is not the star itself, but instead is the actual event horizon. Indeed, he pictures the creation of particle pairs near this event horizon. One particle is emitted towards a distant observer; the antiparticle disappears inside the event horizon. Our picture and computations, though not as general as Hawking's, are very different from them. The event horizon is irrelevant in our analysis. The importance shifts away from the $r = 2M$ surface, and instead the central focus of attention is the resonance radius

$$r^*_{n\,\mathrm{res}}(\omega) = 2M - 2M\ln 4M\omega_n + 2M\ln 4M\omega, \quad (6.22a)$$

or, equivalently, the resonance time

$$t_{n\,\mathrm{res}}(\omega) = v_0 - 2M + 2\ln\ln 4M\omega_n - 2M\ln 4M\omega. \quad (6.22b)$$

There, in a neighborhood of size

$$\Delta r^* = 4\pi M$$

at the vacuum matter interface of the history of the collapsing star, an outgoing wave will get amplified.

It is the nonadiabatic change in the red-shift which is responsible for this amplification and hence the emission of radiation stimulated by the vacuum fluctuations. The nonadiabatic nature of the change is expressed and manifests itself by the difference between Eq. (6.2) and Eq. (6.4).

VII. NATURE OF RADIATION EMITTED BY A BLACK HOLE

It is easy to identify three properties of the radiation emitted from the surface of the collapsing star:

(1) Each zero-point fluctuation mode Φ_{nlm} gives rise to a spectrum that consists of three parts: (i) the Lorentz invariant zero-point spectrum, (ii) the blackbody spectrum, and (iii) the fluctuation spectrum.

(2) The radiation fields associated with their respective vacuum fluctuation modes are statistically identical.

(3) The radiation fields associated with their

respective vacuum fluctuation modes are emitted in a time-sequential order, each radiation field being characterized by its switch-off time, Eq. (5.16b).

The first property has already been identified in the discussion following Eq. (4.8). That the first term should refer to the zero-point radiation spectrum is a proposition that follows from an exact frequency (ω) by frequency cancellation of this outgoing zero-point energy with the spectral zero-point energy going into the incipient black hole. This is shown in Sec. IX and also in Sec. X.

Each spectrum, Eq. (4.8), depends on the random phase angle β_{nlm} of the zero-point fluctuation mode Φ_{nlm}. There are many such modes which contribute to the total seen by the distant observer. The phase β_{nlm} is a random function of the integers (n, l, m). Consequently, the fluctuation term in Eq. (4.8) drops out. Thus the mean squared magnitude of the Fourier amplitude ψ_ω for outgoing radiation due to the vacuum fluctuations of the standing wave Φ_{nlm}, Eq. (4.5), reduces to

$$\langle |\psi_\omega(\omega_n)|^2\rangle = \frac{2\hbar}{a\omega_n}\frac{M^2}{2\pi}\frac{\pi}{4\omega M}\frac{2\cosh 4\pi\omega M}{\sinh 4\pi\omega M}$$

$$= \frac{\hbar}{a\omega_n}\frac{M}{\omega}\left(\frac{1}{2}+\frac{1}{e^{8\pi\omega M}-1}\right). \qquad (7.1)$$

The second property can be identified directly by focusing on the Fourier amplitude of the radiation field caused by a typical mode Φ_{nlm},

$$\psi_\omega(\omega_n, \beta_{nlm}) = \left(\frac{2\hbar}{a\omega_n}\right)^{1/2}\frac{M}{(2\pi)^{1/2}}\Gamma(-4i\omega M) \times (e^{-2\pi\omega M}e^{i\beta_{nlm}} + e^{+2\pi\omega M}e^{-i\beta_{nlm}})e^{i\omega u_{n\,\mathrm{off}}}. \qquad (4.7)$$

The very last factor is the relevant one. It determines the position of the amplified wave field along the history (parametrized by the retarded time coordinate u) of the collapsing surface of the star. Indeed, this positioning follows from the circumstances that if $F(\omega)$ is the Fourier transform of $f(u)$, then $e^{i\omega u_{n\,\mathrm{off}}}F(\omega)$ is the Fourier transform of $f(u-u_{n\,\mathrm{off}})$. In other words, multiplying $F(\omega)$ by $e^{i\omega u_{n\,\mathrm{off}}}$ has the effect of shifting the time-dependent signal $f(u)$ by an amount

$$u_{n\,\mathrm{off}} = v_0 - 4M + 4M\ln 4M\omega$$

into the future (later retarded time). In terms of the Schwarzschild time t, which is used by a distant observer, this shift is, of course

$$t_{n\,\mathrm{off}} = v_0 - 2M + 2M\ln 4M\omega_n.$$

It is now clear why the outgoing packets of radiation caused by their respective Φ_{nlm}'s are statistically identical. Their respective Fourier transforms, Eq. (4.7), are, modulo the time translation factor $e^{i\omega_n u_{n\,\mathrm{off}}}$, identical, i.e., *they are totally independent of the frequency* ω_n [the first factor, $(2\hbar/a\omega_n)^{1/2}$, is merely a normalization factor]. Hence the outgoing packets of radiation are identical. Their only difference lies in the presence of the random phase shift β_{nlm}. This expresses, as is evident from Eq. (4.8), the *statistical* identity of the outgoing radiation packets caused by the Φ_{nlm}'s.

The third property is now also easily identified. By virtue of the presence in the Fourier spectrum, Eq. (4.7), of the time translation factor

$$e^{i\omega u_{n\,\mathrm{off}}}$$

it is evident that the outgoing radiation packets are sequentially ordered along the history of the surface of the collapsing star. Each compound radiation packet starts getting switched off after its switch-off time $u_{n\,\mathrm{off}}$; those packets caused by Φ_{nlm}'s with low frequency ω_n first, those caused by Φ_{nlm}'s with high frequency ω_n later.

VIII. BLACK-HOLE ENTROPY = k ln (NUMBER OF FIELD OSCILLATORS)

An incipient black hole emits radiation packets in consecutive order. It is appropriate therefore to determine the total number emitted during the whole lifetime of the black hole. Each emitted radiation packet is brought into existence by a specific field oscillator whose frequency is

$$\omega_n = \frac{\pi}{a}\left(n+\frac{l}{2}\right).$$

The number of radiation packets brought into existence by all those field oscillators whose frequencies lies in an interval $\Delta\omega_n$ is given by

$$\frac{\Delta n}{n+\frac{1}{2}l} = \frac{\Delta\omega_n}{\omega_n}. \qquad (8.1)$$

(In order for the ingoing waves of these field oscillators to be actually able to enter the star, it is necessary that $\frac{1}{2}l \ll n$. Dropping the angular quantum-number term $\frac{1}{2}l$ will result in a slight inaccuracy in the final result [see also the discussion following Eq. (9.11)]. According to Eq. (5.16b), this happens during a time interval

$$\Delta t = 2M\frac{\Delta\omega_n}{\omega_n}. \qquad (8.2)$$

According to Eq. (1.3) or Eq. (9.11) (to be defined in the next section), the mass emitted in this time interval is

$$\Delta M = \alpha\frac{L_W^2}{M^2}\Delta t,$$

where $\alpha = -0.708\times 10^{-4}$. Combine Eqs. (8.1), (8.2), and (8.3), integrate, and obtain

$$\int_1^n \frac{dn}{n} = \frac{1}{2\alpha}\int_M^0 \frac{MdM}{L_W^{\ 2}}. \qquad (8.3)$$

The result yields the logarithm of the total number of emitted radiation packets, i.e., the total number of field oscillators Φ_{nlm} which cause the emission of blackbody radiation during the finite lifetime [Eq. (1.4)] of the black hole,

$$\ln n = 3.53\times 10^3 \frac{M^2}{L_W^{\ 2}}. \qquad (8.4)$$

The expression on the right-hand side is (modulo a factor ~280) the entropy of the initial incipient black hole[14, 15] (BH)

$$S_{\rm BH} = 4\pi k\frac{M^2}{L_W^{\ 2}}. \qquad (8.5)$$

Recall that the entropy of a black hole is a measure of the number of internal configurations,[16, 17] i.e., possible modes of formation ("hairs") that can give rise to the same macroscopic black hole.[18] In other words, the entropy is (Boltzmann's constant k times) the logarithm of the total number, $n_{\rm int}$, of microscopic (in the statistical-mechanical sense) states any one of which corresponds to the same macroscopic state, the black hole. Thus we are led to the interesting numerical coincidence

$$n \approx n_{\rm internal}; \qquad (8.6)$$

in other words, the number of radiation packets emitted during the lifetime of a black hole equals roughly the total number of internal states ("lost hairs") of the initial black hole. This suggests that in some generalized sense the vacuum fluctuations modes (field oscillators) interact with corresponding internal black-hole states and stimulate them into emitting radiation packets. With the emission of a number of such packets there will be a corresponding decrease in the number of internal states. Although the entropy of the black hole will decrease, the entropy of the total system, black hole plus emitted radiation, presumably will not.[19]

In view of the fact that a black hole has the interrelated attributes of (1) temperature, (2) entropy, and (3) degrees of freedom that cause spontaneous emission, it follows that there is a potentially very consequential way of viewing black holes: Besides, say, compound nuclei, as we know them from nuclear physics,[20] incipient black holes seem to constitute another example of a macroscopic many-level system in a state of excitation, which is best described statistical mechanically in terms of temperature and entropy. Indeed, it is possible to derive the blackbody radiation spectrum together with its fluctuation spectrum from the single assumption that a black hole is endowed with a high-density quantum-level structure.[21]

IX. EMISSION OF MASS-ENERGY BY AN INCIPIENT BLACK HOLE

The outgoing energy spectrum due to the *single* zero-point radiation mode $\Phi_{nlm}^{\rm outgoing}$ being switched off is

$$\frac{\text{(energy)}}{\text{(mode)(unit area)(unit frequency)}}\frac{1}{M}d(M\omega)$$
$$= \frac{1}{4\pi}\frac{1}{r^2}\,|\psi_\omega(\omega_n,\beta_{mlm})|^2\omega^2\frac{1}{M}d(M\omega)$$
$$= \frac{1}{4\pi}\frac{1}{r^2}\frac{\hbar/M}{a\omega_n}(M\omega)$$
$$\times\left(\frac{1}{2}+\frac{1}{e^{8\pi M\omega}-1}+\frac{\cos\beta_{nlm}}{2\sinh 4\pi M\omega}\right)d(M\omega). \qquad (9.1)$$

The factor $(4\pi)^{-1}$ arises from averaging over a sphere surrounding the incipient black hole. The second factor arises from applying Parceval's formula,[22] and hence Eq. (4.8), to the energy flux $\psi_{,t}\psi_{,r}$ $(=|\psi_{,t}|^2)$ associated with the wave field, Eq. (4.6).

The rate at which modes are amplified by the collapsing star is (see Appendix C)

$$\frac{\text{(number of amplified modes)}}{\text{(unit time)}} = \frac{dn}{dt}$$
$$= \frac{a\omega_n}{2\pi}\frac{1}{M}. \qquad (9.2)$$

Multiply this rate by the energy, Eq. (9.1), and obtain thereby the spectral power flux due to the amplification of all those modes Φ_{nlm} that have fixed integers l and m.

For a given frequency ω but for different quantum integers l and m, the number of those modes that are actually able to surmount their centrifugal barriers, Eq. (4.2), and thus are able to escape (or enter) a black hole, is

$$\int_0^l (2l+1)dl = l(l+1)$$
$$= 27(M\omega)^2. \qquad (9.3)$$

The upper limit of integration is determined by Eq. (4.3).

The spectral power flux emitted by an incipient black hole and received by a distant observer is thus the product of Eqs. (9.1), (9.2), and (9.3),

$$\frac{\text{(outgoing energy)}}{\text{(unit time)(unit area)(unit frequency)}}\frac{1}{M}d(M\omega)=\frac{\hbar/M}{2\pi M}\frac{1}{4\pi r^2}\left(\frac{1}{2}+\frac{1}{e^{8\pi\omega M}-1}\right)27(M\omega)^3 d(M\omega). \quad (9.4)$$

Thus a distant observer measures both outgoing thermal radiation as well as zero-point radiation. However, vacuum fluctuations are composed of both ingoing and outgoing zero-point radiation modes, and the modes carrying energy into the black hole must not be ignored.

The ingoing part, for all times, is exhibited in the expression for the standing-wave mode Φ_{nlm}, Eq. (2.4). The squared amplitude of each ingoing mode averaged over a sphere surrounding the black hole and other several modes centered around ω_n is

$$\langle |\Phi_{nlm}^{\text{ingoing}}|^2\rangle=\frac{1}{4\pi r^2}\frac{\hbar}{4a\omega_n}. \quad (9.5)$$

The density of such modes that are able to enter an incipient black hole is

$$\frac{\text{(number of modes)}}{\text{(unit of frequency)}}=\frac{dn}{d\omega_n}$$
$$=\left(\frac{a}{\pi}\right)27(M\omega_n)^2. \quad (9.6)$$

The first factor is the number of modes (of fixed total angular quantum numbers l and m) per unit frequency associated with a sphere of radius a. This factor is multiplied by the total number of those modes of frequency ω_n which are actually able to enter a black hole, Eq. (9.3).

The energy density associated with the mode $\Phi_{nlm}^{\text{ingoing}}$ is $\langle|\Phi_{nlm}^{\text{ingoing}}|^2\omega_n^2\rangle$. Consequently, the spectral energy flux into the black hole is, with the help of Eqs. (9.5) and (9.6),

$$\frac{\text{(ingoing energy)}}{\text{(area)(time)(frequency)}}\frac{1}{M}d(M\omega_n)$$
$$=\langle|\Phi_{nlm}^{\text{ingoing}}|^2\rangle\omega_n^2\frac{dn}{d\omega_n}\frac{1}{M}d(M\omega_n)$$
$$=\frac{1}{4\pi r^2}\frac{\hbar/M}{2\pi M}\left(\frac{1}{2}\right)27(M\omega)^3 d(M\omega). \quad (9.7)$$

The subscript n has been dropped from the last expression.

The net spectral energy flux emitted from an incipient black hole to a distant observer is the difference between Eqs. (9.4) and (9.7),

$$\frac{\text{(net energy)}}{\text{(area)(time)(frequency)}}\frac{1}{M}d(M\omega)$$
$$=\frac{1}{4\pi r^2}\frac{\hbar/M}{2\pi M}\left(\frac{1}{e^{8\pi M\omega}-1}\right)27(M\omega)^3 d(M\omega). \quad (9.8)$$

It follows that the rate at which an incipient black hole is losing energy is

$$\frac{\text{(energy)}}{\text{(time)}}=\frac{dM_{\text{conv}}}{dt}$$
$$=-\frac{\hbar/M}{2\pi M}\int_0^\infty\frac{27(M\omega)^3 d(M\omega)}{e^{8\pi M\omega}-1}$$
$$=-\frac{27}{2\pi\times 15\times 8^4}\frac{\hbar}{M^2}. \quad (9.9)$$

In terms of geometrical units, this equation assumes the form given by Eq. (1.3). Integrating this equation yields the total lifetime, given by Eq. (1.4).

The correctness of the evolution equation, Eq. (9.9), rests upon three implicit assumptions to which one could object:

(1) The correctness of (a) the frequency ω_n expression, Eq. (2.3), for Φ_{nlm} and hence the number of modes amplified per unit time, Eq. (9.2) or Eq. (C4), and (b) the number, Eq. (9.3), of those travelling modes of frequency ω which can surmount the potential barrier Eq. (4.2), and hence of the density of modes, Eq. (9.6).

(2) The correctness of the zero-point energy subtraction process which gives rise to Eq. (9.8).

(3) The correctness of the shift in the time of amplification, Eq. (C2), which gives rise to Eq. (9.2), the rate at which modes get amplified.

There is evidently overlap between objections to assumptions (1) and (3). They have, however, been listed separately because the objections can be made on different grounds. As seen below, objections to assumption (1) are inconsequential, objections to assumption (2) are answered in Sec. X, and objections to assumption (3) need to be answered by means of another paper. Focus on the first assumption first.

(a) Equation (9.2) is based by means of Eq. (C1) on Eq. (2.3). We are implicitly assuming that the wave field Φ_{nlm} has its angular eigenfunction integer l low enough so that inside a sphere of radius $r=2M$ the radial factor (spherical Bessel function) will undergo some oscillations. It can thus be easily shown that the circumstance $2M\ll a$, i.e., collapse taking place inside a very large spherical cavity of radius a, implies $l\ll n$. Hence

$$\omega_n=\frac{\pi}{a}\left(n+\frac{l}{2}\right),$$

the basis for Eq. (C1), is indeed a very accurate expression for the oscillation frequency of Φ_{nlm}.

(b) The power spectrum of the net radiation emitted by an incipient black hole is given by

Eq. (9.8). This blackbody spectrum has its maximum approximately at

$$\omega \approx \frac{1}{8\pi M}. \tag{9.10}$$

The minimum angular momentum carried by a photon is $[l(l+1)]^{1/2}\hbar = \sqrt{2}\hbar$. Consequently, the minimum height of the potential, Eq. (4.2), which separates the black hole from a distant observer, is

$$\omega_{\min} = \frac{1}{3}\left(\frac{2}{3}\right)^{1/2}\frac{1}{M}. \tag{9.11}$$

Emitted radiation of frequencies near and below that given by Eq. (9.13) is not totally transmitted over or through the centrifugal barrier, Eq. (4.2). It follows that Eq. (9.3) does need to be modified for $\omega < \omega_{\min}$ and that the power spectrum seen by a distant observer is therefore not quite the one exhibited by Eq. (9.8), but rather a version which is somewhat distorted and diminished below frequencies of the order given by Eq. (9.11), which is slightly above the frequency, Eq. (9.10), of the blackbody spectrum maximum.

If the black hole is immersed into a blackbody radiation field whose temperature equals that of the black hole, is one to conclude from the existence of this distortion that an incipient black hole cannot stay in thermodynamic equilibrium with its surrounding? The answer is, of course, no. Just as the emerging radiation spectrum is distorted, so is the spectrum of the radiation which surmounts or tunnels through the barrier, Eq. (4.2), in order to be absorbed by the black hole. The transmission and reflection through the barrier is symmetric as far as ingoing and outgoing radiation is concerned; it follows that the radiation which is lost down the black hole is precisely gained coming up out of the black hole. In other words, if the black hole is immersed in blackbody radiation of temperature given by Eq. (1.2), equilibrium is preserved between the black hole and the blackbody radiation outside it. The existence of the barrier merely makes the black hole emit blackbody radiation "masked by a filter."

The second implicit assumption, discussed in more detail in Sec. X touches the more important issue of principle, namely, what unambiguous principles one refers to, in dealing with the seemingly unlimited number of field oscillators; their combined zero-point energies yield an unlimited energy-density permeating space. A superficial but straightforward qualitative application of Einstein's gravitation theory would argue very forcibly for profound global gravitational effects, which are not observed in reality.[23,24] One should note, however, that this infinite energy associated with the unlimited number of Klein-Gordon (KG) degrees of freedom is problematic only to a classically formulated black-hole theory (or, more generally, gravitation theory). It is therefore not so obvious whether the problem persists in a theory that analgamated the KG degrees of freedom with the microscopic internal degrees of freedom[18] of, say, a macroscopic black hole.

The third assumption underlying Eq. (9.9) is Eq. (C2). Starting with the resonance time Eq. (6.22b) for Φ_{nlm}, or better yet with Eq. (B9),

$$\begin{aligned} t &= v_0 - 2M + 2M\ln 4M\omega - 2M\ln 4M\omega \\ &\quad - M\ln\frac{16l(l+1)}{27} \\ &= t_{n\,\mathrm{res}} - M\ln\frac{16l(l+1)}{27}, \end{aligned} \tag{9.12}$$

in order to take into account the centrifugal barrier also, one obtains

$$\begin{aligned} \Delta t &= 2M\frac{\Delta\omega_n}{\omega_n} + 2\Delta M\ln 4M\omega_n - 2\Delta M\ln 4M\omega \\ &\quad - \Delta M\omega\frac{16l(l+1)}{27}. \end{aligned} \tag{9.13}$$

In other words, the approximation that went into Eq. (9.4), the rate at which modes get amplified, is that M stays constant. Evidently the number of modes that get amplified during the time interval Δt is substantially influenced by the amount of mass, ΔM, lost by the incipient black hole during Δt.

Let us see what consequences are entailed by incorporating changes in black-hole mass into Eq. (9.2). Introduce Eq. (9.12) into Eq. (9.13) to eliminate the logarithmic terms, use Eq. (C1), and obtain instead of Eq. (9.2)

$$\begin{aligned} \frac{(\text{number of amplified modes})}{(\text{unit time})} &= \frac{dn}{dt} \\ &= \frac{a\omega_n}{2\pi}\frac{1}{M}\left[1 - \frac{dM}{dt}\frac{1}{M}(t - v_0)\right]. \end{aligned} \tag{9.14}$$

Let us make the tentative assumptions (a) that in spite of variable black-hole mass the outgoing energy spectrum is still given by Eq. (9.1) and (b) that the zero-point energy will not appear in the net-energy ledger. Then the differential equation for the evolution of an incipient black hole, Eq. (9.9), gets replaced by

$$\frac{dM}{dt} = -\alpha\frac{L_W{}^2}{M^2}\left[1 - \frac{dM}{dt}\left(\frac{t}{M} - \frac{v_0}{M}\right)\right]$$

or

$$\frac{dM}{dt}=-\alpha\frac{L_W^2}{M^2}\frac{1}{1-\alpha\frac{L_W^2}{M^2}\left(\frac{t}{M}-\frac{v_0}{M}\right)}.$$

Here $\alpha=27(2\pi\times 15\times 8^4)^{-1}$, $L_W=(\hbar G/c^3)^{-3}$ is the Wheeler length. This modified evolution equation evidently yields a black-hole lifetime different from the one given by Eq. (1.3).

Owing to the present nonexistence of more general well-formulated principles, one may therefore not as yet exclude the possibility that once mass-energy loss has been incorporated into the radiation production mechanism and into the evolution dynamics of a radiating *incipient* black hole, the general properties of such a black hole are substantially different from its first approximation considered in this paper.

X. NET ENERGY NECESSARY TO DEFORM THE ZERO-POINT FLUCTUATIONS IS CUTOFF INDEPENDENT

In the process of identifying the phenomenon of zero-point fluctuations as the common denominator which explains other directly observable phenomena of nature, one is invariably confronted with the problem of dealing with an unlimited number of field oscillators in their ground state and hence with an unlimited amount of zero-point energy. Instead of defining (i.e., identifying) generally applicable principles, one usually deals with the problem on a case-by-case basis as it arises in the phenomenon under examination. The derivation of the Planck radiation formula for an incipient black hole is no exception.

Consider the integrated energies associated respectively with (a) the outgoing energy spectrum, Eq. (9.1), due to a single mode Φ_{nlm},

$$\frac{\hbar/M}{a\omega_n}\int\left(\frac{1}{2}+\frac{1}{e^{8\pi M\omega}-1}+\frac{\cos 2\beta_{nlm}}{2\sinh 4\pi M\omega}\right)(M\omega)d(M\omega)\,, \tag{10.1}$$

(b) the outgoing spectral power, Eq. (9.4), due to the amplification process,

$$\frac{\hbar/M}{2\pi M}\int\left(\frac{1}{2}+\frac{1}{e^{8\pi M\omega}-1}\right)27(M\omega)^3d(M\omega)\,, \tag{10.2}$$

and (c) the ingoing spectral power, Eq. (9.7), due to zero-point radiation absorbed by the black hole,

$$\frac{\hbar/M}{2\pi M}\int\left(\frac{1}{2}\right)27(M\omega)^3d(M\omega)\,. \tag{10.3}$$

The upper limit of integration in all of these expressions should not be set to infinity. It is, however, permitted to set the upper limit of the total net energy, Eq. (9.9), associated with net spectral energy, Eq. (9.8), equal to infinity.

The reason for the imposition of such a double standard is that, strictly speaking, none of the integration limits should be set to infinity. Consider specifically the integration limits of the expression in Eq. (10.1), the total energy emitted by a black hole and caused by a standing wave of frequency ω_n inside the star. According to the discussion in Appendix A the expression for the Fourier amplitude, Eq. (4.7), and hence the total energy, Eq. (10.1), is only applicable for those (conserved) Schwarzschild frequencies ω, for which

$$\omega \ll \omega_n\,.$$

Consequently, the upper integration limit in Eq. (10.1) can be very large as long as $\omega\ll\omega_n$, which is always the case during late stages of collapse. Furthermore, during the whole lifetime of the incipient black hole, ω_n can only assume finite values—finite values because the emission of radiation from an incipient black hole is caused by interior standing waves of successively high frequency [see Eq. (5.16b) or Eq. (9.12)], the highest frequency being the limiting frequency of the sequence $\{\omega_n\}$ arrived at in Sec. VIII. This limit is characterized by the integer n given by Eq. (8.6). It starts entering the picture only near the end of the life of the classical incipient black hole. The conclusion is therefore that a large but finite cut-off frequency should be assigned to the integration limits (10.1) and (10.2). As a result, the emitted radiation, zero-point plus blackbody, is finite. To determine the net radiation emitted, focus on ingoing radiation, Eq. (10.3), with the same cutoff, perform the subtraction frequency by frequency, and obtain the net rate of energy emission, Eq. (9.8). The only frequencies that actually enter into the subtraction process are those that characterize wave fields that have actually been amplified by the collapsing shell star. Other wave fields need never be mentioned in the energy ledger. Furthermore, the expression Eq. (9.8) approaches a finite limit as the upper integration limit approaches infinity, i.e., the final result is cutoff independent.

XI. ENERGY SPECTRUM AND FLUCTUATION SPECTRUM: BLACKBODY VS INCIPIENT BLACK HOLE

(1) Both a blackbody and an incipient black hole emit radiation characterized by Planck's blackbody radiation formula. To facilitate a comparison between the two emitters, follow the arguments of Einstein[25] and de Broglie[26] by recalling the important features of the Planck spectrum,

$$\frac{1}{e^{(\hbar\omega/kT)}-1}=e^{-\hbar\omega/kT}+e^{-2\hbar\omega/kT}+e^{-3\hbar\omega/kT}+\cdots\,, \tag{11.1}$$

and of the mean square of the fluctuations associated with the energy deposited into, say, $\Delta G_\omega = 8\pi a^3\omega^2\Delta\omega/(2\pi)^3$ phase-space cells,

$$\langle\epsilon^2\rangle = \hbar\omega\left(\frac{\Delta G_\omega \hbar\omega}{e^{\hbar\omega/kT}-1} + \frac{\Delta G_\omega \hbar\omega}{(e^{\hbar\omega/kT}-1)^2}\right) \tag{11.2a}$$

$$= \hbar\omega(\Delta G_\omega \hbar\omega e^{-\hbar\omega/kT}) + 2\hbar\omega(\Delta G_\omega \hbar\omega e^{-2\hbar\omega/kT}) + 3\hbar\omega(\Delta G_\omega \hbar\omega e^{-3\hbar\omega/kT}) + \cdots \,. \tag{11.2b}$$

This dispersion $\langle\epsilon^2\rangle$ is that of a mixture of statistically independent Boltzmann gases of single quanta $\hbar\omega$, pairs $2\hbar\omega$, trios $3\hbar\omega$, etc. The Planck spectrum itself refers, of course, to the mean number of quanta in each phase-space cell. In view of the nature of the dispersion $\langle\epsilon^2\rangle$, the occupants of a single phase-space cell are single quanta $\hbar\omega$, or pairs $2\hbar\omega$, or trios $3\hbar\omega$, etc., the mean number of n-tuples being given by the respective terms of the sum in the Planck spectrum, Eq. (11.1).

Thus one is led to consider the blackbody radiation as an ensemble of thermalized quantized field oscillators: The field oscillator whose frequency is ω can be excited only in steps of energy $\hbar\omega$; the nth term in the sum refers to the probability of exciting the field oscillator to its nth energy level by virtue of the simultaneous presence of n photons each of energy $\hbar\omega$. In other words, given the Planck blackbody radiation spectrum, together with its fluctuation spectrum, Einstein's and de Broglie's arguments lead to the ideas that (a) blackbody radiation consists of quantized field oscillators, and that (b) the probability for exciting the oscillator of frequency ω by one step (and thereby creating only one photon) is given by

$$e^{(-\hbar\omega/kT)}\,.$$

An incipient black hole, like a blackbody, also emits radiation characterized by a Planck radiation spectrum. Indeed, as a result of parametric amplification (by the collapsing star) of a vacuum fluctuation mode Φ_{nlm}, the spectral energy [see Eq. (9.1)] emitted into the exterior Schwarzschild vacuum geometry is

$$\frac{\text{(energy)}}{\text{(mode)(unit frequency)}}\Delta\omega = \frac{M\Delta\omega}{a\omega_n}\left\{\frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{8\pi M\omega}-1} + \cos 2\beta_{nlm}\left[\hbar\omega\left(\frac{\hbar\omega}{e^{8\pi M\omega}-1}\right) + \left(\frac{\hbar\omega}{e^{8\pi M\omega}-1}\right)^2\right]^{1/2}\right\} \tag{11.3a}$$

$$= \frac{M\Delta\omega}{\pi n}\left[\frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{8\pi M\omega}-1} + \hbar\omega\cos\beta_{nlm}(e^{-8\pi M\omega} + 2e^{-8\pi M\omega 2} + 3e^{-8\pi M\omega 3} + \cdots)^{1/2}\right]. \tag{11.3b}$$

The first term refers to the unamplified zero-point energy. See Eqs. (6.2) and (6.4). The resemblance of the other terms in this expression to not only the blackbody energy spectrum, Eq. (11.1), but also to its fluctuation spectrum, Eqs. (11.2), is striking. If one wishes to apply to the radiation from an incipient black hole the underlying principles of the theory of blackbody radiation—and the nature of the fluctuation spectrum indicates that one must—then it is necessary to assume that the radiation from an incipient black hole is in the form of quanta. Indeed, the coefficient of $\cos\beta_{nlm}$ in Eq. (11.3a) is the root mean squared fluctuation in the energy of radiation consisting of quanta and waves, or alternately of single quanta, pairs of quanta, trios of quanta, etc. as we have learned from Einstein and de Broglie.

The necessary assumption that the radiation is in the form of quanta carries with it, as seen below, a price (or bonus): One must assume that the emission process is also quantized, and that the emission of quanta due to the vacuum fluctuation modes Φ_{nlm} is therefore governed by probability concepts.

(2) The rate at which the vacuum fluctuation modes Φ_{nlm} play their active role along the star's surface is, Eq. (C4),

$$\frac{dn}{dt} = \frac{n}{2M}\,.$$

It follows from applying this equation to Eq. (11.3a) that the mean energy emitted in a (Schwarzschild) time interval Δt is

$$\frac{\text{(energy)}}{\text{(time)(frequency)}}\Delta t\Delta\omega = \left(\frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{8\pi M\omega}-1}\right)\frac{\Delta t\Delta\omega}{2\pi}\,. \tag{11.4}$$

Consider those modes having fixed angular integers l and m. The density of these modes in phase (i.e., t,ω) space is such that the phase-space volume assigned to one travelling mode (i.e., the volume of one phase-space cell or one field oscillator) is

$$\Delta t\Delta\omega = 2\pi\,. \tag{11.5}$$

Indeed, the notion of phase space and phase-space cells (and the "Planck oscillator") has been made precise by means of the set of orthonormal wave packets $P_{jk}(u)$ in Sec. VI. One such traveling wave is assigned to each phase-space cell. Its frequency interval is $\Delta\omega=\epsilon$. The interval of time within which each traveling wave is localized is $\Delta t=\Delta u/2 = 2\pi/\epsilon$. The phase-space volume of each mode is therefore given by Eq. (11.5).

The mean energy in each phase space cell is thus

$$\frac{(\text{energy})}{(\text{unit phase-space cell})}=\frac{\hbar\omega}{2}+\frac{\hbar\omega}{e^{8\pi M\omega}-1}\ . \tag{11.6}$$

By contrast, in the given frequency interval $\Delta\omega$, each single mode Φ_{nlm} causes the emission of energy which in the mean is

$$\frac{(\text{mean emitted energy})}{(\text{mode})}=\left(\frac{\hbar\omega}{2}+\frac{\hbar\omega}{e^{8\pi M\omega}-1}\right)\frac{M\Delta\omega}{n\pi}\ . \tag{11.7}$$

The square of the deviation from the mean energy, Eq. (11.6), deposited into a unit phase-space cell, is, according to Eq. (11.3a),

$$\cos^2 2\beta_{nlm}\left[\hbar\omega\left(\frac{\hbar\omega}{e^{8\pi M\omega}-1}\right)+\left(\frac{\hbar\omega}{e^{8\pi M\omega}-1}\right)^2\right]\ . \tag{11.8}$$

Having averaged this expression over several vacuum fluctuation modes Φ_{nlm}, one obtains

$$\langle\epsilon^2\rangle=\tfrac{1}{2}[\hbar\omega(h\omega\, e^{-8\pi M\omega})+2\hbar\omega(h\omega\, e^{-8\pi M\omega 2}) + 3\hbar\omega(\hbar\omega\, e^{-8\pi M\omega 3})+\cdots]\ . \tag{11.9}$$

This is the sum of the mean squared energy fluctuations (in one phase-space cell) due to a mixture of statistically independent Boltzmann gases. (For an isolated black hole considered here, the factor $\frac{1}{2}$ expresses the fluctuation of only the outgoing part of the blackbody radiation; in our considerations, ingoing blackbody radiation is absent.) Such a fluctuation spectrum suggests therefore very strongly the following meaning to the mean energy, Eq. (11.6):

$$\frac{\hbar\omega}{2}+\frac{\hbar\omega}{e^{8\pi M\omega}-1}=\frac{\hbar\omega}{2}+\hbar\omega\, e^{-8\pi M\omega}+\hbar\omega\, e^{-8\pi M\omega 2} +\hbar\omega\, e^{-8\pi M\omega 3}+\cdots\ ; \tag{11.10}$$

each exponential factor $e^{-8\pi M\omega r}$ gives the mean number of r-tuples of photons in a unit phase-space cell (i.e., the relative probability that a given field oscillator is excited to the rth state).

One can draw three conclusions from the existence and the nature of the energy fluctuation associated with the amplification of vacuum fluctuations.

First, the radiation from an incipient black hole is thermal radiation in the precise sense of the term, i.e., the outgoing radiation is characterized not only by a Planck-type spectrum, but also by fluctuations which only blackbody radiation has.[27] Such a fluctuation spectrum implies[21] (i) that an incipient black hole is an entity endowed with densely spaced quantum levels and (ii) that therefore the basic laws of statistical mechanics can be applied and, thus, used to explain the thermal radiation emitted from a black hole.

Second, for given angular integers l and m, over a short enough time interval and frequency interval ($\Delta t\,\Delta\omega\sim 2\pi$) the amplified energy is dominated entirely by its fluctuating part, Eq. (11.8), especially at higher frequencies. Indeed, it is evident from Eq. (11.9) that for high frequencies the root mean squared energy fluctuation from one phase-space cell to another phase-space cell,

$$(\Delta\epsilon)_{\text{rms}}=\langle\epsilon^2\rangle^{1/2}\sim\frac{1}{\sqrt{2}}\hbar\omega e^{-4\pi M\omega}\ ,$$

is much larger than the mean amplified (non-zero point) energy, Eq. (11.6), deposited into a phase-space cell,

$$\frac{(\text{amplified energy})}{(\text{unit phase-space cell})}\sim\hbar\omega\, e^{-8\pi M\omega}\ .$$

The radiation, therefore, is emitted in the form of single quanta $\hbar\omega$ in the roughest sense of the term.

These single quanta can be pictured to be responsible for the relatively large energy fluctuations from phase-space cell to phase-space cell. In view of the fact that they are negative as well as positive [$-1\leqslant\cos\beta_{nlm}\leqslant 1$ in Eq. (11.3)], the energy density due to the amplified radiation can be negative at the surface of the collapsing star. Indeed, even though the energy condition[28] is satisfied in the mean, it is likely that it is not satisfied statistically and that for this reason the focusing theorem[29] will not apply and hence gravitational collapse will not evolve into a standard black hole.[28]

Third, a single vacuum fluctuation mode Φ_{nlm} can be said to stimulate the emission of a photon with a probability

$$\frac{M}{\pi n}e^{-8\pi M\omega}=\text{probability for } \Phi_{nlm} \text{ to cause the emission of a single photon.} \tag{11.11}$$

Such an observation is arrived at by first observing that the amplification process is linear and therefore can also be applied to a vacuum fluctuation mode Φ_{nlm}. However, an attempt to do this to many such modes runs into an illuminating contradiction between (a) the mean spectral energy, Eq.

(11.7), produced by a single vacuum fluctuation mode and (b) the fluctuations in this energy. On one hand, the fluctuation spectrum dictates that the amplification and hence the deposition of energy into phase-space cells is in the form of quanta $\hbar\omega$, especially at high frequencies. On the other hand, it is evident from Eq. (11.7) that, classically, a single mode Φ_{nlm} cannot marshall enough energy to fill a single phase-space cell with the mean energy given by Eq. (11.6), to speak nothing about enough energy for the emission of a single quantum $\hbar\omega$. The contradiction can be circumvented by asserting that Eq. (11.7) is understood to hold only in the mean, and that the factor

$$\frac{M}{\pi n}\frac{1}{e^{8\pi M\omega}-1}=\frac{M}{\pi n}(e^{-8\pi M\omega}+e^{-8\pi M\omega 2}+e^{-8\pi M\omega 3}+\cdots)$$

refers to the probability of Φ_{nlm} causing the emission of a single quantum $\hbar\omega$, or a pair $2\hbar\omega$, or a triple $3\hbar\omega$, etc. Thus for high frequencies the first term, which expresses the emission of single quanta, is the dominating one, and is the one given by Eq. (11.11). The first term inside the parenthesis, the probability of finding a single quantum in a a phase-space cell, is the quantity which Hawking[4] determined and from which he found the Planck spectrum.

XII. CASIMIR EFFECT, INCIPIENT BLACK HOLES, AND SAKHAROV'S VIEWPOINT

The relationship between zero-point fluctuations and Einstein's geometrodynamics is not new. Sakharov, evidently referring to the energy necessary to deform the zero-point fluctuations, points out that the "metric elasticity of space" can be considered as the underlying principle from which Einstein's geometrodynamics follows.[30]

It is possible to point to at least two macroscopic phenomena in nature which support Sakharov's idea, namely, the Casimir effect[31,32,33] and the effect discovered by Hawking,[4,5,34,35] which in this paper has been reformulated and extended in terms of zero-point fluctuations. The Casimir effect, we recall, consists of an attractive force between two uncharged capacitor plates at the same potential. This attraction arises from the fact that it takes a negative amount of energy to deform adiabatically the zero-point energy density between two plane conductors. This negative deformation energy has an effective mass and hence should be an active source of gravitation.[36] In other words, one should identify the deformation energy of the zero-point fluctuations with Sakharov's metric elastic energy of space.

When a star undergoes collapse it is possible to draw a sharp distinction between (a) the background geometry and (c) perturbations of this geometry, or other (e.g., electromagnetic) fields evolving in the geometry. This distinction during the evolution of an incipient black hole allows one to assert that although the dynamic geometry deforms the zero-point fluctuations in an adiabatic fashion during the early stages of collapse, the geometry deforms the zero-point fluctuations also decidedly nonadiabatically during the late stages of collapse. Were it not for these nonadiabatic parametric excitations of the zero-point fluctuations the exterior geometry would evolve into the vacuum Schwarzschild geometry as seen by a distant observer. As it stands, however, the existence of the blackbody radiation should be viewed as the extra amount of negative work that the collapsing star is doing in deforming the zero-point fluctuations nonadiabatically. Consequently, the total (adiabatic plus nonadiabatic) deformation of the zero-point fluctuations should be associated not merely with an interior geometry together with an exterior Schwarzschild geometry but rather with a geometry that differs "slightly" from it (possibly a variant of the radiating Vaidya geometry?)[37] From Sakharov's viewpoint the existence of the additional qualitatively different (nonadiabatic) deformation of the zero-point energy is therefore to be associated with the difference between the metric elasticity of the exterior space endowed with the Schwarzschild geometry (corresponding to only "adiabatic deformation of space"), and that of the exterior space endowed with a radiating Vaidya (?) geometry (corresponding to "adiabatic together with nonadiabatic deformation of space").

XIII. EFFECT OF VACUUM FLUCTUATIONS ON BLACK-HOLE FORMATION

The existence of a vacuum black hole formed by a collapsed star is a sufficient condition for the production of blackbody radiation. It is, however, not a necessary condition. Indeed, the assumption that a black hole in the standard sense of the term[28] is actually formed leads to a direct conflict with the resonance mechanism described in Sec. VI. We must accept therefore an alternative picture of blackbody radiation emitted from a collapsing configuration. This picture is based on classical field theory applied to zero-point radiation permeating the spacetime of a collapsing star.

The essential aspect of our picture of the production of blackbody radiation is not the existence of an event horizon but the existence of a radius of resonance, Eq. (6.22), for any two modes Φ_{nlm} (inside the star) and ψ^{ω}_{lm} (outside the star) having the same angular integers l and m. As we have seen in Sec. VI, it is difficult to account for the behavior of the emitted radiation except to attribute it to being manufactured at the surface of the collapsing shell star. Indeed, within the context of classical field theory there is a process by which a zero-point fluctuation mode Φ_{nlm} gives rise to a packet of radiation which in the mean has a Planck spectrum. Each spectral component, regardless of its frequency, is manufactured during a retarded time interval

$$\Delta u = 8\pi M$$

at the retarded resonance time

$$u_{n\,\mathrm{res}}(\omega) = v_0 - 4M + 4M \ln 4M\omega_n - 4M \ln 4M\omega$$

along the history of the star's surface. These processes involve the modes Φ_{nlm} in a time-sequential way and thereby give rise to a well-determined rate at which the packets of radiation are produced:

$$\frac{(\text{number of amplified modes})}{(\text{unit time})} = \frac{(\text{number of switched off modes})}{(\text{unit time})} = \frac{dn}{dt} = \frac{a\omega_n}{2\pi}\,\frac{1}{M}\,.$$

The packets, as we saw in Sec. VII, are, on the average, identical, each one carrying a well-determined amount of energy as given by Eq. (9.1). The energy is carried along outgoing null trajectories from the history of the star's surface towards future null infinity. The fact that the radiative energy has its origin along the history of the star's surface implies that the star will not collapse through its $r = 2M$ surface. Indeed, if one assumed it did, then an unlimited amount of energy would be emitted by the star by the time its surface pases through $r = 2M$. This follows from the fact (a) that the amount of mass-energy radiated per unit Schwarzschild time from the star's surface is a steady nonzero quantity and (b) that as r approaches $2M$, the Schwarzschild time at the star's surface approaches infinity. An unlimited amount of energy would therefore be extracted from the surface of the collapsing star. In view of the conservation of energy-momentum, this extraction would have an unlimited effect on the collapsing matter itself. It is difficult to avoid such a paradox unless one did not make the above assumption, which is that the star will collapse through its $r = 2M$ surface.

An objection might be raised to the effect that the above conclusion cannot be trusted because it is based on an analysis which is done within the framework of merely classical field theory; in view of the fact that the emission of blackbody radiation is a quantum process, an analysis that incorporates this fact into its very foundation might lead one to a different conclusion. In response to this objection the following reply can be made: The amplification process operating on waves emerging from the collapsing star is a linear one and is therefore applicable to classical wave fields of large as well as small amplitudes. As was pointed out in Sec. XI, the thermal fluctuations in the high-frequency part of the emitted Planck spectrum become so large that the characteristic quantum nature of the emission process comes in direct conflict with a strictly classical identification of the process. This conflict we resolve by extrapolating the classical correspondence limit from large amplitude waves to small (zero-point) amplitude waves and thereby asserting that on the average results obtained within the classical framework coincide with those within a quantum-mechanical framework.

XIV. INCIPIENT BLACK HOLE: AN INTERMEDIATE CONFIGURATION IN THE EVOLUTION TOWARDS THE FINAL STATE

If the two simplest and most universal principles of physics are the law of gravitation, which identifies and embodies phenomena of the macroscopic "classical" world, and the law of blackbody radiation, which identifies and embodies the essentials of the microscopic "quantum" world, then the formation and evolution of an incipient black hole (1) evidently concretizes the direct logical link between the two, and (2) leads to a reformulation of the issue of the final state within the context of a collapsing star.

The classical version of the issue of the final state of stellar gravitational collapse focuses attention on the inapplicability of the Einstein field equations in those regions of spacetime which are inside the event horizon. The inevitability of the paradox of gravitational collapse, "the greatest crisis of physics of all time," follows from the universal law of classical gravitation.[38]

By focusing attention on collapse as such, it is easy to identify analogies between the paradox of the collapsing atom formulated by Rutherford and Bohr and the paradox of the gravitationally collapsing star. The former, the greatest crisis of physics during its time, was a dominant factor in the motivation towards the identification and codification of the quantum aspects of matter. The latter nowadays serves the same purpose in regard to similar aspects of space itself. In view of (a) the blackbody radiation emitted by an incipient black hole and (b) the consequent finite lifetime (as seen by a distant observer) of the classical entity, an examination of the total evolution of an incipient black hole should have a high priority.

To gain a qualitative perspective of its evolution and its ultimate ground state, recall that information about a black hole, such as size, position, etc. is contained in scattering (most generally by waves) experiments. The scattering waves may, for example, be matter (Klein-Gordon) waves, radar signals, or even the zero-point fluctuations permeating the universe. These waves must satisfy two necessary conditions in order that one learn the features and hence the existence of a black hole:

First, the wavelength must be short enough so that details can be seen, i.e.,

$$\lambda = \frac{c}{\omega} \leq M . \tag{14.1}$$

Here λ is the reduced wavelength of radiation emerging from the black hole. The quantity ω is the frequency and equality would imply that the black hole is being observed at its diffraction limit. In terms of conventional mass units the mass is $M = M_{\rm conv} G/c^2$. Consequently, the inequality is

$$\left(\frac{c^3}{G}\right)\frac{1}{\omega} \leq M_{\rm conv} . \tag{14.2}$$

Second, the frequency must be low enough so that the black hole suffers no excessive recoil:

$$\left(\frac{\hbar}{c^2}\right)\omega \leq M_{\rm conv} . \tag{14.3}$$

Equality implies that the mass energy of the scattered quantum (scalar particle, photon, etc.) is so large that it is impossible to describe the process in term of a perturbation propagating in a sharply distinguishable background geometry. In

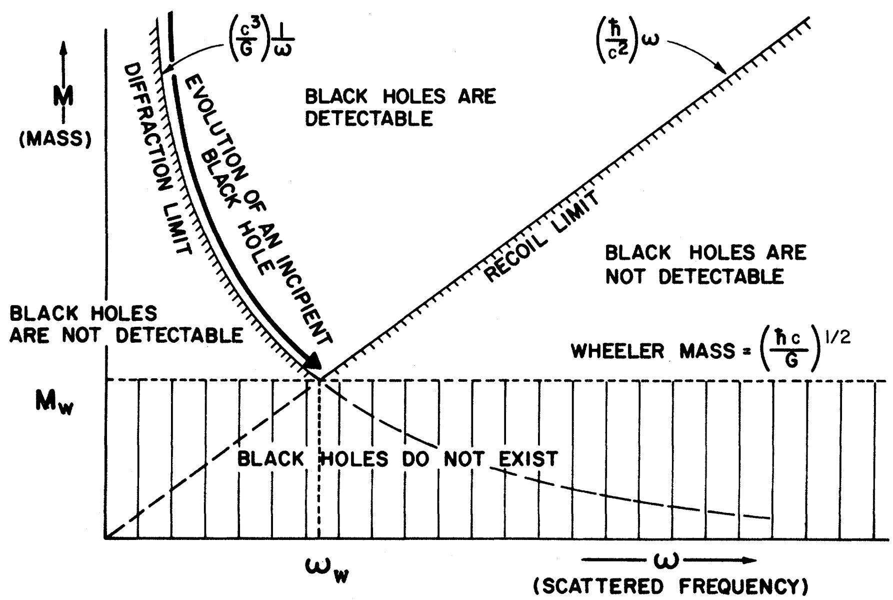

FIG. 4. Realm of observability of classical black holes. The abscissa measures the frequency scattered (or absorbed) by a black hole. The ordinate is a coordinate that allows simultaneous comparison of (1) black-hole mass with the mass-energy of a photon scattered (or absorbed) by that black hole, and of (2) the black-hole size (in mass units) with the wavelength ($\times c^2/G$ = mass units) of the same photon scattered (or absorbed) by the black hole. For a fixed black-hole mass the "recoil limit" curve gives that frequency of a scattered photon beyond which the black-hole curve would suffer an excessive recoil; in other words, the wave field of that single photon no longer constitutes a small perturbation on the black-hole geometry. For a fixed black-hole mass the "diffraction limit" curve gives that frequency of a scattered photon below which the scattered radiation has such long wavelength that it is incapable of furnishing information about the geometrical features of the black hole. It is evident therefore that a black hole is detectable only by scattering off it of radiation of frequencies between the two indicated limits. If one uses frequencies outside these limits a black hole is undetectable. An entity about whose existence one can not learn anything by scattering is not an entity; not being endowed with any distinguishing features, it is not anything in particular. In other words, classical black holes whose mass is smaller than the Wheeler mass do not exist. The solid curved arrow depicts the evolution of an incipient black hole. As discussed in the text, it emits radiation which at any instant of time has a spectrum that is peaked at the frequency $\omega \sim (8\pi M)^{-1}$, which is slightly below the diffraction limit. Evidently the evolution is towards some lowest-mass-energy state near the Wheeler mass.

other words, first-order perturbation analysis is inapplicable. The limits on the detectability of black holes by scattered radiation is elaborated upon in Fig. 4.

What is the path of an evolving incipient black hole along the mass-frequency diagram, Fig. 4? Recall that according to Eq. (9.10) the power spectrum of emitted photons is

$$\frac{(\text{number of photons}\times\hbar\omega)}{(\text{unit time})(\text{unit frequency})} = \frac{27(M\omega)^2}{2\pi}\,\frac{\hbar\omega}{e^{8\pi M\omega}-1} \quad (14.4)$$

The energy of the photons emitted most frequently is

$$\hbar\omega \approx \frac{\hbar c}{8\pi M}. \quad (14.5)$$

It is evident that the incipient black hole radiates blackbody radiation predominantly at a wavelength comparable to the dimension of the black hole. It follows that a black hole radiates its irreducible mass in such a way that most of the radiation emerges from the black hole at the "diffraction limit."

Implicit in the whole analysis of this article is that one can make a sharp distinction between the wave fields and the arena, the background geometry, within which they propagate. This is an excellent approximation provided the emitted blackbody photons are of sufficiently small energy in comparison to the instantaneous mass of the black hole itself. It follows from Eq. (13.5) that this happens as long as

$$\frac{\hbar c}{8\pi M} \ll M_{\text{conv}}c^2 .$$

This inequality guarantees that the black hole is going to suffer a negligible recoil upon the emission of blackbody radiation. In view of this inequality the analysis of the interplay of vacuum fluctuations and geometry becomes in applicable for black-hole masses of the order

$$M_{\text{conv}} \approx \frac{1}{(8\pi)^{1/2}}\left(\frac{\hbar c}{G}\right)^{1/2}, \quad (14.6)$$

i.e., when the mass is of the order of or less than the Wheeler mass. As a matter of fact, as the caption of Fig. 4 indicates, classical black holes below that mass are nonexistent; if a classical incipient black hole is to have a state of lowest energy, it cannot have a mass less than that given by Eq. (14.6).

The updated version of the issue of the final state of stellar gravitational collapse poses therefore the following central question: Given an evolving incipient black hole. Put it into a surrounding which contains no radiation which that black hole can absorb. Let it radiate its mass energy. It will do so at an ever-increasing rate for only a finite total time. Question: *What are the features of the entity that remains after the runaway evolution has run its course?*

Presently, very little is known about the answer to this question. With the recent astronomical discovery of black holes it is more difficult than ever to avoid facing this version of the issue of the final state.

XV. SUMMARY

Those results of this article that should be classified as new are:

(1) a reformulation of the radiation mechanism from an incipient black hole in terms of zero-point fluctuations,

(2) tracing the cause of the emitted blackbody radiation to the amplification of the uncorrelated vacuum fluctuations by the collapsing star,

(3) characterizing the amplification mechanism in terms of a simultaneous resonance time,

(4) determining the sequential order in which the standing-wave modes inside the star cause the emission of statistically identical blackbody radiation packets,

(5) relationship between black-hole entropy and the logarithm of the total number of relevant standing-wave modes inside the star,

(6) the statistical fluctuations in the emitted radiation are those of thermal radiation,[27] and

(7) the conclusion that the emission of blackbody radiation prevents a star from collapsing through its $r=2M$ surface.[39]

APPENDIX A: FOURIER SPECTRUM OF STANDING-WAVE MODE AT THE SURFACE OF A COLLAPSING SHELL STAR

This appendix exhibits the steps leading to the Fourier amplitude, Eq. (4.7), of the radiation emitted from the surface of a collapsing-shell star. The basis functions are the outgoing traveling modes of the separated wave equation $\Box\psi=0$ in Schwarzschild geometry, Eq. (3.1). The equation for that factor of ψ which depends on the radial coordinate only,

$$\frac{1}{r^2}\frac{d}{dr}(r^2-2Mr)\frac{dR}{dr}+\left[\frac{\omega^2}{1-2M/r}-\frac{l(l+1)}{r^2}\right]R=0,$$

becomes upon introducing (a) the new dependent variable $Q=rR(r)$, and (b) the tortoise coordinate $r^*=r+2M\ln(r/2M-1)$ as the new independent variable, simply

$$\frac{d^2Q}{dr^{*2}}+\left\{\omega^2-\left[\frac{l(l+1)}{r^2}+\frac{2M}{r^3}\right]\left(1-\frac{2M}{r}\right)\right\} Q=0\,.$$

It is therefore evident that the traveling-wave modes are correctly expressed by Eq. (4.12) in regions near the star's surface during late collapse, near a distant observer (large r), and also in between these two in the approximation which neglect the effective potential.

The function to be Fourier-analyzed is the wave field of a standing-wave mode, Eq. (2.1) evaluated along the history of the surface of the star near $r=2M$ (see Fig. 1.) The Fourier transform of this function, ψ_ω, is

$$\text{“transform”}=\left(\frac{2\hbar}{a\omega_n}\right)^{1/2}\frac{1}{(2\pi)^{1/2}}\int_{u=u_0}^{u=\infty}\sin[\omega_n t_{\rm in}(u)+\beta_{nlm}]\,\sin\omega_n\left[r(u)-\frac{l\pi}{2}\right]\bigg|_{\text{star's surface}} e^{i\omega u}du\,. \tag{A1}$$

Here u is the retarded Finkelstein time coordinate,

$$u=t-r^*=t-r-2M\ln\left(\frac{r}{2M}-1\right)\,, \tag{A2}$$

which characterized not only the histories of outgoing (expanding) wave fronts, but also parametrizes the history of the collapsing-shell star surface. The Fourier analysis starts at $u=u_0$, which in our analysis is taken to correspond to very late retarded time. This history of the stellar surface is characterized by

$$t_{\rm in}+r=r_0\,, \tag{A3a}$$

$$t+r^*=v_0\,. \tag{A3b}$$

Thus on the star's surface

$$\begin{aligned}u&=v_0-2r^*\\&=v_0-2r-4M\ln\left(\frac{r}{2M}-1\right),\end{aligned} \tag{A4}$$

and very large u characterizes therefore the very late stage of collapse. The field of standing wave inside the star is the superposition of an expanding and a contracting wave front:

$$\begin{aligned}\text{“transform”}=\frac{1}{(2\pi)^{1/2}}\int_{u_0}^{\infty}\left(\frac{2\hbar}{a\omega_n}\right)^{1/2}\left(\frac{-1}{4}\right)&[e^{i\omega_n(t_{\rm in}+r)+i\beta_{nlm}-il\pi/2}+\text{c.c.}\\&-e^{i\omega_n(t_{\rm in}-r)+i\beta_{nlm}+il\pi/2}-\text{c.c.}]|_{t_{\rm in}+r=r_0}e^{i\omega u}du\,.\end{aligned} \tag{A5}$$

Only the expanding wave will contribute to radiation emitted from the surface and traveling to a distant observer. Consequently, the first two terms will be omitted from further discussion. The evaluation of the integral is facilitated by using the radial coordinate r as an integration parameter. As a matter of fact, introduce the dimensionless variable

$$p=\frac{r}{2M}-1\,.$$

Rewrite both the retarded time u on the star's surface [Eq. (A4)] and r in terms of p,

$$u=v_0-4M-4Mp-4M\ln p\,,\quad r=2M+2Mp\,.$$

The integral, Eq. (A5), becomes

$$\begin{aligned}\text{“transform”}=&\frac{1}{(2\pi)^{1/2}}\left(\frac{2\hbar}{a\omega_n}\right)^{1/2}Me^{i\omega(v_0-4M)}\\&\times\int_0^{p_0=r_0/2M-1}(e^{i\beta_{nlm}-4iM(\omega_n+\omega p+\omega\ln p)}+e^{-i\beta_{nlm}-4iM(-\omega_n p+\omega p+\omega\ln p)})(1+p)\frac{dp}{p}\,.\end{aligned} \tag{A6}$$

Here the additive phase constant $\frac{1}{2}l\pi-4\omega_n M+\omega_n r_0$ has been absorbed into the random phase β_{nlm} of the field oscillator Φ_{nlm}.

First, determine the qualitative features of the transform. Later evaluate it analytically. The integration is performed over the whole history of the surface of the collapsing star. This collapse history is divided naturally into three stages:

early stage: $1 \ll p = \frac{r}{2M} - 1$; (A7a)

intermediate stage: $p = \frac{r}{2M} - 1 \sim 1$; (A7b)

late stage: $p = \frac{r}{2M} - 1 \ll 1$. (A7c)

Here we shall discuss only the early and the late stages of collapse. The predominant contributions to the integral come from those ranges of p over which the two exponentials have constant phase.

Early stage. The upper integration limit in Eq. (A6) extends up to $r_0 \gg 2M$. The contributions from the early stages come from $r \gg 2M$, i.e., when

$$\ln p \ll p. \tag{A8}$$

Thus, dropping the logarithm from the exponents, discover that

$$\text{“transform”} \sim e^{i\beta_{nlm}}\delta(\omega_n + \omega) + e^{-i\beta_{nlm}}\delta(\omega_n - \omega). \tag{A9}$$

This result implies that the converging ingoing part of the field oscillator Φ_{nlm} is converted into an outgoing (scattered) travelling wave emerging from the star. The number of field oscillator Φ_{nlm} is unlimited. Hence the present formulation leads to the well-known difficulty that flat space is permeated by zero-point fluctuations whose total energy

$$\sum_{\substack{\text{all field oscillators} \\ \text{characterized by } n,l,m}} \tfrac{1}{2}\hbar\omega_n \tag{A10}$$

is not finite. This difficulty is circumvented by the theory of renormalization.[40] This circumvention embodies the fact that field oscillators above some cutoff frequency do not have a basis in any known facts of reality and that usually only deformations of the (formally infinite) zero-point energy embody observable effects. Thus we conclude that during the early stages of collapse the radiation coming from a collapsing star is due to the nonadiabatic deformation of the zero-point energy of space. This radiation is, however, not what is calculated here. Neither do we determine the radiation emitted during the intermediate stage of collapse. Instead focus on the following:

Late stage. To determine the contribution to the transform from the late stages of collapse set the upper integration limit to

$$p_0 = \frac{r_0}{2M} - 1. \tag{A11}$$

This eliminates in the integral all contributions from the early and late stages. In this limit during the whole integration interval p satisfies

$$p \ll 1 \ll |\ln p|. \tag{A12}$$

Dropping therefore the appropriate linear terms results in a simplified integral:

$$\text{“transform”} = \frac{1}{(2\pi)^{1/2}}\left(\frac{2\hbar}{a\omega_n}\right)^{1/2} M e^{i\omega(v_0 - 4M)}\left(e^{i\beta_{nlm}}\int_0^{p_0} e^{-4iM(\omega_n p + \omega \ln p)}\frac{dp}{p} + e^{-i\beta_{nlm}}\int_0^{p_0} e^{4iM(\omega_n p - \omega \ln p)}\frac{dp}{p}\right). \tag{A13}$$

It is evident therefore that during the late stages of collapse the phases are stationary and the transform therefore differs appreciably from zero only for those frequencies ω which satisfy

$$|\omega| \ll \omega_n. \tag{A14}$$

This is also eminently reasonable from the following direct physical argument:

Consider an outgoing wave packet inside the star,

$$\Phi^{\omega_n}{}_{lm} \propto \int_{-\infty}^{+\infty} e^{i(t_{\text{in}} - r)\omega'_n} f(\omega'_n)\, d\omega'_n,$$

characterized by the mean frequency ω_n [i.e., $f(\omega'_n)$ is nonzero in an interval centered around ω_n]; thus

$$\Phi^{\omega_n}{}_{lm} \sim e^{i(t_{\text{in}} - r)\omega_n}. \tag{A15}$$

This wave packet travels towards the surface of the collapsing star and there gives rise to a corresponding wave packet

$$\psi^{\omega}{}_{lm} \sim e^{i(t - r^*)\omega}, \tag{A16}$$

with a mean frequency ω. This wave packet will travel in the exterior vacuum geometry to some distant observer.

To verify Eq. (A14) by a direct physical argument it is necessary to determine the relationship between frequencies that characterize the two wave packets Eqs. (A15) and (A16). First, recall that the phase difference $\Delta\phi$ between the histories of two successive wave crests of a packet is a continuous function across the shell star's interface

$$\Delta\phi_{\text{in}} = \Delta\phi_{\text{out}}. \tag{A17}$$

Second, recall that the phase difference inside the shell star is

$$\Delta\phi_{\text{in}} = \Delta(t_{\text{in}} - r)\,\omega_n \Big|_{t_{\text{in}} + r = r_0} = -2\omega_n \Delta r. \tag{A18}$$

Outside this difference is

$$\Delta\phi_{\text{out}} = \Delta(t - r^*)\,\omega \Big|_{t + r^* = v_0} = -2\omega\Delta r^* = -2\frac{\Delta r}{1 - 2M/r}\,\omega. \tag{A19}$$

Third, recall that the coordinate r is continuous on spacetime. The above three equations (A17)–(A19) imply therefore that

$$\omega = \left(1 - \frac{2M}{r}\right)\omega_n. \tag{A20}$$

This is the relationship between the frequencies of the wave packet inside and outside the star at the instance when its surface has the radius r. During the late stages of collapse $1 - 2M/r \ll 1$. Thus

$$\omega \ll \omega_n.$$

This verifies Eq. (A14) for positive frequencies ω and also completes the discussion of the qualitative behavior of the integrand of the "transform", Eq. (A16) and Eq. (A13). Additional significant features of the "transform" are discussed in Sec. VII.

The context of the phenomenon under examination (star during late collapse) demands that we evaluate, Eq. (A13), the transform whose upper integration limit is

$$p_0 = \frac{r_0}{2M} - 1 \ll 1. \tag{A21}$$

This Fourier transform becomes a Mellin transform by introducing the new integration variables

$$z = 4\omega_n M p. \tag{A22}$$

The expression for the transform becomes

$$\text{"transform"} = \frac{1}{(2\pi)^{1/2}}\left(\frac{2\hbar}{a\omega_n}\right)^{1/2} M e^{i\omega(v_0 - 4M + 4M\ln 4\omega_n M)} \times \left(e^{i\beta_{nlm}}\int_0^{z_0 = 4M\omega_n p_0} e^{-iz} z^{-4i\omega M - 1} dz + e^{-i\beta_{nlm}}\int_0^{z_0 = 4M\omega_n p_0} e^{iz} z^{-4i\omega M - 1} dz\right). \tag{A23}$$

To evaluate these integrals as Mellin transforms[41] of $e^{\mp iz}$, two conditions are necessary: (1) The integrals must converge, and (2) the error in the integrals must not be too large if one replaces the upper integration limit by $z_0 = +\infty$.

The first condition is satisfied if the integrands have suitable convergence properties at $z = 0$, namely $z^{-4i\omega M} \to 0$ as $z \to 0$. That this actually happens follows from the fact that the "transform", Eq. (A1), is to give a Fourier representation of a causal function, i.e., one which is nonzero only for $u_0 < u < \infty$. Consequently, ω is always understood to have a "vanishingly" small positive imaginary part:

$$\omega \to \omega + i\epsilon, \quad 0 < \epsilon \ll 1.$$

The embodiment of causality in this form guarantees that the integrand has the necessary convergence properties.

The second condition requires, as is evident from Eq. (A22), that ω_n, the frequency, be so large that even though Eq. (A21) holds (i.e., we focus attention on late stages of collapse), nevertheless

$$1 \ll \left(\frac{r_0}{2M} - 1\right) M\omega_n. \tag{A24}$$

In view of the previous discussion about outgoing wave packets the physical meaning of the second condition is that inside the star only those standing waves may be considered which have a high enough frequency ω_n. So high, in fact, must their frequency ω_n be that they give rise to wave packets [Eq. (A16)] at $r = r_0$ which are characterized by Schwarzschild frequency $\omega \gg 1/M$.

Return now to the evaluation of the transform, Eq. (A23). Replace the upper limit of integration by $z_0 = +\infty$ and obtain a Mellin transform, which is

$$\text{"transform"} = \frac{1}{(2\pi)^{1/2}}\left(\frac{2\hbar}{a\omega_n}\right)^{1/2} M e^{i\omega(v_0 - 4M + 4M\ln 4\omega_n M)}\Gamma(-4i\omega M)\left[e^{i\beta_{nlm}} e^{-2\pi\omega M}(1 + \text{error}) + e^{-i\beta_{nlm}} e^{+2\pi\omega M}(1 + \text{error})\right]. \tag{A25}$$

Here Γ is the gamma function. The error, which is obtained by a more detailed examination using confluent hypergeometric functions, is

$$|\text{error}| \sim \frac{\omega}{(r_0/2M - 1)\omega_n}. \tag{A26}$$

Omitting the error terms, obtain the square of this transform by using the identity

$$|\Gamma(4i\omega M)|^2 = \frac{\pi}{4\omega M \sinh 4\pi\omega M};$$

$$|\text{"transform"}|^2 = \frac{2\hbar}{a\omega_n}\frac{M^2}{2\pi}\left(\frac{\pi}{4\omega M \sinh 4\pi\omega M}\right) \times (2\cosh 4\pi\omega M + 2\cos 2\beta_{nlm}) \quad \text{(A27)}$$

$$= \frac{2\hbar}{a\omega_n}\frac{M^2}{2\pi}\frac{\pi}{\omega M} \times \left(\frac{1}{2} + \frac{1}{e^{8\pi\omega M}-1} + \frac{\cos 2\beta_{nlm}}{2\sinh 4\pi\omega M}\right). \quad \text{(A28)}$$

Equations (A25) and (A27) are the expressions used in Eqs. (4.7) and (4.8) of this article.

APPENDIX B: HISTORY OF AN OUTGOING WAVE TRAIN

Consider a shell star during its late stages of collapse. Focus attention on a scattered outgoing wave train, which inside the star is represented by

$$\Phi_{nlm}^{\text{outgoing}} = \left(\frac{2\hbar}{a\omega_n}\right)^{1/2}\frac{1}{2r}\cos[\omega_n(t_{\text{in}} - r) + \beta_{nlm}]Y_l^m(\theta, \rho). \quad \text{(B1)}$$

Here t_{in} refers to the interior time of the star as specified by the flat metric, Eq. (3.2).

Inside the star the frequency ω_n is constant. What is the frequency at the shell star's surface as ultimately seen by a distant observer? To answer this question, evaluate the wave field at the surface in terms of the exterior Schwarzschild time. Note that the events on the star's surface are coordinatized by

$$t = v_0 - r^* = v_0 - r - 2M\ln\left(\frac{r}{2M} - 1\right) \quad \text{(B2)}$$

and

$$t_{\text{in}} = r_0 - r \quad \text{(B3)}$$

in terms of interior and exterior time, respectively. Both v_0 and r_0 characterize an ingoing null cone, the history of the collapsing shell star. Near the "incipient event horizon" ($r = 2M$), Eq. (B2) becomes

$$r = 2M + 2M\exp\left(\frac{v_0 - r}{2M}\right)e^{-t/2M} \approx 2M + 2M\exp\left(\frac{v_0}{2M} - 1\right)e^{-t/2M} \quad \text{(B4)}$$

and Eq. (B3) becomes

$$t_{\text{in}} \approx r_0 - 2M - 2M\exp\left(\frac{v_0}{2M} - 1\right)e^{-t/2M}. \quad \text{(B5)}$$

Consequently, the wave field at the surface of the collapsing shell star is

$$\Phi_{nlm}^{\text{outgoing}}\Big|_{\substack{\text{star's}\\\text{surface}}} = \left(\frac{2\hbar}{a\omega_n}\right)^{1/2}\frac{1}{2r}\cos(b_n e^{-t/2M} + \beta_{nlm}), \quad \text{(B6)}$$

where

$$b_n = 4M\omega_n \exp\left(\frac{v_0}{2M} - 1\right)$$

and an additive constant has been absorbed into the random phase β_{nlm}. Viewing the wave train as a succession of wave packets, assign a frequency to each one, and designate it as the instantaneous frequency of $\Phi_{nlm}^{\text{outgoing}}$ at the star's surface as seen by a distant observer. Evidently this instantaneous frequency is

$$\omega(t) = \frac{1}{2}\frac{d}{dt}b_n e^{-t/2M} = \omega_n \exp\left(\frac{v_0}{2M} - 1\right)e^{-t/2M}. \quad \text{(B7)}$$

This frequency is a function decreasing monotonically with time. Consequently, after a certain time the frequency will be low enough for the potential barrier, Eq. (4.2), to prevent any further transmission of wave packets. This happens when

$$\frac{l(l+1)}{27M^2} = \omega^2(t), \quad \text{(B8)}$$

i.e., when

$$t = (v_0 - 2M) + 2M\ln(4M\omega_n) - M\ln\frac{16l(l+1)}{27} = t_{n\,\text{off}} - M\ln\frac{16l(l+1)}{27}. \quad \text{(B9)}$$

The time $t_{n\,\text{off}}$ is the effective switch-off time for $\Phi_{nlm}^{\text{outgoing}}$. After the time given by Eq. (B9), wave packets of angular quantum number l and mean frequency given by Eq. (B7) have too long a wavelength to surmount the centrifugal barrier. They do not reach a distant observer.

APPENDIX C: RATE AT WHICH SCATTERED OUTGOING MODES ARE AMPLIFIED AND EXTINGUISHED

Each scattered outgoing wave, Eq. (B1), inside the star undergoes a red shift in frequency as it crosses the star's surface. Outgoing modes of successively higher frequency ω_n are red-shifted out of a distant observer's view at successively later times as given by Eq. (B9). This appendix determines the rate at which this happens. Let Δn be the number of scattered outgoing traveling modes of fixed angular quantum number and having frequencies between $\omega_n - \frac{1}{2}\Delta\omega_n$ and $\omega_n + \frac{1}{2}\Delta\omega_n$.

According to Eq. (2.3)

$$\Delta n = \frac{a}{\pi}\Delta\omega_n. \tag{C1}$$

According to Eq. (B9) the time interval necessary to red-shift these modes out of the view of a distant observer is

$$\Delta t = \frac{2M}{\omega_n}\Delta\omega_n. \tag{C2}$$

Consequently, the rate at which scattered outgoing waves of fixed quantum number l and of frequency ω_n are red-shifted out of the view of a distant observer is

$$\frac{(\text{switched-off modes})}{(\text{unit time})} = \frac{dn}{dt} = \frac{a\omega_n}{2\pi}\frac{1}{M}. \tag{C3}$$

Instead of referring to Eq. (B6), consider Eq. (6.22b), the time at which a wave mode gets amplified. Obtain there by a similar expression

$$\frac{(\text{amplified modes})}{(\text{unit time})} = \frac{dn}{dt} = \frac{a\omega_n}{2\pi}\frac{1}{M}. \tag{C4}$$

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*Expansion of a synopsis given at the Seventh Texas Symposium on Relativistic Astrophysics, Dallas, 1974. Summary in the proceedings of the First International Marcell Grossmann Meeting, edited by T. Damour and R. Ruffini (North-Holland, to be published).

[1]R. Price, Phys. Rev. D 5, 2419 (1972).

[2]There is, however, evidence that a black hole may be characterized by an additional fourth desciptor, scalar charge; J. D. Bekenstein, Ann. Phys. (N.Y.) 91, 75 (1975).

[3]This is Price's theorem as stated by K. S. Thorne, in *Magic Without Magic: John Archibald Wheeler*, edited by John R. Klauder (Freeman, San Francisco, 1972).

[4]S. Hawking, Nature 248, 30 (1974).

[5]S. W. Hawking, Commun. Math. Phys. 43, 199 (1975).

[6]For directly observable manifestation of vacuum fluctuations see A. Einstein and O. Stern, Ann. Phys. (Leipz.) 60, 551 (1913); T. A. Welton, Phys. Rev. 74, 1157 (1948); C. P. Enz and A. Thellung, Helv. Phys. Acta 33, 839 (1960); see especially Refs. 31 and also 32.

[7]R. Isaacson, Phys. Rev. 166, 1263 (1969); see also Chap. 22 in Ref. 21.

[8]T. H. Boyer, Phys. Rev. D 11, 793 (1975); see also Phys. Rev. 182, 1374 (1969).

[9]C. W. Misner, in *Astrophysics and General Relativity, 1968 Brandeis Summer Institute in Theoretical Physics*, edited by M. Chrêtien, S. Deser, and J. Goldstein (Gordon and Breach, New York, 1969).

[10]C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).

[11]T. H. Boyer, Phys. Rev. 174, 1764 (1968), discusses zero-point oscillations between spherically symmetric conducting shells.

[12]E. A. Power and J. A. Wheeler, Rev. Mod. Phys. 29, 480 (1957), reprinted in J. A. Wheeler's *Geometrodynamics* (Academic, New York, 1962).

[13]G. Baym, *Lectures on Quantum Mechanics* (Benjamin, Reading, Mass., 1974), 3rd edition.

[14]Black-hole entropy was first introduced by J. D. Bekenstein, Phys. Rev. D 7, 2333 (1973). He computed it to be approximately $96k\,(M/L_W^2)$.

[15]As first pointed out by S. Hawking [Commun. Math. Phys. 43, 199 (1975)], the exact value of the black-hole entropy follows from the second law of thermodynamics, $dU = TdS$, applied to a black hole, $dM_{\text{conv}} = c^{-2}TdS$; here T is given by Eq. (1.2). Obtain the entropy S by integrating the differential dS.

[16]F. Reif, *Fundamentals of Statistical and Thermal Physics* (McGraw-Hill, New York, 1965), p. 63.

[17]C. Kittel, *Thermal Physics* (Wiley, New York, 1969), Chap. 4.

[18]J. D. Bekenstein, thesis, Princeton University, 1972 (unpublished), p. 137.

[19]J. D. Bekenstein, Phys. Rev. D 9, 3292 (1974).

[20]J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (Wiley, New York, 1952).

[21]U. Gerlach, Bull. Am. Phys. Soc. 21, 614 (1976).

[22]See, for example, J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962), Chap. 14.

[23]R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965), Chap. 9, Sec. 3.

[24]J. A. Wheeler, *Geometrodynamics* (Academic, New York, 1962), see the chapter on neutrinos, gravitation, and geometry.

[25]A. Einstein, Phys. Z. 10, 185 (1909).

[26]L. V. de Broglie, quoted in E. Whittaker, *A History of the Theories of Aether and Electricity* (Harper, New York, 1960), Vol. II, Chap. III, pp. 102–103.

[27]R. M. Wald [Commun. Math. Phys. 43, 9 (1975)] has obtained the stronger result that the density matrix for particle emission is that of a blackbody. One can deduce the fluctuation spectrum from his density matrix.

[28]S. Hawking, in *Black Holes*, 1972 Les Houches Lectures, edited by B. S. DeWitt (Gordon and Breach, New York, 1973).

[29]See for example Ref. 10, Chap. 22.

[30]A. Sakharov, Dokl. Acad. Nauk. S. S. S. R. 117, 70 (1967) [Sov. Phys.—Dokl. 12, 1040 (1968)]; also referred to and summarized in Ref. 10, Box 17.2, part 6. C. W. Misner, K. S. Thorne, and J. A. Wheeler (Ref. 10), Box 17.2, part 6.

[31]H. B. G. Casimir, K. Ned. Akad. Wet. Versl. Gewone Vergad. Afd. Natuurkd. 51, 793 (1948).

[32]M. Fierz, Helv. Phys. Acta 33, 855 (1960).

[33]L. H. Ford, Ph.D. thesis, Princeton University, 1974 (unpublished).

[34]D. G. Boulware, Phys. Rev. D 13, 2169 (1976).

[35]B. S. DeWitt, Phys. Rep. 19C, 297 (1975).

[36]As a matter of fact, B. DeWitt (Ref. 35) has actually exhibited and discussed the stress-energy tensor associated with the Casimir effect.

[37]For a discussion and references, see R. W. Lindquist, R. A. Schwartz, and C. W. Misner, Phys. Rev. 137,

B1364 (1965).

[38]C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Ref. 10), Chap. 44.

[39]D. G. Boulware, Ref. 34, has independently arrived at the same conclusion.

[40]The chief issues for our context are mentioned by J. D. Bjorken and Sidney D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965), Chap. 12.4; H. Feshbach, in *Electromagnetic Theory: Proceedings, Vol. 2,* Symposia in Applied Mathematics, Cambridge, Mass., 1948, edited by A. H. Taub (American Mathematical Society, Providence, 1950); see also T. A. Welton, mentioned in Ref. 6.

[41]See, for example, P. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), Part I, p. 485.