Statistical bootstrap duality*

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Direct-channel resonances are summed to build the two-body inelastic scattering amplitudes. The distribution of direct-channel resonances over energy and angular momentum is given by the Chiu and Heimann statistical bootstrap equation. By comparing with experimental data a phenomenological test of the Chiu-Heimann results is made which discriminates between their various solutions and which determines the parameters of their distributions. A good description of 27 differential cross sections corresponding to 23 differential inelastic reactions is obtained.

I. INTRODUCTION

Hagedorn,[1] utilizing the statistical bootstrap concept, was the first to calculate the density of hadronic states as a function of energy. He found that the number of states essentially increases exponentially with energy. Phenomenological tests[2] have revealed the experimental validity of the exponential energy behavior of the hadronic density of states. Chiu and Heimann[3] have extended Hagedorn's results by calculating the density of states both as a function of energy and angular momentum. We hope to exploit their results by describing the two-body hadronic inelastic scattering amplitude as a sum of direct-channel resonances whose distribution is given by the Chiu-Heimann density of states. Through this application of duality we hope to provide a phenomenological test of the Chiu-Heimann result, to discriminate among the three sets of solutions they found, to determine the parameters of these solutions, and, in addition, to describe the general two-body inelastic scattering reactions.

II. KINEMATICS

The differential cross section is expressed by the spin-flip and the spin-non-flip amplitudes, g and f, as

$$\frac{d\sigma}{d\Omega} = |f|^2 + |g|^2. \tag{1}$$

The differential cross sections $d\sigma/d\Omega$ and $d\sigma/dt$ have the relation

$$\frac{d\sigma}{d\Omega} = \frac{\lambda(s, m_1^2, m_2^2)\lambda(s, m_3^2, m_4^2)}{4\pi s}\,\frac{d\sigma}{dt}, \tag{2}$$

where

$$\lambda(s, m_1^2, m_2^2) = (s^2 + m_1^4 + m_2^4 - 2sm_1^2 - 2sm_2^2 - 2m_1^2 m_2^2)^{1/2}$$

and m_i is a mass of the ith particle.

The spin-flip and the spin-non-flip amplitudes are related to the conventional f_1 and f_2 amplitudes by

$$f = f_1 + f_2\cos\theta,\qquad g = f_2\sin\theta, \tag{3}$$

where θ is the scattering angle.

Finally f_1 and f_2 can be expanded by the parity-conserving amplitudes f_{l+} and f_{l-},

$$f_1 = \sum_l (f_{l+}P'_{l+1} - f_{l-}P'_{l-1}), \tag{4}$$

$$f_2 = \sum_l (f_{l-} - f_{l+})P'_l,$$

where P'_l is a first derivative of the Legendre function of lth order,

$$P'_l = \frac{d}{dZ}P_l(z). \tag{5}$$

III. METHOD AND DISCUSSION

We assume that two-body inelastic hadronic scattering can be built in the manner of duality by summing all of the direct-channel resonances. We assume that the Chiu-Heimann distribution $\rho(E,l)$ describes the density of direct-channel resonances, and that summing over all the states is equivalent to integrating $\rho(E,l)$ over energy, E, and angular momentum, l.

We assume that each resonance is described by a Breit-Wigner form so that the spinless scattering amplitude for $AB \to CD$ is given by

$$f(E,\cos\theta) = -\frac{1}{2k}\int dE' \int dl(2l+1)\rho(E',l) \times \frac{\gamma_{AB}\gamma_{CD}}{E - E' + \frac{1}{2}i\Gamma_T}P_l(\cos\theta), \tag{6}$$

where γ_{AB}, γ_{CD}, and Γ_T are functions of E and l and represent the partial and total widths of the resonances. We then assume that the narrow-width approximation is valid so that the Breit-Wigner denominator becomes a δ function in energy. The scattering amplitude after integrating over energy then reduces to

$$f(E,\cos\theta) = -\frac{i\pi\gamma_{AB}\gamma_{CD}}{2k}\int dl(2l+1)\rho(E,l) \times P_l(\cos\theta), \qquad (7)$$

where γ_{AB} and γ_{CD} are functions of energy only. In the narrow-width approximation we are using we ignore the real part of the amplitude.

In summing the direct-channel resonances we have assumed that the resonances add coherently. In addition to the coherent sum there is also a contribution to the cross section from the incoherent contribution of the resonances. This contribution depends on the partial widths and the density of states such that the incoherent contribution falls off exponentially with energy; we have ignored it and restricted ourselves to high-energy data.[4]

We shall attempt to describe all two-body inelastic reactions for which there exist high-energy differential cross-section data. Because we have made the narrow-width approximation, our amplitudes are purely imaginary and hence we cannot describe the polarization data. Most of the two-body inelastic differential cross sections that have been measured are either initiated by pions or antikaons on nucleons. There exist some data, for proton-antiproton initiated reactions as well. All of the πN inelastic scattering amplitudes will be built from the same density of states, i.e., the πN, ηN, $K\Sigma$, $K\Lambda$, etc. resonances. All of the KN inelastic scattering amplitudes will be built from the same KN density of states which in principle might be different from the πN density of states.

Both the πN and KN reactions are described by two partial-wave amplitudes f_{l^+} and f_{l^-} corresponding to the even- and odd-parity states. The density of states for the parity-even and parity-odd states will be essentially the same aside from a scale factor since the bootstrap equation for the two sets of states is identical. The coupling of the initial and final states to the individual parity-even and parity-odd resonances may vary from reaction to reaction. However, we shall assume for the sake of simplicity that for a given reaction the ratio of the coupling to the parity-even states and parity-odd states is the same for all the resonances, so that for a given reaction f_{l^+} and f_{l^-} in Eq. (4) are proportional, viz., $f_{l^-} = \tan\alpha f_{l^+}$.[2] The ratio $\tan\alpha$ may vary from reaction to reaction and is varied in order to fit the two-body inelastic differential cross sections. When considering nucleon-antinucleon reactions we will assume that the five spin amplitudes needed to describe these reactions can be approximated by f_{l^+} and f_{l^-}.

Chiu and Heimann[3] were unable to find a unique solution to their bootstrap equation. Instead they found that the following three forms satisfied their equation:

$$\text{I} \quad \rho(E,l_z) = \frac{\rho(E)}{(\pi D E)^{1/2}}\exp\left(\frac{-l_z^2}{DE^\gamma}\right), \quad \gamma \le 1, \quad 0<D<B \qquad (8)$$

$$\text{II} \quad \rho(E,l_z) = \frac{\rho(E)}{2DE}\left[\cosh\left(\frac{\pi l_z}{2DE}\right)\right]^{-1}, \quad 0<D<B \qquad (9)$$

$$\text{III} \quad \rho(E,l_z) = \frac{\rho(E)}{2DE}\left[\cosh\left(\frac{\pi l_z}{2DE}\right)\right]^{-2}, \quad 0<D<B \qquad (10)$$

where $B \approx 7$ GeV$^{-\gamma}$ in (8) and $B \approx 7$ GeV^{-1} in (9) and (10) and $\rho(E) = (\rho_0/E^3)\exp(EB)$ is the density of states Frautschi[5,6] obtains by refining Hagedorn's equations. The density of states, $\rho(E,l)$, is related to $\rho(E,l_z)$ by

$$\rho(E,l) \approx -\frac{d}{dl_z}\rho(E,l_z)\Big|_{l_z=l}. \qquad (11)$$

Of the three distributions the first, Eq. (8), can be eliminated on the basis of energy considerations. Experimental evidence indicates that the diffraction peaks of the inelastic reactions tend to shrink logarithmically. The first distribution gives rise to a diffraction peak which expands linearly with energy for $\gamma = 1$ and more than linearly with energy for $\gamma < 1$. If one substitutes Eq. (8) into Eq. (7), defines the impact parameter $b = l/k \approx l/E$, and approximates $P_l(\cos\theta)$ by $J_0(b\sqrt{-t})$ then Eq. (7) becomes

$$f \approx \frac{ik\rho(E)}{(\pi DE)^{1/2}}E^2\int db^2\exp\left(\frac{-b^2E^{2-\gamma}}{D}\right)J_0(b\sqrt{-t}) \approx \frac{ik\rho(E)}{(\pi DE)^{1/2}}\exp\left(\frac{Dt}{4E^{2-\gamma}}\right). \qquad (12)$$

The other two solutions are proportional to $[\cosh(\pi/2D)b]^{-1}$ and $[\cos(\pi/2D)b]^{-2}$ and hence give rise to diffraction peaks that do not expand with energy. Unfortunately, these solutions do not give rise to diffraction peaks which shrink logarithmically. In order to reproduce the correct energy dependence expected on the basis of Regge theory, one would want to find a solution to the bootstrap equation which behaved like

$$\rho(E,l_z) = \frac{\rho(E)}{2DE}\left[\cosh\left(-\frac{\pi l_z}{2E(D+D'\ln E)}\right)\right]^{-1}. \qquad (13)$$

There is no reason to believe that Eq. (13) would not satisfy the bootstrap equation, but this must be carefully examined first.[7] Although the second and third solutions of Chiu and Heimann do not provide the detailed logarithmic energy dependence required by experiment, these distributions do give a good approximation to the data and hence will be utilized. We found no qualitative and little quantitative difference between solutions II and III and chose to concentrate our effort on solution II. The limitation of the energy dependence (no shrinkage) of the Chiu and Heimann solutions restricts our aim in this work to a description of only the angular dependence of differential cross sections. These points can be summarized in Fig. 1 in which we display $\pi^- p \to \eta n$ data at three energies normalized at $t=0$. We also show the best fit of the three Chiu-Heimann solutions at these energies. As expected from Eq. (12), the Gaussian solution does not shrink, but broadens, providing a poor fit while the $\cosh^{-1}$ and $\cosh^{-2}$ solutions are very similar, energy-independent, and consistent with the data.

Because of our ignorance of the scale factor, ρ_0, and the partial widths $\gamma_{AB}(E)$, $\gamma_{CD}(E)$, $\tilde{\gamma}_{AB}(E)$, and $\tilde{\gamma}_{CD}(E)$ (γ and $\tilde{\gamma}$ refer to the parity-even and parity-odd couplings, respectively), we have two parameters corresponding to the scale of the parity-even and parity-odd amplitudes. These scale factors change from reaction to reaction. One of

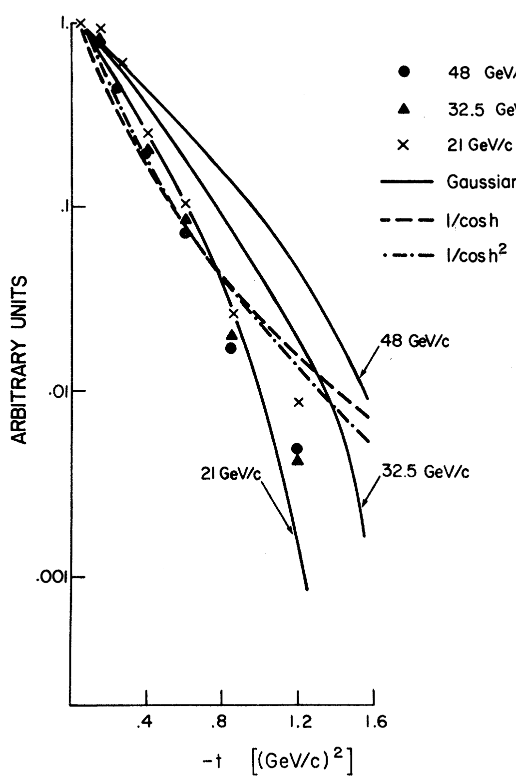

FIG. 1. Comparison of three Chiu-Heimann models with $\pi^- p \to \eta n$ differential cross-section data at 21, 30.5, and 48 GeV/c.

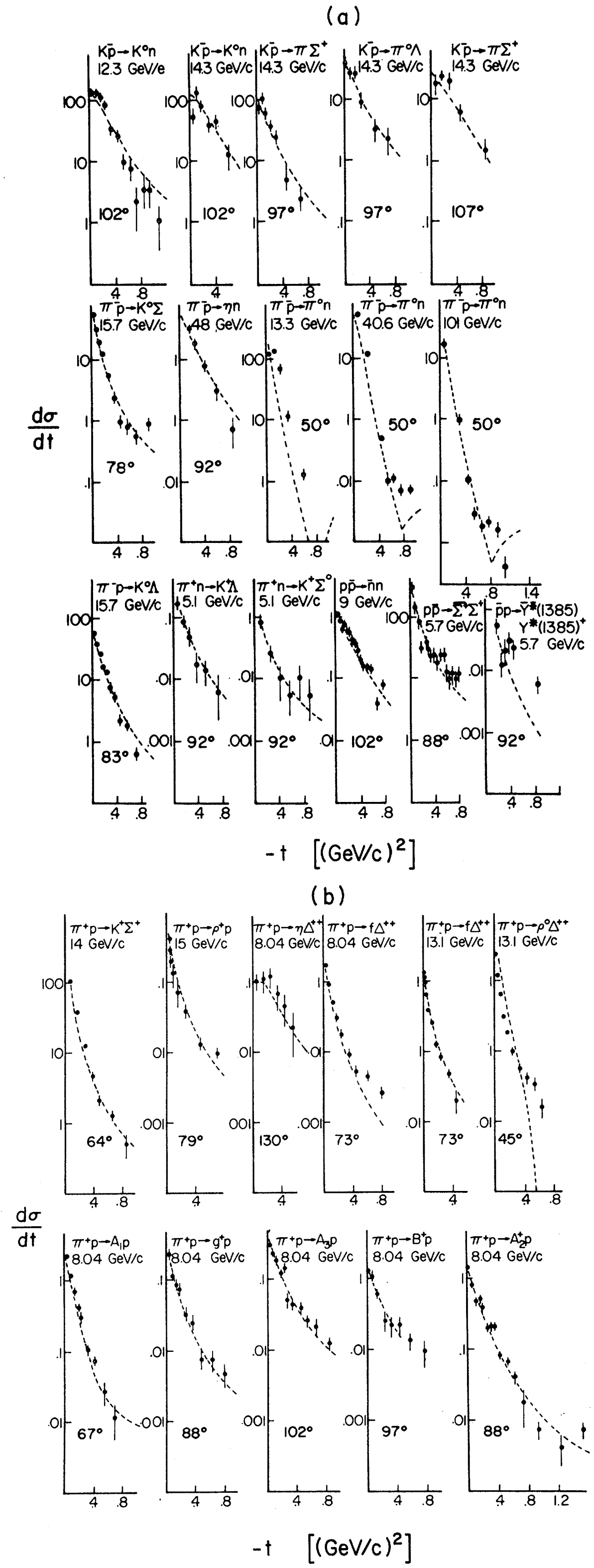

FIG. 2. Statistical-bootstrap-duality description of two-body inelastic cross sections compared with the data of Refs. 8–21. The ordinate units are arbitrary.

the scale factors, proportional to ρ_0 and the partial widths, is determined by demanding that the calculated cross section at $t=0$ agree with experiment. The other scale factor, related to $\tan\alpha$, is determined by a χ^2 fit in which the highest-energy data are weighted most heavily. $\tan\alpha$ is the one free parameter available per reaction. Hence for $\pi^- p \to \pi^0 n$, for example $\alpha = 50°$ for the three energies, the fit deteriorating at the lowest energy. The only other quantity to be determined is D which must be the same for all the πN reactions or the same for all $\bar{K}N$ reactions. The value of D is not necessarily the same for πN, $\bar{K}N$, and $\bar{N}N$ induced reactions. However, we found that we were able to describe the data best with the same value of D for all reactions, namely $D = 1.6$ GeV^{-1}. In Figs. 2(a) and 2(b) we display the 27 differential cross sections[8-21] describing the 23 reactions we obtained by fixing $D = 1.6$ GeV^{-1} and varying α from reaction to reaction. We feel that our results are exceedingly encouraging considering the scope of the reactions we describe and the crudeness of our model. Our least satisfactory fit is to $\pi^- p \to \pi^0 n$. We are confident that the density of states, $\rho(E, l_z)$, in Eq. (13) will improve this fit. The presence or absence of dip bump structure in our model depends on $\tan\alpha$, the ratio of parity-odd to parity-even couplings of the initial and final states to the direct-channel resonances. The width of the diffraction peak is determined to zeroth order by the constant, D. However, the ratio $\tan\alpha$ can effect this zeroth-order value of the diffraction peak. This explains why we are able to describe so many different diffraction peaks using only one value of D. It is interesting to note that for a majority of the reactions the parameter α falls within the narrow range 95° to 133° (see Table I). We have no explanation at this time for this apparent coincidence.

TABLE I. Table of two-body inelastic reactions.

Reaction	p_{lab} (GeV/c)	α (deg)	Reference
$K^- p \to K^0 n$	12.3	133	17
$K^- p \to K^0 n$	14.3	133	16
$Kp^- \to \pi^- \Sigma^+$	14.3	133	16
$K^- p \to \pi^0 \Lambda$	14.3	123	16
$K^- p \to \pi^- \Sigma^+(1385)$	14.3	114	16
$\pi^- p \to K^0 \Sigma^0$	15.7	95	13
$\pi^- p \to \eta n$	48	133	14
$\pi^- p \to \pi^0 n$	13.3	30	21
$\pi^- p \to \pi^0 n$	40.6	30	15
$\pi^- p \to \pi^0 n$	101	30	15
$\pi^- p \to K^0 \Lambda$	15.7	171	13
$\pi^+ n \to K^+ \Lambda$	5.1	114	18
$\pi^+ n \to K^+ \Sigma^0$	5.1	114	18
$p\bar{p} \to \bar{n}n$	9	114	17
$p\bar{p} \to \bar{\Sigma}^+ \Sigma^+$	5.7	95	19
$p\bar{p} \to \bar{Y}^*(1385)^- Y^*(1385)^+$	5.7	19	20
$\pi^+ p \to K^+ \Sigma^+$	14	9	8
$\pi^+ p \to \rho^+ p$	15	95	12
$\pi^+ p \to \eta \Delta^{++}$	8.04	133	9
$\pi^+ p \to g^+ p$	8.04	104	9
$\pi^+ p \to f \Delta^{++}$	8.04	88	9
$\pi^+ p \to f \Delta^{++}$	13.1	88	11
$\pi^+ p \to \rho^0 \Delta^{++}$	13.1	66	10
$\pi^+ p \to A_1 p$	8.04	0	9
$\pi^+ p \to A_3 p$	8.04	114	9
$\pi^+ p \to B^+ p$	8.04	114	9
$\pi^+ p \to A_2 p$	8.04	104	9

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