Nucleon polarizabilities and $\pi^0 \to 2\gamma$, $\eta^0 \to 2\gamma$ decay rates from Compton-scattering amplitudes

I. Guiaşu and E. E. Radescu
Institute of Atomic Physics, P.O. Box 5206, Bucharest, Romania
(Received 17 December 1975)

The difference of the dynamic electric and magnetic polarizabilities of the nucleon $\alpha - \beta$ is estimated on the basis of sum rules obtained from unsubtracted backward dispersion relations for the relevant nucleon Compton-scattering amplitudes in the framework of the two-particle unitarity approximation. The s- and u-channel contributions are then evaluated in terms of pion photoproduction data while the contribution of the two-pion state in the t channel is shown to contain a large positive piece, explicitly computed, which could bring the theoretical prediction into qualitative agreement with the existing experimental results. It is consequently pointed out that if the ordering $\alpha > \beta$ is further confirmed experimentally, the annihilation-channel exchanges should be strongly responsible for this fact. The importance of the two-pion continuum for the dispersion determination of the $\eta^0 \to 2\gamma$ decay rate from the superconvergence of certain nucleon Compton-scattering amplitudes for fixed s and large t is also particularly emphasized.

I. INTRODUCTION

According to the well-known low-energy theorem,[1] the nucleon Compton-scattering amplitude at fixed angle is completely determined up to second order in the photon energy by the mass, charge, and magnetic moment of the target and two structure-dependent constants α and β, usually called the dynamic electric and magnetic polarizabilities of the nucleon. The sum $\alpha + \beta$ can be theoretically found in a model-independent way from the Gell-Mann–Goldberger–Thirring[2] dispersion relation for the forward spin-averaged amplitude in terms of the total photoabsorption cross-section σ^T. In their analysis of forward Compton scattering, Damashek and Gilman[3] found

$$(\alpha + \beta)_{\text{proton}} = \frac{1}{2\pi^2}\int_{\omega_{\text{thr}}}^{\infty} \frac{d\omega}{\omega^2}\,\sigma^T_{\text{proton}}(\omega) = (14.1 \pm 0.3) \times 10^{-4}\ \text{fm}^3. \tag{1.1}$$

The theoretical calculation of α and β separately is a much more difficult task, requiring many other dynamical assumptions than those needed to derive Eq. (1.1).

Experimentally, after the first measurement of these parameters made many years ago by Goldansky *et al.*,[4] a better determination (which essentially confirmed the old results) has been recently done by Baranov *et al.*[5] These authors, using the expression of the differential cross-section as given by the low-energy theorem for a least-squares fit of experimental data in the laboratory photon-energy range $80 < \omega < 110$ MeV obtained the values

$$\alpha^{\text{exp.}}_{\text{proton}} = (10.7 \pm 1.1) \times 10^{-4}\ \text{fm}^3,$$
$$\beta^{\text{exp.}}_{\text{proton}} = (-0.7 \pm 1.6) \times 10^{-4}\ \text{fm}^3. \tag{1.2}$$

The recent work of Ericson and collaborators[6] renewed the interest in both experimental and theoretical investigation of nuclear and elementary-particle polarizabilities. In a paper devoted to nucleon polarizabilities, Bernabeu, Ericson, and Ferro Fontan[7] calculated α and β starting from an approximate sum rule obtained from a backward dispersion relation for a certain combination of Compton-scattering amplitudes (neglecting the contributions to the dispersion integral coming from the annihilation channel) and found $\alpha_{\text{proton}} \simeq 4 \times 10^{-4}\ \text{fm}^3$, $\beta_{\text{proton}} \simeq 10 \times 10^{-4}\ \text{fm}^3$. The ordering $\beta > \alpha$ would be in agreement with simple but not compulsory expectations based either on quark-model arguments or on the fact that the most important excited states of the nucleon are of magnetic nature [the $N^*(3/2, 3/2)$ 1236 MeV resonance appears, for instance, in a magnetic multipole].

In this paper we shall present an alternative dispersion analysis of the nucleon dynamic electric and magnetic polarizabilities arguing that if the ordering $\alpha > \beta$ is further confirmed experimentally, the consideration of the annihilation channel exchanges in a dispersion approach could help understanding the situation. The need for taking into account the contributions of the singularities in the t channel to the dispersive integrals comes from the bad asymptotic behavior in the forward direction (at least as far as the Regge-pole model is concerned) of the Compton amplitudes which, at threshold, define the combination $\alpha - \beta$. Thus, unlike the case of $\alpha + \beta$, the model dependence encountered in the dispersive determination of $\alpha - \beta$ makes the theoretical result for this quantity much more uncertain. In any case, working in the framework of fixed-angle ($\theta = 180°$) dispersion relations for which a better asymptotic behavior is expected for the relevant amplitudes, we shall show that the first thing which comes in mind with respect to the t-channel singularities, the consideration of the two-pion state, is capable by itself of giving a large positive contribution to $\alpha-\beta$, thus ensuring the ordering $\alpha>\beta$.

The needed kinematics is presented in Sec. II. In Secs. III (which also contains a detailed account of a previous short communication[8]) and IV two different sum rules for $\alpha-\beta$ based on backward dispersion relations are presented and evaluated within the two-particle unitarity scheme. The treatment of s- and u-channel contributions is given, as usual, in terms of pion-photoproduction data while, as far as the t-channel singularities are concerned, we give here an explicit calculation of the two-pion-state continuum contribution in the approximation of keeping for the matrix elements of the processes $N\bar{N}\to\pi\pi$ and $\pi\pi\to\gamma\gamma$ only the corresponding Born poles plus the constants dictated by the existing low-energy theorem for $\pi\pi\to\gamma\gamma$. The results of both approaches appear to be very close to each other and agree qualitatively with Eqs. (1.2). In order to see how some of the approximations for the evaluation of s- and u-channel contributions work, in Sec. V an estimation of $\alpha+\beta$ using only the pion + nucleon intermediate states in the absorptive part of the forward spin-averaged amplitude is presented.

As a byproduct, the same two-pion model for the annihilation-channel exchanges employed in Secs. III and IV in connection with the nucleon dynamic polarizabilities is used in Sec. VI for the calculation of a piece so far disregarded in a sum rule[9] for the $\eta^0\to 2\gamma$ decay amplitude coming from the supposed superconvergence[10] of a certain nucleon Compton-scattering invariant amplitude at fixed $s=0$. It is shown that the theoretical prediction for the $\eta^0\to 2\gamma$ decay rate,[9] known to be much lower than the experimental value, is thereby considerably increased.

The conclusions of this paper and the results of other related approaches are briefly discussed in Sec. VII. Technical details which also establish notations and conventions needed for the evaluation of the s- and u-channel contributions to the dispersive integrals considered in this work are given in Appendix A, while Appendix B contains details about the two-pion model for the t-channel contributions. Because at times in this paper the pion + nucleon continuum contributions to the sum rules have also been compared with the results of a simple saturation with the $N^*(3/2,\ 3/2)$ resonance in the zero-width approximation, we found it useful to collect in Appendix C the N^* contributions to the s-channel absorptive parts of the six invariant Compton-scattering amplitudes belonging to the particular set we were working with.

II. KINEMATICAL PRELIMINARIES

The scattering matrix for the nucleon Compton effect is

$$S_{fi}=\delta_{fi}-i(2\pi)^4\delta(p'+k'-p-k)(2\pi)^{-6}m(4k_0k_0'p_0p_0')^{-1/2}\times\epsilon_\mu'^*(k')\bar{u}(p')M_{\mu\nu}(p',k';p,k)u(p)\epsilon_\nu(k). \tag{2.1}$$

$k(p)$ and $k'(p')$ are the photon (nucleon) momenta before and after the collision; m is the nucleon mass. In terms of the three independent vectors

$$Q=k'-k=p-p',\quad K=\tfrac{1}{2}(k+k'),\quad P=\tfrac{1}{2}(p+p'), \tag{2.2}$$

the usual Mandelstam invariants are

$$\begin{aligned} s&=(k+p)^2=m^2+2P\cdot K-\tfrac{1}{2}Q^2,\\ t&=(k-k')^2=Q^2\\ u&=(k-p')^2=m^2-2P\cdot K-\tfrac{1}{2}Q^2 \end{aligned} \tag{2.3}$$

$$K^2=-\frac{t}{4},\quad P^2=m^2-\frac{t}{4},$$

$$P\cdot K\equiv\nu=\tfrac{1}{4}(s-u),\quad P\cdot Q=K\cdot Q=0\ .$$

We shall work here with the six scalar invariant amplitudes $A_i(s,t)$ $(i=1,2,\ldots,6)$ of Bardeen and Tung,[11] known to be free of any kinematical singularities and zeros:

$$M_{\mu\nu}=\sum_{i=1}^{6}\mathfrak{L}^i_{\mu\nu}A_i(s,t),$$

where

$$\begin{aligned}
\mathfrak{L}_1&=K^2g_{\mu\nu}-2K_\mu K_\nu,\\
\mathfrak{L}_2&=\tfrac{1}{2}K^2[\gamma_\mu(\gamma\cdot K)\gamma_\nu-\gamma_\nu(\gamma\cdot K)\gamma_\mu]-(P\cdot K)(K_\mu\gamma_\nu+\gamma_\mu K_\nu)+(\gamma\cdot K)(K_\mu P_\nu+P_\mu K_\nu),\\
\mathfrak{L}_3&=m(\gamma\cdot K)g_{\mu\nu}-(P\cdot K)g_{\mu\nu}-\tfrac{1}{2}K^2(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)+K_\mu\tfrac{1}{2}[(\gamma\cdot K),\ \gamma_\nu]+\tfrac{1}{2}[\gamma_\mu,(\gamma\cdot K)]K_\nu-m(K_\mu\gamma_\nu+\gamma_\mu K_\nu)+(K_\mu P_\nu+P_\mu K_\nu)\\
\mathfrak{L}_4&=K^2(\gamma_\mu P_\nu+P_\mu\gamma_\nu)-(P\cdot K)(K_\mu\gamma_\nu+\gamma_\mu K_\nu)-(\gamma\cdot K)(K_\mu P_\nu+P_\mu K_\nu)+(P\cdot K)g_{\mu\nu}(\gamma\cdot K)-mK^2g_{\mu\nu}+2mK_\mu K_\nu,\\
\mathfrak{L}_5&=K^2P_\mu P_\nu-(P\cdot K)(K_\mu P_\nu+P_\mu K_\nu)-\tfrac{1}{2}[P^2K^2-(P\cdot K)^2]g_{\mu\nu}+P^2K_\mu K_\nu,\\
\mathfrak{L}_6&=P_\mu P_\nu(\gamma\cdot K)-\tfrac{1}{2}(P\cdot K)(\gamma_\mu P_\nu+P_\mu\gamma_\nu)+\tfrac{1}{4}(P\cdot K)[\gamma_\mu(\gamma\cdot K)\gamma_\nu-\gamma_\nu(\gamma\cdot K)\gamma_\mu]\\
&\quad+\tfrac{1}{4}mK^2(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)+\tfrac{1}{2}m(P\cdot K)g_{\mu\nu}-\tfrac{1}{2}P^2g_{\mu\nu}(\gamma\cdot K)+K_\mu K_\nu(\gamma\cdot K)\\
&\quad+\tfrac{1}{2}m^2(K_\mu\gamma_\nu+\gamma_\mu K_\nu)-\tfrac{1}{2}m(K_\mu P_\nu+P_\mu K_\nu)-\tfrac{1}{4}mK_\mu[(\gamma\cdot K),\gamma_\nu]-\tfrac{1}{4}m[\gamma,(\gamma\cdot K)]K_\nu.
\end{aligned} \tag{2.4}$$

The amplitudes $A_{1,2,4,5}$ are even while $A_{3,6}$ are odd under s-u crossing. The structure of the s- and u-channel Born pole terms of the amplitudes A_i is given by

$$A_i^B(s,t,u) = R_i^+[(m^2-s)^{-1}+(m^2-u)^{-1}] + R_i^-[(m^2-s)^{-1}-(m^2-u)^{-1}] + R_i^{su}(m^2-s)^{-1}(m^2-u)^{-1}, \tag{2.5}$$

where

$$\begin{aligned} &R_1^+ = -(2\hat{\kappa}\hat{e}+\hat{\kappa}^2)/2m, \quad R_1^{su}=4m\hat{e}^2, \quad R_1^-=0,\\ &R_2^+ = (2\hat{\kappa}\hat{e}+\hat{\kappa}^2)/2m^2, \quad R_2^{su}=4\hat{e}(\hat{e}+\hat{\kappa}), \quad R_2^-=0,\\ &R_3^- = (2\hat{\kappa}\hat{e}+\hat{\kappa}^2)/2m, \quad R_3^{su}=R_3^+=0,\\ &R_4^+ = \hat{\kappa}^2/2m^2, \quad R_4^{su}=4\hat{e}(\hat{e}+\hat{\kappa}), \quad R_4^-=0,\\ &R_5^{su} = -8\hat{\kappa}\hat{e}m, \quad R_5^+=R_5^-=0,\\ &R_6^- = -\hat{\kappa}^2/m^2, \quad R_6^+=R_6^{su}=0. \end{aligned} \tag{2.6}$$

The charge and magnetic moment matrices $\hat{e}$ and $\hat{\kappa}$ are defined as

$$\hat{e} = \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{\kappa} = \begin{pmatrix} e\kappa_p & 0 \\ 0 & e\kappa_n \end{pmatrix},$$

with $e^2/4\pi = 1/137$, $\kappa_p = 1.79$, $\kappa_n = -1.91$. The pseudoscalar (π^0 and η^0) meson poles in the t channel only contribute to A_2:

$$A_2^{(\text{pole})} = -(2/m)g_{\pi NN}\tau_3 F_\pi(\mu^2-t)^{-1} + (2/m)g_{\eta NN}F_\eta(\mu_\eta{}^2-t)^{-1}, \tag{2.7}$$

where F_π, F_η and $g_{\pi NN}$, $g_{\eta NN}$ are the meson decay constants and couplings with the nucleon while μ and μ_η denote the mass of the pion and η meson [$g_{\pi NN} \equiv g_r \simeq (4\pi\times 14.5)^{1/2}$].

In the barycentric system, denoting by q_c the magnitude of the photon (or particle) three-momentum and by θ the scattering angle, one has the following kinematical relations:

$$\begin{aligned} &t = -2q_c{}^2(1-\cos\theta),\\ &q_c = (s-m^2)/2s^{1/2},\\ &t(s,\cos\theta) = -\frac{(s-m^2)^2}{2s}(1-\cos\theta). \end{aligned} \tag{2.8}$$

In particular, at $\theta = 180°$

$$t = -\frac{(s-m^2)^2}{s}, \quad u = \frac{m^4}{s}. \tag{2.9}$$

The photon energy in the laboratory system ω is related to s through the relation

$$s = 2m\omega + m^2. \tag{2.10}$$

The connection between the invariant amplitudes A_i and the six independent helicity amplitudes describing nucleon Compton scattering is given by the following relations:

$$A_1 = -\frac{2ms^{1/2}}{\sin(\frac{1}{2}\theta)(s-m^2)^2} f_{-1/2\,-1,1/2\,1} + \frac{4s-2(s+m^2)\sin^2(\frac{1}{2}\theta)}{(s-m^2)^2\sin^2(\frac{1}{2}\theta)\cos(\frac{1}{2}\theta)} f_{1/2\,-1,1/2\,1} + \frac{2ms^{1/2}}{\sin(\frac{1}{2}\theta)(s-m^2)^2} f_{1/2\,-1,-1/2\,1}, \tag{2.11}$$

$$A_2 = \frac{2s^{1/2}}{\sin(\frac{1}{2}\theta)(s-m^2)^2} f_{-1/2\,-1,1/2\,1} - \frac{2(s+m^2)[2s-(s-m^2)\sin^2(\frac{1}{2}\theta)]}{m(s-m^2)^3\sin^2(\frac{1}{2}\theta)\cos(\frac{1}{2}\theta)} f_{1/2\,-1,1/2\,1} + \frac{2s^{1/2}[4s-(s-m^2)\sin^2(\frac{1}{2}\theta)]}{(s-m^2)^3\sin^3(\frac{1}{2}\theta)} f_{1/2\,-1,-1/2\,1}, \tag{2.12}$$

$$A_3 = \frac{2ms^{1/2}}{\sin(\frac{1}{2}\theta)(s-m^2)^2} f_{-1/2\,-1,1/2\,1} + \frac{2(s+m^2)}{(s-m^2)^2\cos(\frac{1}{2}\theta)} f_{1/2\,-1,1/2\,1} - \frac{2ms^{1/2}}{\sin(\frac{1}{2}\theta)(s-m^2)^2} f_{1/2\,-1,-1/2\,1}, \tag{2.13}$$

$$A_4 = \frac{2m}{(s-m^2)^2\cos(\frac{1}{2}\theta)} f_{1/2\,1,1/2\,1} + \frac{4m^2[2s-(s-m^2)\sin^2(\frac{1}{2}\theta)]}{s^{1/2}(s-m^2)^3\sin(\frac{1}{2}\theta)\cos^2(\frac{1}{2}\theta)} f_{-1/2\,1,1/2\,1} + \frac{2m[s(s+3m^2)-m^2(s-m^2)\sin^2(\frac{1}{2}\theta)]}{s(s-m^2)^3\cos^3(\frac{1}{2}\theta)} f_{-1/2\,1,-1/2\,1}, \tag{2.14}$$

$$A_5 = -\frac{16ms^{1/2}}{\sin(\frac{1}{2}\theta)\cos^2(\frac{1}{2}\theta)(s-m^2)^3} f_{-1/2\,1,1/2\,1} - \frac{16m^2}{\cos^3(\frac{1}{2}\theta)(s-m^2)^3} f_{-1/2\,1,-1/2\,1}, \tag{2.15}$$

$$A_6 = -\frac{4m}{(s-m^2)^2\cos(\frac{1}{2}\theta)} f_{1/2\,1,1/2\,1} + \frac{8m^2\sin(\frac{1}{2}\theta)}{s^{1/2}(s-m^2)^2\cos^2(\frac{1}{2}\theta)} f_{-1/2\,1,1/2\,1} + \frac{4m[s+m^2\sin^2(\frac{1}{2}\theta)]}{s(s-m^2)^2\cos^3(\frac{1}{2}\theta)} f_{-1/2\,1,-1/2\,1}. \tag{2.16}$$

The helicity amplitudes appearing in the above relations are those of Ref. 11 divided by $2m$ with a different sign definition only for $f_{-1/2\,-1,\,1/2\,1}$. Some sign errors inside the curly brackets in the first of Eqs. (51)

and the first of Eqs. (52) of Ref. 11 have been corrected in our Eqs. (2.11)–(2.16).

The continuum parts of the invariant amplitudes are, by construction,

$$A_i^c \equiv A_i - A_i^B \tag{2.17}$$

and the dynamic electric and magnetic polarizabilities α and β are defined by

$$\alpha - \beta = \frac{1}{4\pi} A_1^c(s=m^2, t=0)\,, \tag{2.18}$$

$$\alpha + \beta = \frac{m}{4\pi}\left[A_4^c(s=m^2, t=0) + \frac{m}{2} A_5^c(s=m^2, t=0)\right]. \tag{2.19}$$

III. BACKWARD DISPERSION RELATION FOR A_1

In this section we shall derive and evaluate a dispersion sum rule for $\alpha-\beta$ involving the amplitude A_1 from the decomposition displayed in Eqs. (2.4). Our main hypothesis is that the amplitude A_1 goes to zero when $s\to\infty$ at fixed $\theta=180°$, so that it satisfies an unsubtracted backward dispersion relation. The amplitude A_1 cannot be expected to satisfy a forward unsubtracted dispersion relation, at least as far as the Regge-pole prediction is concerned. As a matter of fact, we simply explore in this section what would come about from the supposed convergence of a backward dispersion representation for the amplitude A_1 which directly defines $\alpha-\beta$. Forgetting for the time being the contribution of the single-nucleon state (which will be discussed later on, when we shall have already specialized our considerations to $\theta=180°$), an unsubtracted fixed-angle dispersion relation for an invariant Compton amplitude even under s-u crossing looks as follows[12] (see also Ref. 13 for details):

$$\begin{aligned} A(s,c) = {} & \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} ds' \frac{A^{\mathrm{I}}[s', t(s',c)]}{s'-s} + \frac{2}{\pi}\int_{(m+\mu)^2}^{\infty} du' \frac{1}{\overline{s}_+(u') - \overline{s}_-(u')} \\ & \times\left\{\frac{\overline{s}_+(u')}{s(1+c)-\overline{s}_+(u')} A^{\mathrm{I}}\left[u', t\left(\frac{\overline{s}_+(u')}{1+c}, c\right)\right] - \frac{\overline{s}_-(u')}{s(1+c)-\overline{s}_-(u')} A^{\mathrm{I}}\left[u', t\left(\frac{\overline{s}_-(u')}{1+c}, c\right)\right]\right\} \\ & + \left(\frac{2}{\pi}\int_{4\mu^2}^{2m^2(1-c)} dt' + \frac{2}{\pi}\int_{2m^2(1-c)}^{\infty} dt'\right) \frac{1}{t_+(t') - t_-(t')}\left\{\frac{t_+(t')}{s(1-c)-t_+(t')} A^{\mathrm{III}}\left[\frac{t_+(t')}{1-c}, t'\right]\right. \\ & \left. - \frac{t_-(t')}{s(1-c)-t_-(t')} A^{\mathrm{III}}\left[\frac{t_-(t')}{1-c}, t'\right]\right\}, \end{aligned} \tag{3.1}$$

where c stands for $\cos\theta$, and

$$t_\pm(t') = m^2(1-c) - t' \pm \{[m^2(1-c)-t']^2 - m^4(1-c)^2\}^{1/2}\,, \tag{3.2}$$

$$\overline{s}_\pm(u') = m^2(1+c) - u' \pm \{[m^2(1+c)-u']^2 + m^4(1-c^2)\}^{1/2}\,. \tag{3.3}$$

The absorptive parts A^{I}, A^{III} are given by the unitarity condition in the direct and annihilation channels.

At $\theta=180°$ one has

$$\overline{s}_+(u') = 0,\quad \lim_{c\to -1}\frac{\overline{s}_+(u')}{1+c} = \frac{m^4}{u'},\quad \overline{s}_-(u') = -2u'\,, \tag{3.4}$$

$$t_+(t') = 2m^2 - t' + i[t'(4m^2-t')]^{1/2}\,,\quad t_-(t') = 2m^2 - t' - i[t'(4m^2-t')]^{1/2}\,. \tag{3.5}$$

The second term of the integral over u' in Eq. (3.1) at $\theta=180°$ contributes at most a subtraction constant, which in the case considered here is zero by hypothesis. We supposed here that the amplitude $A_1(s,\cos\theta=-1)$ tends to zero as $s\to\infty$. Recalling the definition Eq. (2.18), we are interested in having a dispersion representation for A_1^c, so that we must now carefully analyze the Born term A_1^B. At $\cos\theta=-1$, from Eqs. (2.5) and (2.6) we find

$$A_1^B(s, c=-1) = -\frac{4m\hat{e}^2 s}{m^2(s-m^2)^2} - \frac{(2\hat{\kappa}\hat{e}+\hat{\kappa}^2)}{2m^3}\,. \tag{3.6}$$

This means that our assumption

$$A_1(s,\cos\theta=-1) \xrightarrow[s\to\infty]{} 0$$

implies

$$A_1^c(s,\cos\theta=-1) \equiv A_1(s,\cos\theta=-1) - A_1^B(s,\cos\theta=-1) \xrightarrow[s\to\infty]{} +\frac{(2\hat{\kappa}\hat{e}+\hat{\kappa}^2)}{2m^3}\,.$$

In view of all these considerations, the needed representation reads

$$A_1^c(s, c=-1) = \frac{(2\hat{\kappa}\hat{e}+\hat{\kappa}^2)}{2m^3} + \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} ds' \frac{A_1^{\rm I}[s', t(s', c=-1)]}{s'-s} + \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} du' \frac{m^4}{u'(u's-m^4)} A_1^{\rm I}\left[u', t\left(\frac{m^4}{u'}, c=-1\right)\right]$$

$$+\left(\frac{2}{\pi}\int_{4\mu^2}^{4m^2} dt' + \frac{2}{\pi}\int_{4m^2}^{\infty} dt'\right) \frac{1}{t_+(t')-t_-(t')}\left\{\frac{t_+(t')}{2s-t_+(t')} A_1^{\rm III}\left[\frac{t_+(t')}{2}, t'\right] - \frac{t_-(t')}{2s-t_-(t')} A_1^{\rm III}\left[\frac{t_-(t')}{2}, t'\right]\right\}, \quad (3.7)$$

which for $s=m^2$ implies the following formula for the difference of the dynamic electric and magnetic polarizabilities:

$$(\alpha-\beta) = (\alpha-\beta)^{(1)} + (\alpha-\beta)^{(2)} + (\alpha-\beta)^{(3)}, \quad (3.8)$$

$$(\alpha-\beta)^{(1)} = \frac{1}{4\pi}\,\frac{2\hat{\kappa}\hat{e}+\hat{\kappa}^2}{2m^3}, \quad (3.9)$$

$$(\alpha-\beta)^{(2)} = \frac{1}{4\pi^2}\int_{(m+\mu)^2}^{\infty} ds' \left(1+\frac{m^2}{s'}\right)\frac{A_1^{\rm I}[s', t=-(s'-m^2)^2/s']}{s'-m^2} \quad (3.10)$$

$$(\alpha-\beta)^{(3)} = \frac{1}{2\pi^2}\left(\int_{4\mu^2}^{4m^2} dt' + \int_{4m^2}^{\infty} dt'\right)\frac{1}{t_+(t')-t_-(t')}\left\{\frac{t_+(t')}{2m^2-t_+(t')} A_1^{\rm III}\left[\frac{t_+(t')}{2}, t'\right] - \frac{t_-(t')}{2m^2-t_-(t')} A_1^{\rm III}\left[\frac{t_-(t')}{2}, t'\right]\right\}. \quad (3.11)$$

The contributions of the different singularities in the complex s plane (at fixed $\theta = 180°$), as given by Eq. (3.7), can be read off on Fig. 1, where the corresponding cut structure is displayed. So, as can be immediately checked, the integral over s' in Eq. (3.7) represents the contribution of the right-hand cut in Fig. 1, the u' integral represents the contribution of the segment $[0, m^4/(m+\mu)^2]$ of the other cut on the real axis, while the first of the two integrals over t' (the one which extends over t' values in the interval $[4\mu^2, 4m^2]$) expresses the contribution of the circular cut (along the curve $|s| = m^2$) and the last integral over t' (for $t' > 4m^2$) corresponds to the cut $(-\infty, 0]$ along the negative real axis in the s plane. As noted in the figure, the circular cut extends up to the points $m^2 e^{\pm i\chi_0}$, where

$$\chi_0 = \arctan\frac{2\rho(1-\rho^2)^{1/2}}{1-2\rho^2} = \arccos(1-2\rho^2), \quad (3.12)$$

$$\rho \equiv \frac{\mu}{m}. \quad (3.13)$$

The threshold of the three-pion state ($t' = 9\mu^2$) corresponds to the points $m^2 e^{\pm i\chi_1}$ on the circular cut, where

$$\chi_1 = \arctan\frac{3\rho(4-9\rho^2)^{1/2}}{2-9\rho^2}.$$

The s' integral in Eq. (3.10) by means of s-u crossing takes into account both the contributions of the direct s-channel right-hand cut and of the cut on the interval $[0, m^4/(m+\mu)^2]$.

We proceed now to evaluate numerically the different contributions to $\alpha-\beta$ according to Eqs. (3.8)–(3.11). For $(\alpha-\beta)^{(1)}$ we have from Eq. (3.9)

$$(\alpha-\beta)^{(1)}_{\rm proton} = +2.40\times 10^{-4}\ {\rm fm}^3, \quad (3.14a)$$

$$(\alpha-\beta)^{(1)}_{\rm neutron} = +1.29\times 10^{-4}\ {\rm fm}^3. \quad (3.14b)$$

We shall estimate the quantity $(\alpha-\beta)^{(2)}$ as given by Eq. (3.10), restricting ourselves to the consid-

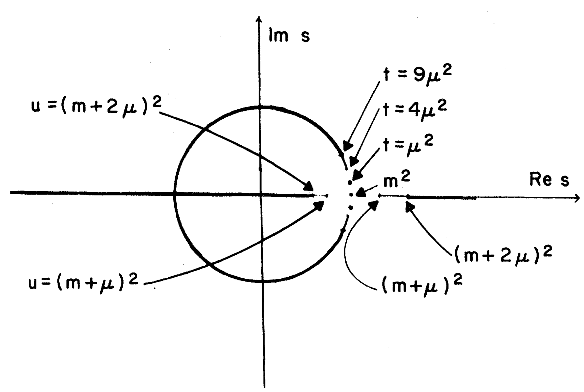

FIG. 1. Cut structure in the complex s plane.

eration of only two-particle (pion + nucleon) intermediate states in the unitarity sum. We then use pion-photoproduction data, and these are taken from the analyses of Refs. 14 and 15. The integration over energy will be extended well outside the threshold of double pion photoproduction, considering that this extrapolation does not introduce drastic modifications. In terms of tabulated photoproduction multipoles (see Appendix A for notations) our formula for $(\alpha-\beta)^{(2)}$ reads

$$
\begin{aligned}
(\alpha-\beta)^{(2)}_{\text{proton(neutron)}} &= \frac{1}{4\pi^2}\int_{(m+\mu)^2}^{\infty}\frac{\operatorname{Im}A_1(s,\theta=180°)}{s-m^2}\left(1+\frac{m^2}{s}\right)\\
&\simeq \frac{10^{-4}}{\pi}\int_{\omega_{\text{thr}}}\frac{d\omega}{\omega^4}\left(1+\frac{\omega}{m}\right)\Bigg\{\left(2\frac{\omega}{m}+1\right)^{1/2}\Big[3|A^{p(n)}_{S1/2}|^2-3|A^{p(n)}_{P1/2}|^2\\
&\qquad -6|A^{p(n)}_{P3/2}|^2+6|A^{p(n)}_{D3/2}|^2-\frac{9}{2}|B^{p(n)}_{P3/2}|^2+\frac{9}{2}|B^{p(n)}_{D3/2}|^2+2|A^{V3}_{S1/2}|^2\\
&\qquad -2|A^{V3}_{P1/2}|^2-4|A^{V3}_{P3/2}|^2+4|A^{V3}_{D3/2}|^2-3|B^{V3}_{P3/2}|^2+3|B^{V3}_{D3/2}|^2\Big]\\
&\qquad +6\frac{\omega}{m}\operatorname{Re}\left[2(B^{V3*}_{P3/2}A^{V3}_{P3/2}-B^{V3*}_{D3/2}A^{V3}_{D3/2})+3(B^{p(n)*}_{P3/2}A^{p(n)}_{P3/2}-B^{p(n)*}_{D3/2}A^{p(n)}_{D3/2})\right]\Bigg\}\\
&\qquad (s=2m\omega+m^2,\ \omega_{\text{thr}}=\mu+\mu^2/2m), \qquad (3.15)
\end{aligned}
$$

where only waves with the total angular momentum $J=\frac{1}{2},\frac{3}{2}$ have been retained.

For the proton, the integration in Eq. (3.15) over the low-energy region (the photon laboratory energy ω from 180 MeV up to 250 MeV) using the analysis from the first paper of Ref. 14) yields a contribution of 0.13×10^{-4} fm^3, the integration over the domain of ω from 250 MeV up to 750 MeV gives -7.83×10^{-4}fm^3, while the region $750<\omega<1210$ MeV contributes with $+0.30\times10^{-4}$ fm^3 (for $\omega>250$ MeV the data of Ref. 15 have been used). So, neglecting contributions coming from energies higher than 1210 MeV, we found the result

$$(\alpha-\beta)^{(2)}_{\text{proton}}\simeq-7.40\times10^{-4}\ \text{fm}^3. \qquad (3.16a)$$

For the neutron, corresponding to the three energy regions mentioned above, we found

$$
\begin{aligned}
(\alpha-\beta)^{(2)}_{\text{neutron}} &\simeq (+0.17-6.14+0.26)\times10^{-4}\ \text{fm}^3\\
&=-5.71\times10^{-4}\ \text{fm}^3. \qquad (3.16b)
\end{aligned}
$$

(Here we used the second paper of Refs. 14) for $\omega<250$ MeV, and Ref. 15 for $\omega>250$ MeV).

We observe that if one tries to saturate the integral in Eq. (3.15) only with the $N^*(\frac{3}{2},\frac{3}{2})$ ($M=1236$ MeV) resonance in the zero-width approximation, one would find

$$
\begin{aligned}
(\alpha-\beta)^{(2)(N*)}_{\text{proton}} &= (\alpha-\beta)^{(2)(N*)}_{\text{neutron}}\\
&=-\frac{e^2}{4\pi}\,\frac{1}{8m^3}\,\frac{M^2+m^2}{M^2-m^2}\\
&\quad\times\left(3+4\frac{m}{M}-3\frac{m^2}{M^2}\right)G_M{}^2\\
&\simeq-12.3\times10^{-4}\ \text{fm}^3. \qquad (3.17)
\end{aligned}
$$

G_M is the magnetic N^*N transition form factor ($G_M\simeq3$). The other transition form factor, G_E, which is very small with respect to G_M (see the analysis of Jones and Scadron[16]) has been set equal to zero. The resonance-saturation scheme can sometimes represent an useful laboratory in which one simply gets a rough estimation of different continuum contributions otherwise hard to compute; so we give in Appendix C some details about this procedure and display there the N^* resonance contributions to the s-channel absorptive parts of the amplitudes A_i in terms of the NN^* transition form factors G_M and G_E.

We start now estimating $(\alpha-\beta)^{(3)}$ as defined in Eq. (3.11). For this purpose, we shall consider only the contributions to the absorptive part A_1^{III} coming from the two-pion intermediate states. Because our lack of knowledge of all but a few of the matrix elements in the sum over intermediate states precludes any exact calculation of the absorptive part A_1^{III}, we shall use two-particle unitarity in Eq. (3.11) all over the t' integration region. We are then left with the needed absorptive part expressed in terms of the product of the amplitudes of the processes $N\bar{N}\to\pi\pi$ and $\pi\pi\to\gamma\gamma$, which will be both taken in the Born approximation (that is the covariant matrix element for $N\bar{N}\to\pi\pi$ will be approximated by nucleon-pole terms alone, while the reaction $\pi\pi\to\gamma\gamma$ will be approximated by pole terms plus the constants required by the existing low-energy theorem). This simple model for the annihilation-channel contributions has been previously employed by Holliday[13] and by Baranov, Filkov and Sokol[17], among others, in their dispersion-theoretical treatments of nucleon Compton scattering. We hope that it can also give some qualitative indications about the structure-dependent constants α and β. We note first that within this (2π) model one gets

$$(\alpha-\beta)^{(3)}_{\text{proton}} = (\alpha-\beta)^{(3)}_{\text{neutron}} .$$

Using the reality condition

$$A_1^{\text{III}}[\tfrac{1}{2}t_-(t'), t'] = (A_1^{\text{III}}[\tfrac{1}{2}t_+(t'), t'])^*$$

for $4\mu^2 \leq t' \leq 4m^2$ in Eq. (3.11) one arrives at

$$(\alpha-\beta)^{(3)} = \frac{1}{4\pi^2}\int_{4\mu^2}^{4m^2} dt' \left\{ \frac{\text{Re}A_1^{\text{III}}[\frac{1}{2}t_+(t'), t']}{t'} - \frac{\text{Im}A_1^{\text{III}}[\frac{1}{2}t_+(t'), t']}{[t'(4m^2-t')]^{1/2}} \right\} + \frac{1}{4\pi^2}\int_{4m^2}^{\infty} dt' \frac{A_1^{\text{III}}[\frac{1}{2}t_+(t'), t']}{t'} .$$

In the particular model considered for A_i it turns out that $A_1^{\text{III}(2\pi)}[\frac{1}{2}t_+(t'), t']$ is a purely real quantity in the t' integration region from $4\mu^2$ to $4m^2$ and so we are left with

$$(\alpha-\beta)^{(3)} \simeq \frac{1}{4\pi^2}\int_{4\mu^2}^{4m^2} dt' \frac{\text{Re}A_1^{\text{III}(2\pi)}[\frac{1}{2}t_+(t'), t']}{t'} + \frac{1}{4\pi^2}\int_{4m^2}^{\infty} \frac{dt'}{t'} A_1^{\text{III}(2\pi)}[\tfrac{1}{2}t_+(t'), t'] . \tag{3.18}$$

The different (2π) contributions to the t-channel absorptive parts $A_i^{\text{III}}(s,t)$ of the invariant amplitudes A_i are found in Appendix B. In the t' integration regions from $4\mu^2$ to $4m^2$ (corresponding to the circular cut in the s plane) and from $4m^2$ to ∞ [corresponding to the cut $(-\infty, 0)$ in Fig. 1], the integrand of Eq. (3.18) is expressed through Eqs. (B17), (B19), and (B20). After also evaluating the integrals over z in Eqs. (B19) and (B20) and substituting their expressions into Eq. (B17), one finds the result

$$(\alpha-\beta)^{(3)}_{\text{proton}} = (\alpha-\beta)^{(3)}_{\text{neutron}} \simeq \left(\frac{e^2}{4\pi}\frac{g_r^2}{4\pi}\frac{2\rho^2}{\pi m^3}\right)\int_{\rho}^{\infty} dx \left\{ \frac{(x^2-\rho^2)^{1/2}}{x^2[\rho^4+4(1-\rho^2)x^2]} + \frac{-4x^4+2x^2(2-\rho^2+\frac{1}{2}\rho^4)+(\rho^4-\frac{1}{2}\rho^6)}{x^3[\rho^4+4(1-\rho^2)x^2]^2}\ln\frac{x+(x^2-\rho^2)^{1/2}}{x-(x^2-\rho^2)^{1/2}} - \frac{4(x^2-\frac{1}{2}\rho^2)(\frac{1}{4}\rho^4+x^2)}{x^4[\rho^4+4(1-\rho^2)x^2]^2} Y(x) \right\}, \tag{3.19}$$

where

$$Y(x) = \begin{cases} 2(1-x^2)^{1/2}\arctan\dfrac{2[(1-x^2)(x^2-\rho^2)]^{1/2}}{2x^2-\rho^2} & \text{for } \rho \leq x \leq 1 \\ -(x^2-1)^{1/2}\ln\dfrac{2x^2-\rho^2+2[(x^2-1)(x^2-\rho^2)]^{1/2}}{2x^2-\rho^2-2[(x^2-1)(x^2-\rho^2)]^{1/2}} & \text{for } x>1 \end{cases} \tag{3.20}$$

[$\rho = \mu/m$ and the integration variable t' has been changed to $x = (t'/4m^2)^{1/2}$].

The numerical value of $(\alpha-\beta)^{(3)}_{\text{proton(neutron)}}$ as given by Eq. (3.19) is

$$(\alpha-\beta)^{(3)} \simeq 17.51\times10^{-4}\ \text{fm}^3 . \tag{3.21}$$

The result of the integration over $x>1$ in Eq. (3.19) is practically negligible.

Summing up the three contributions to $\alpha-\beta$ found in Eqs. (3.14a), (3.14b); (3.16a), (3.16b); and (3.21) one finally obtains

$$(\alpha-\beta)_{\text{proton}} \simeq (2.40-7.40+17.51)\times10^{-4}\ \text{fm}^3 = +12.51\times10^{-4}\ \text{fm}^3 \tag{3.22a}$$

$$(\alpha-\beta)_{\text{neutron}} \simeq (1.29-5.71+17.51)\times10^{-4}\ \text{fm}^3 = +13.09\times10^{-4}\ \text{fm}^3 . \tag{3.22b}$$

If we join to Eq. (3.22a) the determination of Damashek and Gilman, Eq. (1.1), we get

$$\alpha_{\text{proton}} \simeq 13.3\times10^{-4}\ \text{fm}^3, \quad \beta_{\text{proton}} \simeq 0.8\times10^{-4}\ \text{fm}^3 , \tag{3.23}$$

to be compared with the experimental result given in Eqs. (1.2).

We note that even if the t integration in Eq. (3.19) is extended only over the small region from $4\mu^2$ up to $9\mu^2$ (the threshold of the three-pion state) where two-particle unitarity is exact, one would still get a large positive contribution to $\alpha-\beta$ amounting to $(\alpha-\beta)^{(3)} = +11.0\times10^{-4}$ fm^3, the ordering $\alpha > \beta$ still being preserved.

The presence in $\alpha-\beta$ [as given by Eq. (3.8)] of the piece $(\alpha-\beta)^{(1)}$ expressed in terms of nucleon anomalous magnetic moments seems surprising in view of the fact that the dynamic polarizabilities are intimately related to continuum and not to pole contributions. The quantity $(\alpha-\beta)^{(1)}$ should be taken only as indicating a small but non-negligible amount to $\alpha-\beta$ (as calculated in this section) introduced by a presumably needed subtraction at infinity hard to handle by more precise considerations.

IV. BACKWARD DISPERSION RELATION FOR A_1-A_3

We present and evaluate here another sum rule for $\alpha-\beta$, analogous to the one derived in the previous section but obtained using the combination A_1-A_3 instead of the amplitude A_1. The main assumption here is that A_1-A_3 vanish when $s\to\infty$ at fixed $\theta=180°$ (as u-channel Regge-pole arguments would suggest), so that an unsubtracted backward dispersion relation for A_1-A_3 can be written down. Noting that A_3, unlike A_1, is odd under s-u crossing, and using the same procedure and notations as in Sec. III, the following sum rule for $\alpha-\beta$ emerges:

$$\alpha-\beta=\frac{1}{4\pi}\left[(A_1-A_3)-(A_1^B-A_3^B)\right](s=m^2,c=-1)$$
$$=(\alpha-\beta)^{(2')}+(\alpha-\beta)^{(3')}, \tag{4.1}$$

with

$$(\alpha-\beta)^{(2')}=\frac{1}{4\pi^2}\int_{(m+\mu)^2}^{\infty}\frac{ds'}{s'-m^2}\times\left[\frac{s'+m^2}{s'}A_1^{\mathrm{I}}(s',c=-1)-\frac{s'-m^2}{s'}A_3^{\mathrm{I}}(s',c=-1)\right], \tag{4.2}$$

$$(\alpha-\beta)^{(3')}=\frac{1}{4\pi^2}\int_{4\mu^2}^{4m^2}dt'\left[\frac{\mathrm{Re}(A_1^{\mathrm{III}}-A_3^{\mathrm{III}})}{t'}-\frac{\mathrm{Im}(A_1^{\mathrm{III}}-A_3^{\mathrm{III}})}{[t'(4m^2-t')]^{1/2}}\right]+\frac{1}{4\pi^2}\int_{4m^2}^{\infty}dt'\left[\frac{A_1^{\mathrm{III}}}{t'}+\frac{A_3^{\mathrm{III}}}{[t'(t'-4m^2)]^{1/2}}\right], \tag{4.3}$$

where the arguments of the amplitudes in the integrand of Eq. (4.3) are the same as in Eq. (3.18). The absence, in this approach, of any contribution in terms of nucleon anomalous magnetic moments, similar to $(\alpha-\beta)^{(1)}$ of Sec. III, is a consequence of the following structure of the nucleon Born term $(A_1-A_3)^B$ at $\theta=180°$:

$$(A_1^B-A_3^B)^{\mathrm{proton}}(s,c=-1)=-\frac{4me^2}{(s-m^2)^2}-\frac{[4-(2\kappa_p+\kappa_p^2)]e^2}{m(s-m^2)}, \tag{4.4a}$$

$$(A_1^B-A_3^B)^{\mathrm{neutron}}(s,c=-1)=\frac{\kappa_n^2e^2}{m(s-m^2)}, \tag{4.4b}$$

which imply $(A_1-A_3)^B(s,\theta=180°)\to 0$ when $s\to\infty$.

We evaluate the sum rule Eqs. (4.1)–(4.3) using the same model and approximations as in Sec. III. In the same notations, we find

$$(\alpha-\beta)^{(2')}_{\mathrm{proton(neutron)}}\simeq\frac{10^{-4}}{\pi}\int_{\omega_{\mathrm{thr}}}\frac{d\omega}{\omega^4}\left(2\frac{\omega}{m}+1\right)^{3/2}\Big[\big(2|A^{V3}_{S\,1/2}|^2+4|A^{V3}_{D\,3/2}|^2+3|B^{V3}_{D\,3/2}|^2+3|A^{p(n)}_{S\,1/2}|^2+6|A^{p(n)}_{D\,3/2}|^2+\tfrac{9}{2}|B^{p(n)}_{D\,3/2}|^2\big)$$
$$-\big(2|A^{V3}_{P\,1/2}|^2+4|A^{V3}_{P\,3/2}|^2+3|B^{V3}_{P\,3/2}|^2+3|A^{p(n)}_{P\,1/2}|^2+6|A^{p(n)}_{P\,3/2}|^2+\tfrac{9}{2}|B^{p(n)}_{P\,3/2}|^2\big)\Big] \tag{4.5}$$

and

$$(\alpha-\beta)^{(3')}_{\mathrm{proton}}=(\alpha-\beta)^{(3')}_{\mathrm{neutron}}$$
$$\simeq\frac{e^2}{4\pi}\frac{g_r^2}{4\pi}\frac{2\rho^2}{\pi m^3}\int_{\rho}^{\infty}\frac{dx}{x^3[\rho^4+4x^2(1-\rho^2)]}\left\{\ln\frac{x+(x^2-\rho^2)^{1/2}}{x-(x^2-\rho^2)^{1/2}}-\frac{2x^2-\rho^2}{2x(1-x^2)}Y(x)\right\}. \tag{4.6}$$

The necessary steps to get Eqs. (4.5) and (4.6) are given respectively in Appendixes A and B. Numerically one gets

$$(\alpha-\beta)^{(2')}_{\mathrm{proton}}=-4.92\times10^{-4}\ \mathrm{fm}^3, \tag{4.7a}$$

$$(\alpha-\beta)^{(2')}_{\mathrm{neutron}}=-4.32\times10^{-4}\ \mathrm{fm}^3, \tag{4.7b}$$

and

$$(\alpha-\beta)^{(3')}_{\mathrm{proton}}=(\alpha-\beta)^{(3')}_{\mathrm{neutron}}=17.03\times10^{-4}\ \mathrm{fm}^3, \tag{4.8}$$

so that finally, in this approach, there results

$$(\alpha-\beta)_{\mathrm{proton}}\simeq12.11\times10^{-4}\ \mathrm{fm}^3, \tag{4.9a}$$

$$(\alpha-\beta)_{\mathrm{neutron}}\simeq12.71\times10^{-4}\ \mathrm{fm}^3. \tag{4.9b}$$

Equation (4.9a) together with Eq. (1.1) gives

$$\alpha_{\mathrm{proton}}\simeq13.1\times10^{-4}\ \mathrm{fm}^3,\qquad \beta_{\mathrm{proton}}\simeq1.0\times10^{-4}\ \mathrm{fm}^3. \tag{4.10}$$

We observe that the above numbers are quite close to those obtained in Eqs. (3.22a), (3.22b), and (3.23).

Here we also give the result of $N^*(\frac{3}{2},\frac{3}{2})$ saturation in the zero-width approximation for $(\alpha-\beta)^{(2')}$ [Eq. (4.2)], obtained using the projections from

Appendix C:

$$(\alpha-\beta)^{(2')(N^*)}_{\text{proton(neutron)}} = -\frac{e^2}{4\pi}\,\frac{M}{m^2(M^2-m^2)}\,(G_M{}^2+3G_E{}^2)$$

$$\simeq -10.9\times 10^{-4}\ \text{fm}^3 \quad \text{for } G_M\simeq 3,\ G_E\simeq 0$$

If we restrict the integration in Eq. (4.6) to the small region ($\rho \leq x \leq \frac{3}{2}\rho$) where the two-pion states saturate the t-channel unitarity, we get instead of Eq. (4.8)

$$(\alpha-\beta)^{(3')} \simeq 10.9\times 10^{-4}\ \text{fm}^3\,,$$

e.g., a large positive contribution to $\alpha-\beta$ is present in any case and therefore the ordering $\alpha > \beta$ is still obtained.

We note that the sum rule for $\alpha-\beta$ represented by Eqs. (4.1)–(4.3) can be written as

$$\alpha-\beta = \frac{1}{2\pi^2}\int_{\omega_{\text{thr}}} \frac{d\omega}{\omega^2}\left(1+2\frac{\omega}{m}\right)^{1/2} [\sigma(\Delta P=\text{yes})-\sigma(\Delta P=\text{no})] + (\text{annihilation-channel contribution})\,, \tag{4.11}$$

where, as seen from Eq. (4.5), $\sigma(\Delta P=\text{yes})$ and $\sigma(\Delta P=\text{no})$ contain respectively the parity-flip and non-parity-flip multipoles. By writing an unsubtracted backward dispersion relation for A_1-A_3 we then recover the sum rule from Ref. 7), with the difference, however, that no supplementary contribution in terms of nucleon anomalous magnetic moments actually appears.

V. CALCULATION OF $(\alpha+\beta)$

From the exact relation (1.1) the theoretically numerical value of $\alpha+\beta$ for the proton is already known in terms of total photoabsorption data. In this section we shall compute the sum $\alpha+\beta$ of the nucleon polarizabilities along the same lines as done in Secs. III and IV for the s- and u-channel contributions, in order to see how the different approximations work. We shall make the same two-particle (pion + nucleon) unitarity approximation and we shall also employ the photoproduction data not only up to the threshold of double pion photoproduction, but well outside it. So we are calculating $\alpha+\beta$ defined in Eq. (2.19) from a forward ($t=0$) unsubtracted dispersion relation for the combination of amplitudes $A_4+\frac{1}{2}mA_5$:

$$\begin{aligned}(\alpha+\beta) &= \frac{m}{4\pi}\left[A_4^c(s=m^2,t=0)+\tfrac{1}{2}mA_5^c(s=m^2,t=0)\right]\\ &= \frac{m}{(2\pi)^2}\int_{(m+\mu)^2}^{\infty} ds'\,\frac{\text{Im}(A_4+\frac{1}{2}mA_5)(s',t=0)}{s'-m^2}\\ &= \frac{1}{2\pi^2}\int_{\omega_{\text{thr}}}^{\infty} d\omega\,\frac{\sigma^T(\omega)}{\omega^2}\,,\end{aligned} \tag{5.1}$$

where, within our approximations and in terms of quantities defined in Appendix A, we put in Eq. (5.1), instead of $\sigma^T(\omega)$, the total photoproduction cross section $\sigma^{\text{proton(neutron)}}_{(\gamma N\to\pi N)}$, with

$$\begin{aligned}\sigma^{\text{proton(neutron)}}_{(\gamma N\to\pi N)}(\omega) \simeq \frac{10^{-4}\pi(2\omega+m)}{m\omega^2}\{&6|A^{p(n)}_{S\,1/2}|^2+6|A^{p(n)}_{P\,1/2}|^2+12|A^{p(n)}_{P\,3/2}|^2+12|A^{p(n)}_{D\,3/2}|^2+9|B^{p(n)}_{P\,3/2}|^2+9|B^{p(n)}_{D\,3/2}|^2\\ &+4|A^{V3}_{S\,1/2}|^2+4|A^{V3}_{P\,1/2}|^2+8|A^{V3}_{P\,3/2}|^2+8|A^{V3}_{D\,3/2}|^2+6|B^{V3}_{P\,3/2}|^2+6|B^{V3}_{D\,3/2}|^2\}\,.\end{aligned} \tag{5.2}$$

The multipoles appearing above are again those to be found in tables in Refs. 14 and 15. The result of numerical integration up to the photon laboratory energy $\omega = 1210$ MeV gives

$$(\alpha+\beta)_{\text{proton}} = 11.50\times 10^{-4}\ \text{fm}^3, \tag{5.3a}$$

$$(\alpha+\beta)_{\text{neutron}} = 13.05\times 10^{-4}\ \text{fm}^3. \tag{5.3b}$$

The region from near threshold up to 250 MeV contributes 2.82 for the proton and 2.89 for the neutron, the region 250–750 MeV contributes 8.47 for the proton and 9.95 for the neutron, and the region 750–1210 MeV contributes 0.20 for the proton and 0.21 for the neutron, in units of 10^{-4} fm^3. The smallness of the contribution coming from the last energy region is to be noted.

The result expressed by Eq. (5.3a) is not exceedingly far from the value obtained by Damashek and Gilman,[3] Eq. (1.1).

It is perhaps instructive to note that if one saturates Eq. (5.1) only with the $N^*(\frac{3}{2},\frac{3}{2})$ ($M=1236$ MeV) resonance (see again Appendix C for notations and procedure), one gets

$$(\alpha+\beta)^{(N^*)}_{\text{proton(neutron)}} = \frac{e^2}{4\pi}\,(3G_E{}^2+G_M{}^2)\,\frac{1}{m(M^2-m^2)}\,, \tag{5.4}$$

that is

$$(\alpha+\beta)^{(N^*)}_{\text{proton(neutron)}} \simeq 8.6\times 10^{-4}\ \text{fm}^3 \tag{5.5}$$

for $G_M=3$ and $G_E=0$.

VI. $\pi^0\to 2\gamma$, $\eta^0\to 2\gamma$ DECAY RATES

The (supposed) superconvergence of certain Compton-scattering amplitudes at fixed s and large t implies sum rules for the residues of the t-channel pseudoscalar π^0 and η^0 meson poles F_π and F_η, related to the $\pi^0 \to 2\gamma$ and $\eta^0 \to 2\gamma$ decay rates τ_π and τ_η by

$$\tau_{\pi^0} = \frac{64\pi}{\mu^3 F_\pi^{\,2}}, \quad \tau_{\eta^0} = \frac{64\pi}{\mu_\eta^{\,3} F_\eta^{\,2}}. \tag{6.1}$$

In this section we shall reconsider the sum rule analyzed by Choudhury and Rajaraman,[9] which in terms of the Bardeen and Tung amplitudes A_i [Eqs. (2.4)] reads

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} du' \,\mathrm{Im}\left(A_2 - \frac{1}{m}A_3\right)(s=0,\, t'=2m^2-u',\, u')=0, \tag{6.2}$$

and we shall point out that the consideration of the two-pion continuum in the t channel could improve the agreement with the experimental number for the η-meson decay rate.

Separating in Eq. (6.2) the contributions coming from the t- and u-channels single-particle poles, and using s-u crossing, the sum rule to be evaluated can be rewritten as

$$2mg_{\pi NN}F_\pi \tau_3 - 2mg_{\eta NN}F_\eta + \frac{1+\tau_3}{2}C_1^{(p)} + \frac{1-\tau_3}{2}C_1^{(n)} + C_2^{p(n)} - C_3^{p(n)} = 0, \tag{6.3}$$

where

$$C_1^{(p)} = e^2[4 + 6\kappa_p + \kappa_p^{\,2}] = 1.65, \tag{6.4}$$

$$C_1^{(n)} = e^2\kappa_n^{\,2} = 0.33, \tag{6.5}$$

$$C_2^{p(n)} = \frac{m}{\pi}\int_{(m+\mu)^2}^{\infty} du' [mA_2^{\mathrm{I}} + A_3^{\mathrm{I}}]^{p(n)} (u',\, t'=2m^2-u',\, s=0), \tag{6.6}$$

$$C_3^{p(n)} = \frac{m}{\pi}\int_{4\mu^2}^{\infty} dt' [mA_2^{\mathrm{III}} - A_3^{\mathrm{III}}]^{p(n)} (s=0,\, t',\, u'=2m^2-t'). \tag{6.7}$$

The superscripts $p(n)$ in the absorptive parts under the above integrals refer to proton Compton scattering [corresponding to $\tau_3=+1$ in Eq. (6.3)] and to neutron Compton scattering ($\tau_3=-1$) in Eq. (6.3).

The continuum integrals (6.6) and (6.7) are estimated within the same two-particle unitarity scheme and using the same approximations and model as in Secs. III–V for the polarizabilities, with the only difference that now we are not working at fixed angle but at a fixed value of s (specifically at $s=0$). With the notations and definitions of Appendix A already used in the previous sections one finds

$$\begin{aligned} C_2^{p(n)} \simeq 10^{-4}\int_{\omega_{\mathrm{thr}}} \frac{d\omega}{m}\left(\frac{m}{\omega}\right)^5 \left(2\frac{\omega}{m}+1\right)^{3/2} \Bigg\{ & 8\left(\frac{\omega}{m}\right)^2 (-2|A_{S\,1/2}^{V3}|^2 + 2|A_{P\,1/2}^{V3}|^2 - 3|A_{S\,1/2}^{p(n)}|^2 + 3|A_{P\,1/2}^{p(n)}|^2) \\ & + 4\left(3 - 4\frac{\omega^2}{m^2}\right)(-2|A_{P\,3/2}^{V3}|^2 + 2|A_{D\,3/2}^{V3}|^2 - 3|A_{P\,3/2}^{p(n)}|^2 + 3|A_{D\,3/2}^{p(n)}|^2) \\ & - 12\left(\frac{\omega}{m}+1\right)\left(2\frac{\omega}{m}+1\right)^{1/2} [2\,\mathrm{Re}(B_{P\,3/2}^{V3\,*}A_{P\,3/2}^{V3} - B_{D\,3/2}^{V3\,*}A_{D\,3/2}^{V3}) \\ & \qquad + 3\,\mathrm{Re}(B_{P\,3/2}^{p(n)*}A_{P\,3/2}^{p(n)} - B_{D\,3/2}^{p(n)*}A_{D\,3/2}^{p(n)})] \\ & + 3\left(2\frac{\omega}{m}+1\right)^2 (-2|B_{P\,3/2}^{V3}|^2 + 2|B_{D\,3/2}^{V3}|^2 - 3|B_{P\,3/2}^{p(n)}|^2 + 3|B_{D\,3/2}^{p(n)}|^2)\Bigg\}. \end{aligned} \tag{6.8}$$

The result of numerical integration up to $\omega = 1210$ MeV is

$$C_2^p \simeq -0.90, \tag{6.9}$$

$$C_2^n \simeq -1.59. \tag{6.10}$$

We start now considering the t-channel contribution as given by Eq. (6.7). If only the two-pion state is retained in the corresponding unitarity sum, one finds $C_3^p = C_3^n$ and so, within this approximation, the value of F_π (unlike that of F_η) is not affected by the annihilation-channel exchanges.

The model for the evaluation of $C_3^{p(n)}$ is the same as the one used in Secs. III and IV (see also Appendix B for details). From Eqs. (B5), (B6), and (B10) one gets for the needed integrand the following expression:

$$[mA_2^{\mathrm{III}} - A_3^{\mathrm{III}}](s=0,\, t') = \left(\frac{m}{t'}\right)\left[\frac{\delta_1}{N^2} + \frac{\delta_2}{P'^2} - \frac{P\cdot K}{K^2}\left(\frac{\beta_1}{N^2} + \frac{\beta_2}{P'^2}\right)\right]_{s=0}. \tag{6.11}$$

Working out the above relation using Eqs. (B.11)–(B14) one gets, setting $t' = 4m^2 z$, $\rho = \mu/m$,

$$[mA_2^{\rm III} - A_3^{\rm III}](s=0, t'=4m^2 z) = \left(\frac{e^2}{4\pi}\frac{g_r^2}{4\pi}\right)\left(\frac{\rho^2}{m^3}\right)\frac{\pi}{z^{3/2}}\Bigg\{[(3+\rho^2)z - 4z^2 - \tfrac{1}{2}\rho^2]w_1(z) + [(1-\rho^2)z + \tfrac{1}{2}\rho^2]w_2(z) - \frac{(1-2z)}{z^{1/2}}\ln\frac{z+[z(z-\rho^2)]^{1/2}}{z-[z(z-\rho^2)]^{1/2}}\Bigg\}, \quad (6.12)$$

where

$$w_1(z) = \begin{cases} \dfrac{1}{\{-\rho^2 + z[4z-(1+\rho^2)]^2\}^{1/2}}\ln\dfrac{[4z^2 - z(3\rho^2+1)+\rho^2] + \{(z-\rho^2)[-\rho^2 + z(4z-1-\rho^2)^2]\}^{1/2}}{[4z^2 - z(3\rho^2+1)+\rho^2] - \{(z-\rho^2)[-\rho^2 + z(4z-1-\rho^2)^2]\}^{1/2}}, \\ \quad \text{if } \{-\rho^2 + z[4z-(1+\rho^2)]^2\} > 0 \\ \dfrac{2}{[4z^2 - z(3\rho^2+1)+\rho^2]}, \quad \text{if } \{-\rho^2 + z[4z-(1+\rho^2)]^2\} = 0 \\ \dfrac{2}{\{\rho^2 - z[4z-(1+\rho^2)]^2\}^{1/2}}\arctan\dfrac{\{(z-\rho^2)[\rho^2 - z(4z-1-\rho^2)^2]\}^{1/2}}{[4z^2 - z(3\rho^2+1)+\rho^2]}, \quad \text{if } \{-\rho^2 + z[4z-(1+\rho^2)]^2\} < 0 \end{cases} \quad (6.13)$$

and

$$w_2(z) = \begin{cases} \dfrac{1}{[z(1-\rho^2)^2 - \rho^2]^{1/2}}\ln\dfrac{[z(1+\rho^2)-\rho^2] - \{(z-\rho^2)[z(1-\rho^2)^2 - \rho^2]\}^{1/2}}{[z(1+\rho^2)-\rho^2] + \{(z-\rho^2)[z(1-\rho^2)^2 - \rho^2]\}^{1/2}}, \\ \quad \text{if } z(1-\rho^2)^2 - \rho^2 > 0 \\ \dfrac{2}{[z(1+\rho^2)-\rho^2]}, \quad \text{if } z(1-\rho^2)^2 - \rho^2 = 0 \\ -\dfrac{2}{[\rho^2 - z(1-\rho^2)^2]^{1/2}}\arctan\dfrac{\{(z-\rho^2)[\rho^2 - z(1-\rho^2)^2]\}^{1/2}}{[z(1+\rho^2)-\rho^2]}, \quad \text{if } z(1-\rho^2)^2 - \rho^2 < 0. \end{cases} \quad (6.14)$$

The numerical integration up to $t' = 9\mu^2$ (the threshold of the three-pion state) gives

$$C_3^p = C_3^n \equiv C_3 = 0.40. \quad (6.15)$$

Collecting the results one has

$$mg_{\pi NN}F_\pi \simeq [\tfrac{1}{4}(C_1^n - C_1^p) + \tfrac{1}{4}(C_2^n - C_2^p)] \simeq (-0.33 - 0.17) = -0.50, \quad (6.16)$$

$$mg_{\eta NN}F_\eta \simeq \left[\tfrac{1}{4}(C_1^p + C_1^n) + \tfrac{1}{4}(C_2^p + C_2^n) - \frac{C_3}{2}\right] \simeq (0.49 - 0.62 - 0.22) = -0.33. \quad (6.17)$$

Taking for purposes of orientation $g_{\eta NN} = g_{\pi NN}$, Eqs. (6.16) and (6.17) imply

$$\tau_{\pi^0} = 0.34 \times 10^{-16}\ \text{sec}, \quad (6.18)$$

$$\tau_{\eta^0} = 1.2 \times 10^{-18}\ \text{sec}, \quad (6.19)$$

to be compared with

$$\tau_{\pi^0}^{\rm exp} = 0.85 \times 10^{-16}\ \text{sec} \quad (6.20)$$

for $\pi^0 \to 2\gamma$ and the experimental values

$$\tau_{\eta^0}^{\rm exp} = 0.63 \times 10^{-18}\ \text{sec}, \quad (6.21)$$

given in Ref. 18, and

$$\tau_{\eta^0}^{\rm exp} = 2.06 \times 10^{-18}\ \text{sec}, \quad (6.22)$$

given in Ref. 19.

Using the approximation of taking in the photoproduction integral all multipoles except E_{0+} the same for proton and neutron, the authors of Ref. 15 found for τ_{π^0} a value ($\tau_{\pi^0} = 0.53 \times 10^{-16}$ sec) closer than ours to the experimental number, while the result of their analysis for τ_{η^0} was $\tau_{\eta^0} = 10.3 \times 10^{-18}$ sec. So we see that the contribution of the continuum to the superconvergence sum rule Eq. (6.2) is not negligible and, moreover, helps lower the theoretical prediction for the $\eta^0 \to 2\gamma$ lifetime.

VII. DISCUSSION OF THE RESULTS AND CONCLUSIONS

We have seen that the calculations of $\alpha - \beta$ presented in Secs. III and IV gave results very close to each other and both are in qualitative agreement with the actual experimental values, Eqs. (1.2). Apart from the asymptotic behavior of the relevant Compton amplitudes, taken as given, which ensures the validity of the dispersion representations used, an important dynamical ingredient to get numbers has been the model adopted for the annihilation-channel contributions for the absorptive parts of the corresponding amplitudes. The need for a careful consideration of the t-channel (2π) continuum in the case of nucleon polarizabilities received further support from the results of Sec. VI where a great existing discrepancy between the prediction (based on a superconvergence sum rule) for the decay rate of $\eta^0 \to 2\gamma$ and the experimental data has been mitigated.

However, in spite of the approximations made, our main conclusion, e.g., the presence of a large positive contribution to $\alpha - \beta$ from the annihilation channel-exchanges remains. Indeed, it has been shown that if one restricts the integration region over the t-channel singularities only to the interval from $4\mu^2$ to $9\mu^2$ where the t-channel two-particle unitarity becomes exact, one still gets the ordering $\alpha > \beta$. Of course, even in this small integration region we had to make the strong approximation of keeping in the amplitudes for $N\bar{N} \to \pi\pi$ and $\pi\pi \to \gamma\gamma$ only their Born poles plus the needed constants (to ensure the low-energy theorem for the last process), but in any case, we computed exactly an important piece which is there and found out that at least this exactly computable quantity cannot be blindly neglected. A more detailed investigation of the t-channel exchanges, including the contribution of meson resonances, would be desirable, although such an attempt should perhaps be postponed until better experimental determinations of the nucleon dynamic electromagnetic polarizabilities have been performed. The experimental number Eqs. (1.2) taken for reference in this work could still be quite uncertain. We note, for instance, the slight disagreement existing between the determination of $\alpha + \beta$ in terms of total photoabsorption data and Eqs. (1.2). Also in this context we mention that because the energies considered by the authors of Ref. 5 may still be too high to allow a precise determination of α and β from the third-order low-energy expression of the differential cross section alone, in a subsequent paper Baranov, Filkov, and Starkov[20] extracted the proton dynamic polarizabilities from a fit of the data from Ref. 5 with a modified formula for $d\sigma/d\Omega$ which includes also the contribution to the differential cross section of the π^0 meson pole. They found in this way

$$\alpha_{\text{proton}} = (13.9 \pm 2.1) \times 10^{-4}\ \text{fm}^3,$$

$$(\alpha - \beta)_{\text{proton}} = (19.9 \pm 4.4) \times 10^{-4}\ \text{fm}^3.$$

Considering the present experimental situation as well as the difficulties of the theoretical approaches to determine α and β separately, it would be perhaps safer now to look for model-independent results involving these structure-dependent parameters, either as done by Bernabeu and Tarrach[21] or by trying to find rigorous analyticity bounds for them.[22] For the time being we conclude that if the dynamic electric polarizability α is indeed greater than the magnetic one β, then the t-channel exchanges should be strongly responsible for this fact. The question of possible large contributions coming from the singularities in the annihilation channel has also been touched on by Ericson,[6] who considered the exchange of a $\epsilon(0^+)$ resonance.

Our above-mentioned conclusion is also in agreement with the results of a different analysis recently done by Akhmedov and Filkov[23] in which dispersion relations for the relevant amplitudes in s at fixed t and in s and t at fixed u are consistently used to eliminate unknown subtractions. The more complicated analytical structure at fixed angle in our approach has the advantage that at $\theta = 180°$ the pion photoproduction amplitudes appear under the dispersion integral at physical values, the evaluation of s- and u-channel contributions being so less affected by extrapolations.

ACKNOWLEDGMENTS

We are much indebted to Dr. S. B. Gerasimov (J.I.N.R., Dubna) and to Dr. L. V. Filkov (Moscow, Lebedev Physical Institute) for very useful discussions concerning this work.

APPENDIX A

From the two-particle (pion + nucleon) unitarity condition in the s channel, one has the following relation between the imaginary parts of the Compton-scattering helicity amplitudes and the partial-wave amplitudes for pion photoproduction:

$$\begin{aligned}\operatorname{Im} f_{\lambda_c \lambda_d, \lambda_a \lambda_b} = 2\pi\, |\vec{q}|\, \frac{(s)^{1/2}}{m k_0^{\,2}} & \\ \times \sum_{\lambda' \tau'} \sum_{J} & (J+\tfrac{1}{2}) \langle J\lambda_c \lambda_d |\, {}^{\tau'}T^{+}(s)\, | J\lambda' 0\rangle \\ & \times \langle J\lambda' 0 |\, {}^{\tau'}T(s)\, | J\lambda_a \lambda_b \rangle\, d^{J}_{\lambda,\mu}(\theta)\,, \end{aligned} \tag{A1}$$

where $\lambda = \lambda_a - \lambda_b$, $\mu = \lambda_c - \lambda_d$, τ' is the isospin superscript, and $\vec{q}$ is the pion momentum in the barycentric system of the intermediate πN state. This exact relation up to the threshold of double pion photoproduction $[s = (m + 2\mu)^2]$ has been, however, used all over the integration region, considering that this extrapolation could not drastically modify the results. Also, in the sum over the total angular momentum J, only waves with $J = \frac{1}{2}$ and $J = \frac{3}{2}$ have been retained.

To meet the notations and units used by the authors of Ref. 15 in their tables for the pion-photoproduction partial waves, we introduce the following definition:

$$\langle J\lambda' 0 | {}^{\tau'}T(s) | J\lambda_a \lambda_b \rangle = 10^{-2} \times 2 \left(\frac{k_0}{|\vec{q}|}\right)^{1/2} \times A^J_{-\lambda', -\lambda}(s) . \quad \text{(A2)}$$

Noting that in the center-of-mass system

$$k_0 = \frac{s - m^2}{2s^{1/2}},$$

one has then instead of Eq. (A1) the relation

$$\operatorname{Im} f_{\lambda_c \lambda_d, \lambda_a \lambda_b} = 10^{-4} \times \frac{16\pi s}{m(s - m^2)} \times \sum_{\lambda' \tau'} \sum_J (J + \tfrac{1}{2}) [{}^{\tau'}A^J_{-\lambda', -\mu}(s)]^* \times [{}^{\tau'}A_{-\lambda', -\lambda}(s)] \, d^J_{\lambda, \mu}(\theta). \quad \text{(A3)}$$

We mention also the phase rule

$${}^{\tau'}A^J_{-\mu, -\lambda} = -{}^{\tau'}A^J_{\mu \lambda} . \quad \text{(A4)}$$

To perform the isotopic spin summations one uses for proton Compton scattering the prescription

$$\sum_{\tau = +, 0} | {}^{\tau}A |^2 = |A_+|^2 + |A_0|^2 = \tfrac{1}{2}[2|A^{V3}|^2 + 3|A^p|^2], \quad \text{(A5)}$$

vhere, in the notations of Ref. 15,

$$\begin{aligned} A_+ &\equiv \langle \pi^+ n | A | \gamma p \rangle = -(\tfrac{1}{3})^{1/2} A^{V3} + A^p, \\ A_0 &\equiv \langle \pi^0 p | A | \gamma p \rangle = (\tfrac{2}{3})^{1/2} A^{V3} + (\tfrac{1}{2})^{1/2} A^p, \end{aligned} \quad \text{(A6)}$$

and for neutron Compton scattering

$$\sum_{\tau = -, n0} | {}^{\tau}A |^2 = |A_-|^2 + |A_{n0}|^2 = \tfrac{1}{2}[2|A^{V3}|^2 + 3|A^n|^2], \quad \text{(A7)}$$

where

$$\begin{aligned} A_- &\equiv \langle \pi^- p | A | \gamma n \rangle = (\tfrac{1}{3})^{1/2} A^{V3} + A^n, \\ A_{n0} &\equiv \langle \pi^0 n | A | \gamma n \rangle = (\tfrac{2}{3})^{1/2} A^{V3} - (\tfrac{1}{2})^{1/2} A^n . \end{aligned} \quad \text{(A8)}$$

The connection between the quantities appearing in the s-channel integrands in this paper and $A^J_{\mu\lambda}$ is furnished (for $J = \frac{1}{2}, \frac{3}{2}$) by the following set of linear combinations:

$$\begin{aligned} &A^{1/2}_{1/2,1/2} = -(\tfrac{1}{2})^{1/2}(A_{S1/2} - A_{P1/2}), && A^{3/2}_{1/2,3/2} = (\tfrac{3}{8})^{1/2}(B_{P3/2} - B_{D3/2}), \\ &A^{3/2}_{1/2,1/2} = -(\tfrac{1}{2})^{1/2}(A_{P3/2} - A_{D3/2}), && A^{3/2}_{-1/2,3/2} = (\tfrac{3}{8})^{1/2}(B_{P3/2} + B_{D3/2}), \\ &A^{1/2}_{-1/2,1/2} = -(\tfrac{1}{2})^{1/2}(A_{S1/2} + A_{P1/2}), && A^{1/2}_{1/2,3/2} = A^{1/2}_{-1/2,3/2} = 0, \\ &A^{3/2}_{-1/2,1/2} = -(\tfrac{1}{2})^{1/2}(A_{P3/2} + A_{D3/2}) . \end{aligned} \quad \text{(A9)}$$

The relationship between the quantities $A_{S1/2}$, $A_{P1/2}$, $A_{P3/2}$, $B_{P3/2}$, etc. (that is, $A_{l, J=l\pm1/2} \equiv A_{l\pm}$, $B_{l, J=l\pm1/2} \equiv B_{l\pm}$) tabulated in Ref. 15 and the Chew-Goldberger-Low-Nambu[24] multipoles $E_{l\pm}$, $M_{l\pm}$ is given by

$$\begin{aligned} &[10^{-2}/(|\vec{q}| k_0)^{1/2}] A_{l+} = \tfrac{1}{2}[(l+2)E_{l+} + lM_{l+}], \\ &[10^{-2}/(|\vec{q}| k_0)^{1/2}] A_{(l+1)-} = \tfrac{1}{2}[(l+2)M_{(l+1)-} - lE_{(l+1)-}], \\ &[10^{-2}/(|\vec{q}| k_0)^{1/2}] B_{l+} = -M_{l+} + E_{l+}, \\ &[10^{-2}/(|\vec{q}| k_0)^{1/2}] B_{(l+1)-} = M_{(l+1)-} + E_{(l+1)-} . \end{aligned} \quad \text{(A10)}$$

APPENDIX B

In the framework of the model for the t-channel contributions to the different sum rules considered in this paper, the quantities $A_i^{\text{III}(2\pi)}$ we are interested in should be picked up[13] from the following expression for the t-channel absorptive part of the tensor $\bar{u}M_{\mu\nu}u$ defined in Eq. (2.1):

$$\begin{aligned} \operatorname{Abs}(\bar{u}M^{(2\pi)}_{\mu\nu}u) = {}& \tfrac{1}{2}(2\pi)^7 m^{-1/2}(2k_0' p_0)^{1/2} \bar{u}(p') \\ & \times \sum (2\pi)^{-9/2}(2k_0' 2r_0 2l_0)^{-1/2}(-2e^2)\left[-g_{\mu\nu} + \frac{2l_\nu r_\mu}{\mu^2 - (l+k)^2} + \frac{2l_\mu r_\nu}{\mu^2 - (l-k')^2}\right] \\ & \times (-1)(2\pi)^{-9/2} m^{1/2}(2r_0 2l_0 p_0)^{-1/2} g_r^{\,2} \left[\frac{\gamma\cdot r\,(1+\tau_3)}{(p-r)^2 - m^2} + \frac{\gamma\cdot l(1-\tau_3)}{(p-l)^2 - m^2}\right] u(p)\, \delta^4(r + l - Q) \end{aligned} \quad \text{(B1)}$$

where r and l are the four-momenta of the intermediate π^+, π^-, g_r is the pion-nucleon coupling constant ($g_r^2/4\pi \simeq 14.5$), τ_3 the usual isospin Pauli matrix, and the sum extends over the allowed momenta of the intermediate pions.

If one works in the basis of the four mutually orthogonal vectors K_μ, Q_μ, P'_μ, and N_μ, with

$$P'_\mu = P_\mu - \frac{P \cdot K}{K^2} K_\mu\,, \quad N_\mu = \epsilon_{\mu\nu\delta\lambda} P'_\nu K_\delta Q_\lambda\,, \qquad \text{(B2)}$$

$A_1^{\text{III}(2\pi)}$ can be quickly found by a straightforward projection using certain properties of $M_{\mu\nu}^{(2\pi)}$. Projecting and identifying the coefficients in the relations

$$\operatorname{Abs} N_\mu M_{\mu\nu}^{(2\pi)} N_\nu = \beta_1(\gamma \cdot P') + \delta_1(\gamma \cdot K) + \rho_1(\gamma \cdot Q)\,,$$

$$\operatorname{Abs} P'_\mu M_{\mu\nu}^{(2\pi)} P'_\nu = \beta_2(\gamma \cdot P') + \delta_2(\gamma \cdot K) + \rho_2(\gamma \cdot Q)\,, \qquad \text{(B3)}$$

$$\operatorname{Abs} P'_\mu M_{\mu\nu}^{(2\pi)} N_\nu = \alpha_3(\gamma \cdot N) = \operatorname{Abs} N_\mu M_{\mu\nu}^{(2\pi)} P'_\nu\,,$$

one gets

$$A_1^{\text{III}(2\pi)}(s,t) = \frac{1}{t}\left[2(g_1+g_2) - \frac{(s-m^2)+\frac{1}{2}t}{m^2}(g_3+g_4)\right], \qquad \text{(B4)}$$

$$A_2^{\text{III}(2\pi)}(s,t) = \frac{1}{mt}\left[4g_5 + \frac{2(s-m^2)+t}{2m^2}(g_3+g_4)\right], \qquad \text{(B5)}$$

$$A_3^{\text{III}(2\pi)}(s,t) = \frac{g_3+g_4}{2m^3} \qquad \text{(B6)}$$

$$A_4^{\text{III}(2\pi)}(s,t) = \frac{2(4m^2-t)g_6 - [2(s-m^2)+t](g_3-g_4)}{2m[(s-m^2)^2+st]}\,, \qquad \text{(B7)}$$

$$A_5^{\text{III}(2\pi)}(s,t) = \frac{4(g_1-g_2)-2g_6}{(s-m^2)^2+st}\,, \qquad \text{(B8)}$$

$$A_6^{\text{III}(2\pi)}(s,t) = \frac{t(g_3-g_4)+2[2(s-m^2)+t]g_6}{m[(s-m^2)^2+st]}\,, \qquad \text{(B9)}$$

where

$$g_1 = m\frac{\beta_2}{P'^2}\,, \quad g_2 = \frac{m\beta_1}{N^2}\,,$$

$$g_3 = -\frac{m}{P'^2}\left(\delta_2 - \beta_2\frac{P\cdot K}{K^2}\right), \quad g_4 = -\frac{m}{N^2}\left(\delta_1 - \beta_1\frac{P\cdot K}{K^2}\right), \qquad \text{(B10)}$$

$$g_5 = 0\,, \quad g_6 = \frac{mQ^2}{(P'^2)^{1/2}(N^2)^{1/2}}\left(\frac{\alpha_3}{2}\right),$$

and

$$\beta_1 = \left(-\frac{e^2}{\pi}\frac{g_r^2}{4\pi}\right)\frac{1}{P'^2}\int\frac{d^3l}{2l_0}\,\delta(Q^2-2Ql)\left[N^2-(l\cdot N)^2\left(\frac{1}{l\cdot k}-\frac{1}{l\cdot k'}\right)\right]$$
$$\times(l\cdot P')\left[-\frac{(1+\tau_3)}{\mu^2-2p\cdot Q+2p\cdot l}+\frac{(1-\tau_3)}{\mu^2-2p\cdot l}\right], \qquad \text{(B11)}$$

$$\beta_2 = \left(-\frac{e^2}{\pi}\frac{g_r^2}{4\pi}\right)\frac{1}{P'^2}\int\frac{d^3l}{2l_0}\,\delta(Q^2-2Ql)\left[N^2-(l\cdot N)^2\left(\frac{1}{l\cdot k}-\frac{1}{l\cdot k'}\right)\right]$$
$$\times(l\cdot P')\left[-\frac{(1+\tau_3)}{\mu^2-2p\cdot Q+2p\cdot l}+\frac{(1-\tau_3)}{\mu^2-2p\cdot l}\right], \qquad \text{(B12)}$$

$$\delta_1 = \left(-\frac{e^2}{\pi}\frac{g_r^2}{4\pi}\right)\frac{1}{K^2}\int\frac{d^3l}{2l_0}\,\delta(Q^2-2Ql)\left[N^2-(l\cdot N)^2\left(\frac{1}{l\cdot k}-\frac{1}{l\cdot k'}\right)\right],$$
$$\times(l\cdot K)\left[-\frac{(1+\tau_3)}{\mu^2-2p\cdot Q+2p\cdot l}+\frac{(1-\tau_3)}{\mu^2-2p\cdot l}\right], \qquad \text{(B13)}$$

$$\delta_2 = \left(-\frac{e^2}{\pi}\frac{g_r^2}{K^2}\right)\frac{1}{K^2}\int\frac{d^3l}{2l_0}\,\delta(Q^2-2Ql)\left[P'^2-(l\cdot P')^2\left(\frac{1}{l\cdot k}-\frac{1}{l\cdot k'}\right)\right]$$
$$\times(l\cdot K)\left[-\frac{(1+\tau_3)}{\mu^2-2p\cdot Q+2p\cdot l}+\frac{(1-\tau_3)}{\mu^2-2p\cdot l}\right], \qquad \text{(B14)}$$

$$\alpha_3 = \left(+\frac{e^2}{\pi}\frac{g_r^2}{4\pi}\right)\frac{1}{N^2}\int\frac{d^3l}{2l_0}\,\delta(Q^2-2Ql)\;(l\cdot P')(l\cdot N)\left(\frac{1}{l\cdot k}-\frac{1}{l\cdot k'}\right)\Bigg]$$
$$\times(l\cdot N)\left[-\frac{(1+\tau_3)}{\mu^2-2p\cdot Q+2p\cdot l}+\frac{(1-\tau_3)}{\mu^2-2p\cdot l}\right]. \qquad \text{(B15)}$$

In the backward dispersion relations considered in Secs. III and IV one needs some of the above expressions of the absorptive parts $A_i^{\mathrm{III}(2\pi)}(s,t)$ for the values

$$s \to \frac{t_+(t')}{2} = m^2 - \frac{t'}{2} + \frac{i}{2}\,[t'(4m^2 - t')]^{1/2}, \qquad t \to t' \tag{B16}$$

of their arguments (s,t). From Eqs. (B4) and (B10) we then have for $A_1^{\mathrm{III}(2\pi)}$

$$A_1^{\mathrm{III}(2\pi)}\left[\frac{t_+(t')}{2}, t'\right] = \left(\frac{1}{2m}\right)\left\{\left(\frac{\beta_1}{N^2} + \frac{\beta_2}{P'^2}\right) + i\frac{[t'(4m^2 - t')]^{1/2}}{t'}\left(\frac{\delta_1}{N^2} + \frac{\delta_2}{P'^2}\right)\right\}, \tag{B17}$$

and from Eqs. (B4), (B6), and (B10) we have for $A_1^{\mathrm{III}(2\pi)} - A_3^{\mathrm{III}(2\pi)}$ (at the same arguments)

$$A_1^{\mathrm{III}(2\pi)} - A_3^{\mathrm{III}(2\pi)} = \left(\frac{1}{2m}\right)\left[1 + i\frac{[t'(4m^2 - t')]^{1/2}}{t'}\right]\left[\left(\frac{\beta_1}{N^2} + \frac{\beta_2}{P'^2}\right) + \left(\frac{\delta_1}{N^2} + \frac{\delta_2}{P'^2}\right)\right]. \tag{B18}$$

Working in the center-of-mass frame of the annihilation channel, we obtain from Eqs. (B11)–(B14) the following expressions:

$$\frac{\beta_1}{N^2} + \frac{\beta_2}{P'^2} = -\left(\frac{e^2}{4\pi}g_r^{\,2}\right)\frac{(t' - 4\mu^2)^{1/2}}{(t')^{1/2}(4m^2 - t')}\int_{-1}^{+1} dz\left[\frac{(1 - z^2)(z^2 - \eta_2^{\,2})}{(z^2 + \eta_2^{\,2})^2} + \frac{(1 - z^2)^2(z^2 - \eta_2^{\,2})}{(z^2 - \eta_1^{\,2})(z^2 + \eta_2^{\,2})}\right], \tag{B19}$$

$$\frac{\delta_1}{N^2} + \frac{\delta_2}{P'^2} = -i\left(\frac{e^2}{4\pi}g_r^{\,2}\right)\frac{2(t' - 4\mu^2)^{1/2}}{t'(4m^2 - t')^{1/2}}\int_{-1}^{+1} dz\,\frac{z^2}{z^2 + \eta_2^{\,2}}\left(1 + \frac{1 - z^2}{z^2 - \eta_1^{\,2}}\right), \tag{B20}$$

where

$$\eta_1 = \left(\frac{t'}{t' - 4\mu^2}\right)^{1/2}, \tag{B21}$$

$$\eta_2 = \frac{2\mu^2 - t'}{[(4m^2 - t')(t' - 4\mu^2)]^{1/2}}. \tag{B22}$$

APPENDIX C

We shall display here the $N^*(\frac{3}{2},\frac{3}{2})$(1236 MeV) resonance contributions to the s-channel absorptive parts of the nucleon Compton-scattering amplitudes A_i defined in Eqs. (2.4). They are picked up by a straightforward but tedious projection from the expression

$$\begin{aligned}\mathrm{Abs}\bar{u}(p')M_{\mu\nu}^{(N^*)}(p',k',p,k)u(p) = {}& \tfrac{2}{3}e^2\pi\delta(s - m^2)\bar{u}(p')\{[-k'_\alpha\gamma_\mu + (k'\gamma)g_{\alpha\mu}]G_1(0) \\ & + \tfrac{1}{2}[k'_\alpha(2p'_\mu + k'_\mu) - k'(2p' + k')g_{\alpha\mu}]G_2(0)\}\gamma_5(M + \hat{P} + \hat{K}) \\ & \times\left\{-g_{\alpha\beta} + \tfrac{1}{3}\gamma_\alpha\gamma_\beta + \frac{1}{3M}[\gamma_\alpha(P_\beta + K_\beta) - \gamma_\beta(P_\alpha + K_\alpha)] + \frac{2}{3M^2}(P_\alpha + K_\alpha)(P_\beta + K_\beta)\right\}\gamma_5 \\ & \times\{[-k_\beta\gamma_\nu + (k\cdot\gamma)g_{\beta\nu}]G_1(0) + \tfrac{1}{2}[k_\beta(2p_\nu + k_\nu) - k(2p + k)g_{\beta\nu}]G_2(0)\}u(p),\end{aligned} \tag{C1}$$

where M is the mass of the resonance and the transition form factors G_i are defined through the following form of the NN^* electromagnetic current vertex[16]:

$$\begin{aligned}\langle N^*(p + k)|j_\mu(0)|N(p)\rangle = {}& -ie\bar{u}_\beta(p + k)\{G_1(k^2)[k_\beta\gamma_\mu - (k\cdot\gamma)g_{\beta\mu}] + G_2(k^2)\tfrac{1}{2}[k_\beta(2p + k)_\mu - k(2p + k)g_{\beta\mu}] \\ & + G_3(k^2)(k_\beta k_\mu - k^2 g_{\beta\mu})\}\gamma_5 u(p).\end{aligned} \tag{C2}$$

The isospin convention is such that $(\frac{2}{3})^{1/2}$ should be introduced in the above relation for γpN^{*+} and this accounts for the factor $\frac{2}{3}$ in Eq. (C1). Our γ_5 is $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$. The form factor G_3 does not contribute to real $(k^2 = k'^2 = 0)$ Compton scattering and this is why it does not appear in Eq. (C1).

Below, for completeness, we list the N^* resonance contributions to all the six absorptive parts of the amplitudes A_i:

$$\mathrm{Im}A_1^{(N^*)}(s,t)=-\tfrac{2}{3}e^2\pi\delta(s-M^2)\Big[G_1^2(0)\frac{2}{3M}(3M^2-m^2)+G_1(0)G_2(0)\Big(2M^2-\frac{2m^2}{3}-\frac{mM}{3}-\frac{M^3}{m}-\frac{Mt}{2m}\Big)$$
$$+G_2^2(0)\Big(\frac{5M^3}{6}-\frac{M^4}{2m}-\frac{m^2M}{6}-\frac{mM^2}{6}-\frac{M^2t}{4m}\Big)\Big], \tag{C3}$$

$$\mathrm{Im}A_2^{(N^*)}(s,t)=-\tfrac{2}{3}e^2\pi\delta(s-M^2)\Big\{G_1^2(0)\frac{2}{3Mm}(3M^2+m^2)+G_1(0)G_2(0)\Big[\frac{1}{3m^2}(3M^2+6M^2m-5Mm^2+2m^3)+\frac{Mt}{2m^2}\Big]$$
$$+G_2^2(0)\frac{M}{2m^2}\Big[\frac{Mt}{2}+\tfrac{1}{3}(M-m)(3M^2+4mM-m^2)\Big]\Big\}, \tag{C4}$$

$$\mathrm{Im}A_3^{(N^*)}(s,t)=-\frac{2e^2}{3}\pi\delta(s-M^2)\Big[G_1^2(0)\,\frac{2m^2}{3M}+G_1(0)G_2(0)\Big(\frac{2m^2}{3}-\frac{2mM}{3}+\frac{Mt}{2m}\Big)$$
$$+G_2^2(0)\,\frac{M}{6}\Big(M^2+m^2-2mM+\frac{3Mt}{2m}\Big)\Big], \tag{C5}$$

$$\mathrm{Im}A_4^{(N^*)}(s,t)=-\frac{2e^2}{3}\pi\delta(s-M^2)\Big\{G_1^2(0)\Big(-\frac{1}{3M^2}\Big)(3M^2+m^2)+G_1(0)G_2(0)\Big[-\frac{M^2+m^2}{3M}-\frac{2}{3}(M+m)\Big]$$
$$+G_2^2(0)\Big(-\frac{M^2}{3}+\frac{2m^2}{3}-\frac{mM}{3}-\frac{t}{4}\Big)\Big\}, \tag{C6}$$

$$\mathrm{Im}A_5^{(N^*)}(s,t)=-\frac{2e^2}{3}\pi\delta(s-M^2)[4G_1(0)G_2(0)+G_2^2(0)2(M-m)], \tag{C7}$$

$$\mathrm{Im}A_6^{(N^*)}(s,t)=-\frac{2e^2}{3}\pi\delta(s-M^2)\Big\{G_1^2(0)\Big(-\frac{2}{3M^2}\Big)(3M^2-m^2)+G_1(0)G_2(0)\Big(\frac{2m^2}{3M}+\frac{4m}{3}-2M\Big)$$
$$+G_2^2(0)\Big(-\frac{M^2+m^2}{3}+\frac{2mM}{3}+\frac{t}{2}\Big)\Big\}. \tag{C8}$$

The relation between the form factors G_1, G_2 and those appearing in the paper, G_M and G_E, is

$$G_M=\frac{m(3M+m)}{3M}G_1+\frac{m(M-m)}{3}G_2,\quad G_E=\frac{m(M-m)}{3M}G_1+\frac{m(M-m)}{3}G_2. \tag{C9}$$

We also note here that from the analysis of Jones and Scadron[16] the value of G_M is around 3 while G_E is very small with respect to G_M.

[1]F. E. Low, Phys. Rev. 96, 1428 (1954); M. Gell-Mann and M. L. Goldberger, *ibid.* 96, 1433 (1954); S. Rai Choudhury and D. Z. Freedman, *ibid.* 168, 1739 (1968).

[2]M. Gell-Mann, M. L. Goldberger, and W. Thirring, Phys. Rev. 95, 1612 (1954).

[3]M. Damashek and F. J. Gilman, Phys. Rev. D 1, 1319 (1970).

[4]V. I. Goldansky *et al.*, Nucl. Phys. 18, 473 (1960).

[5]P. Baranov *et al.*, Phys. Lett. B52, 122 (1974).

[6]See the review of T. E. O. Ericson, CERN Report No. TH/2021 (1975), which also contains other references to previous work in this field.

[7]J. Bernabeu, T. E. O. Ericson, and C. Ferro Fontan, Phys. Lett. 49B, 381 (1974).

[8]I. Guiaşu and E. E. Radescu, Phys. Lett. B (to be published).

[9]S. Rai Choudhury and R. Rajaraman, Phys. Rev. 169, 1218 (1968).

[10]H. D. I. Abarbanel and M. L. Goldberger, Phys. Rev. 165, 1594 (1968).

[11]W. A. Bardeen and Wu-Ki Tung, Phys. Rev. 173, 1423 (1968).

[12]A. C. Hearn and E. Leader, Phys. Rev. 126, 789 (1962).

[13]D. Holliday, Ann. Phys. (N.Y.) 24, 289 (1963); 24, 319 (1963).

[14]P. Noelle, W. Pfeil, and D. Schwela, Nucl. Phys. B26, 461 (1971); P. Noelle and W. Pfeil, Nucl. Phys. B31, 1 (1971).

[15]R. G. Moorhouse, H. Oberlack, and A. H. Rosenfeld, Phys. Rev. D 9, 1 (1974).

[16]H. F. Jones and M. D. Scadron, Ann. Phys. (N.Y.) 81, 1 (1973).
[17]P. S. Baranov, L. V. Filkov, and G. A. Sokol, Fortschr. Phys. 16, 595 (1968).
[18]Particle Data Group, Phys. Lett. 50B, 1 (1974).
[19]A. Browman *et al.*, Phys. Rev. Lett. 32, 1067 (1974).
[20]P. Baranov, L. V. Filkov, and Shtarkov, Zh. Eksp. Teor. Fiz. Pis'ma Red. 20, 762 (1974) [JETP Lett. 20, 353 (1974)].
[21]J. Bernabeu and R. Tarrach, Phys. Lett. 55B, 183 (1975).
[22]I. Guiaşu and E. E. Radescu, Phys. Rev. D 10, 357 (1974).
[23]D. M. Akhmedov and L. V. Filkov, Kratk. soobshch. Fiz. 1, 13 (1975). In the notations used in our paper, the correct version of the sum rule for $\alpha-\beta$ analyzed by these authors comes from a fixed-u ($u=m^2$) unsubtracted dispersion relation for $A_1^c - A_3^c$.
[24]G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. 106, 1345 (1957).