Diffraction scattering and group contraction

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We employ the method of group contraction in order to study diffraction scattering at high energies. The elastic scattering amplitude is assumed to be pure imaginary. We then obtain that $d\sigma/dt = (d\sigma/dt)_0 e^{bt}$, where $b = \sigma_t^{\,2}/(16\pi\sigma_{el})$. This simple result explains in a universal manner $\pi^{\pm}p$, $K^{\pm}p$, $\bar{p}p$, and pp scattering data over a wide energy range.

I. INTRODUCTION

Recently several models have been proposed in order to explain high-energy collisions of hadrons. Some of the interesting models are the Regge-pole model and the impact-parameter model. The former is a consequence of the S-matrix theory of hadrons whereas the latter follows from geometrical considerations and receives support from field-theoretic calculations. The impact-parameter representation has been useful in the study of high-energy diffraction scattering, and it is useful in the forward direction at high energies.

The purpose of this paper is to study high-energy diffraction scattering using the method of group contraction. As we shall see later, it turns out that at high energies the group O(3) is contracted to E(2) in an appropriate kinematical region. This region corresponds to the near-forward direction in the scattering process. Thus, one is able to study the scattering amplitude in the E(2) plane and make an expansion in suitable variables in this plane. We feel that this also provides an alternative basis for the impact-parameter representation.

This paper is arranged as follows: In Sec. II we briefly recapitulate the method of group contraction. The necessary results are derived in Sec. III. We discuss various aspects of this model and compare the predictions of the model with experimental data in Sec. IV.

II. GROUP CONTRACTION

At high energies, for small scattering angles, the polar region of the projectile, i.e., the near-forward direction in the c.m. frame of reference, can be viewed as a plane. If we take a sphere of radius c, with polar angles θ and φ, and consider the near-forward directions, with $c\theta$ remaining finite as $c\to\infty$ and $\theta\to 0$, we obtain the Euclidean plane in two dimensions. In this manner, we can obtain the representations of the Euclidean group E(2) by the contraction[1] of the group O(3). We shall use this group contraction in the context of the little group O(3) of O(3,1) going over to the little group E(2) of O(3,1) at high energies. Let J_i be the generators of O(3) and define $J_2 = cQ_a$, $J_1 = -cQ_b$, with J_3 unaltered. Then the standard commutation relations $[J_i, J_j] = i\epsilon_{ijk}J_k$ are transformed to $[J_3, Q_a] = iQ_b$, $[J_3, Q_b] = -iQ_a$, and $[Q_a, Q_b] = 0$ in the limit of $c\to\infty$. Clearly, Q_a and Q_b are momentum operators in E(2) and J_3 is the usual rotation operator. Let us next consider the rotation $R_3(\varphi)R_2(\theta)R_3(-\varphi)$ and take the limit $\theta\to 0$ and $c\to\infty$, such that $c\theta$ remains finite. Then we get

$$R_3(\varphi)R_2(\theta)R_3(-\varphi) \simeq T(c\theta\cos\varphi, c\theta\sin\varphi)\,, \qquad (1)$$

where $T(c\theta\cos\varphi, c\theta\sin\varphi)$ denotes translation by $(c\theta\cos\varphi, c\theta\sin\varphi)$ in E(2). Now consider a particle of mass m at rest. The corresponding little group of O(3,1) is O(3). Let us apply the Lorentz boost L_3 such that $L_3(m,0,0,0) = (p^0, 0, 0, p)$, where p is the magnitude of the three-momenta. We see that the above four-vector approximates $p(1,0,0,1)$ when $p\to\infty$ and (1,0,0,1) has E(2) as its little group.[2] Now consider the Lorentz transformation

$$L_3^{-1}R_3(\varphi)R_2(\theta)R_3(-\varphi)L_3 \qquad (2)$$

in the limit $\theta\to 0$, $p/m\to\infty$, but with $p\theta/m$ remaining finite. Direct simplification of expression (2) is straightforward in the above limit. We shall now relate $p/m\to\infty$ with $c\to\infty$ of Eq. (1). For this purpose we verify that the Lorentz transformation (2) corresponds by Eq. (1) to the Lorentz transformation of translation by $(c\theta\cos\varphi, c\theta\sin\varphi)$, which keeps the vector (1,0,0,1) unaltered,[2] provided we take $p/m = c$ while taking the limits. In fact, we get for (2)

$$L_3^{-1}R_3(\varphi)R_2(\theta)R_3(-\varphi)L_3=\begin{bmatrix} p^0/m^2-\frac{p^2}{m^2}\cos\theta & \frac{p}{m}\sin\theta\cos\varphi & \frac{p}{m}\sin\theta\sin\varphi & \frac{pp^0}{m^2}(1-\cos\theta) \\ \frac{p}{m}\sin\theta\cos\varphi & \cos\theta & 0 & \frac{p^0}{m}\sin\theta\sin\varphi \\ \frac{p}{m}\sin\theta\sin\varphi & (\cos\theta-1)\cos\varphi\sin\varphi & \cos\theta\sin^2\varphi+\cos^2\varphi & \frac{p^0}{m}\sin\theta\cos\varphi \\ \frac{pp^0}{m^2}(\cos\theta-1) & -\frac{p^0}{m}\sin\theta\cos\varphi & -\frac{p^0}{m}\sin\theta\sin\varphi & -\frac{p^2}{m^2}+\frac{p^{0^2}}{m^2}\cos\theta \end{bmatrix},$$

which we identify with $T(c\theta\cos\varphi, c\theta\sin\varphi)$ given as

$$\begin{bmatrix} 1+\frac{1}{2}c^2\theta^2 & c\theta\cos\varphi & c\theta\sin\varphi & \frac{1}{2}c^2\theta^2 \\ c\theta\cos\varphi & 1 & 0 & c\theta\sin\varphi \\ c\theta\sin\varphi & 0 & 1 & c\theta\cos\varphi \\ -\frac{1}{2}c^2\theta^2 & -c\theta\sin\varphi & -c\theta\cos\varphi & 1-\frac{1}{2}c^2\theta^2 \end{bmatrix}$$

obtained with group contraction[1] giving rise to Lorentz transformation with E(2) as the little group.[2] With this identification we utilize the contraction of O(3) to E(2).

As defined earlier the momenta Q_a and Q_b are generators of E(2). We further have for the spinless case

$$J_1^2+J_2^2+J_3^2=c^2(Q_a^2+Q_b^2)+J_3^2\,.$$

Let the momenta (q_a, q_b) specify a representation of the translation subgroup in E(2). We then obtain that

$$c^2(q_a^2+q^{\ 2})=(p^2/m^2)(q_a^2+q_b^2)\simeq(l+\tfrac{1}{2})^2\,, \tag{3}$$

where we have retained only leading terms in (3). Thus, with $q=(q_a^2+q_b^2)^{1/2}$, we get

$$(p/m)q_l=(l+\tfrac{1}{2})\,, \tag{4}$$

where q_l is the magnitude of momentum in E(2) corresponding to l.

III. ELASTIC SCATTERING

The partial-wave expansion for the scattering amplitude $F(s,t)$ in the spinless case is taken as

$$F(s,t)=\frac{\sqrt{s}}{p}\sum_{l=0}^{\infty}(2l+1)a_l(s)p_l(\cos\theta)\,. \tag{5}$$

We make use of (4) and write $a_l(s)=a(q_l,s)$. We now make the assumption that the scattering amplitude is dominantly pure imaginary. Present high-energy data seem to support this assumption in the near-forward direction,[3] which is our region of interest. We note that $a(q,s)$ for fixed s is a function of momenta in E(2), with $-\infty<q_a<\infty$ and $-\infty<q_b<\infty$. We thus take

$$a(q,s)=i\,\alpha\exp[-\beta^2(q_a^2+q_b^2)] \tag{6}$$

as the first term in an expansion on E(2) in terms of Hermite polynomials. In Eq. (6), α and β are real constants which may carry an s dependence. Notice that unitarity demands that

$$0\le\alpha\le1\,. \tag{7}$$

Also using Eq. (4) we get for large l and fixed $p\theta/m$

$$P_l(\cos\theta)\simeq J_0((p/m)q_l\theta) \tag{8}$$

in the limit $p/m\to\infty$ and $\theta\to0$ with $p\theta/m$ remaining finite. Now application of the optical theorem leads to the relation

$$\begin{aligned}\sigma_t(s)&=\frac{4\pi}{p^2}\sum_{l=0}^{\infty}(2l+1)\mathrm{Im}a_l(s)\\ &=\frac{8\pi}{m^2}\int_0^{\infty}\alpha\exp(-\beta^2q^2)q\,dq\\ &=\frac{4\pi}{m^2}\frac{\alpha}{\beta^2}\,.\end{aligned} \tag{9}$$

Note that in converting the above summation to an integral we have used $\Delta q_l=m/p$ and have retained the leading contribution.

Similarly, we derive

$$\begin{aligned}\sigma_{\rm el}&=\frac{4\pi}{p^2}\sum_{l=0}^{\infty}(2l+1)\,|a_l(s)|^2\\ &=\frac{2\pi}{m^2}\frac{\alpha^2}{\beta^2}\,.\end{aligned} \tag{10}$$

Combining Eqs. (9), (10), and (7) we get

$$\sigma_{\rm el}/\sigma_t=\alpha/2\le\tfrac{1}{2} \tag{11}$$

and that

$$\beta^{-2}=\frac{m^2\sigma_t^{\ 2}}{8\pi\sigma_{\rm el}}\,. \tag{12}$$

Notice that if α is independent of s, as seems to be the case, we get $\beta^{-2}\propto\sigma_t(s)$.

Again, Eq. (5) with approximation (6) yields

$$F(s,t)=\frac{2i\sqrt{s}\,p}{m^2}\int_0^{\infty}\alpha\exp(-\beta^2q^2)q\,dq\,J_0(p\theta q/m) \tag{13}$$

$$=\frac{i\sqrt{s}\,p\,\alpha}{m^2\beta^2}\exp[t/(4m^2\beta^2)]\,. \tag{14}$$

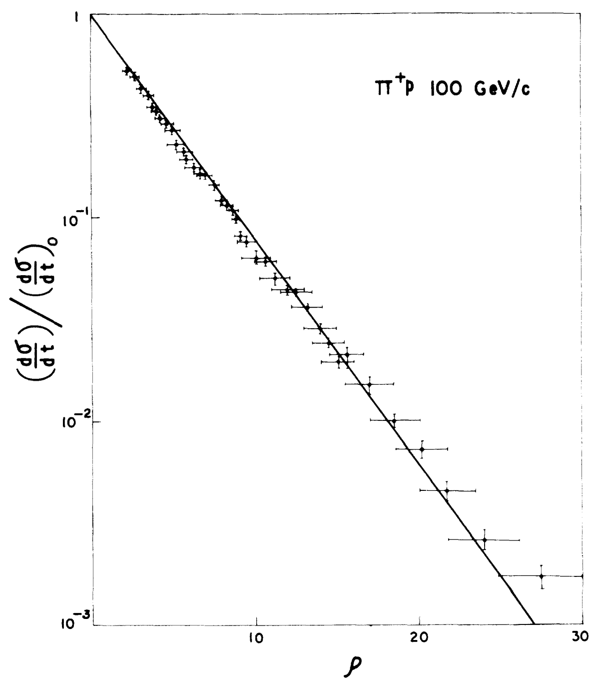

FIG. 1. π^+p elastic scattering data are compared with Eq. (16): $\rho=-t\sigma_t{}^2/(4\pi\sigma_{el})$. Pion laboratory momentum is 100 GeV/c.

If we consider the general case of $a_t(s)=a(q_t s)$ then the above integral becomes

$$F(s,t)=\frac{2\sqrt{s}\,p}{m^2}\int_0^\infty q\,dq\,a(q,s)J_0(\sqrt{-t}\,q/m)\,, \tag{15}$$

which easily illustrates the impact-parameter-like representation of the scattering amplitude with the impact parameter as $b=q/m$. However, the present analysis gives us a different understanding of the problem. The differential cross section in our approximation is now given by

$$\frac{d\sigma}{dt}=\frac{\pi}{p^2}\left|\frac{F(s,t)}{\sqrt{s}}\right|^2$$

$$=\left(\frac{d\sigma}{dt}\right)_0 \exp(-\rho/4)\,, \tag{16}$$

where

$$\rho=-t\sigma_t{}^2/(4\pi\sigma_{el})\,. \tag{17}$$

This corresponds to slope

$$b=\sigma_t{}^2/(16\pi\,\sigma_{el})\,. \tag{18}$$

Furthermore, the optical theorem together with the assumption that the amplitude is purely imaginary automatically gives that $(d\sigma/dt)_0=\sigma_t{}^2/(16\pi)$. We notice that Eq. (16) describes all elastic scattering processes in the near-forward region in terms of a universal variable ρ.

IV. DISCUSSIONS

It is interesting to note that (16) resembles the hypothesis of geometrical scaling.[4] Indeed, since $\sigma_t=O(R^2(s))$ and $\sigma_{el}=O(R^2(s))$ we get

$$\frac{d\sigma}{dt}\propto R^4\exp(+R^2t)\,. \tag{19}$$

Barger *et al.*[5] have found that recent Fermilab data for $\pi^\pm p$, $K^\pm p$, pp, and $\bar{p}p$ are consistent with the hypothesis of geometrical scaling. It is amazing that we are led to a similar conclusion from a different approach which takes into account variation of σ_{el}/σ_t with energy and process. Thus the prescription in Ref. 5 to take the slope as proportional to $\sigma_t{}^2/\sigma_{el}$ receives some justification here. We find that experimental data are in good agreement with (16).

We compare in Figs. 1–6 the experimental data[6] for $\pi^\pm p$, $K^\pm p$, pp, and $p\bar{p}$ from Fermilab at laboratory energies 100 and 200 GeV/c with our predictions. We find that the present data support our prediction within experimental errors. The solid line is the universal curve exp $(-\rho/4)$, and the data points are taken from Ref. 6. It may be

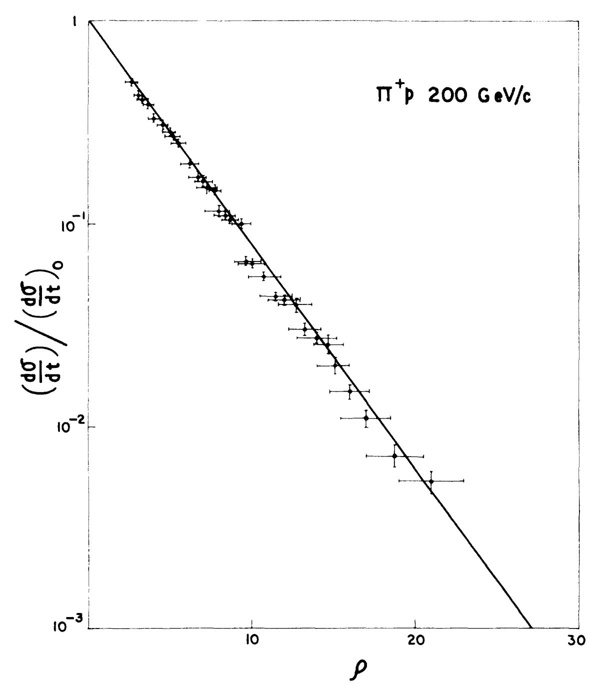

FIG. 2. π^+p data at 200 GeV/c.

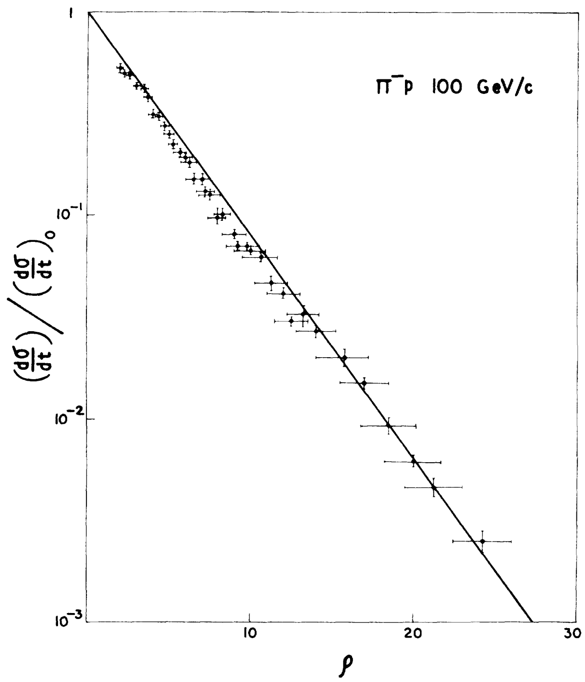

FIG. 3. $\pi^- p$ data at 100 GeV/c.

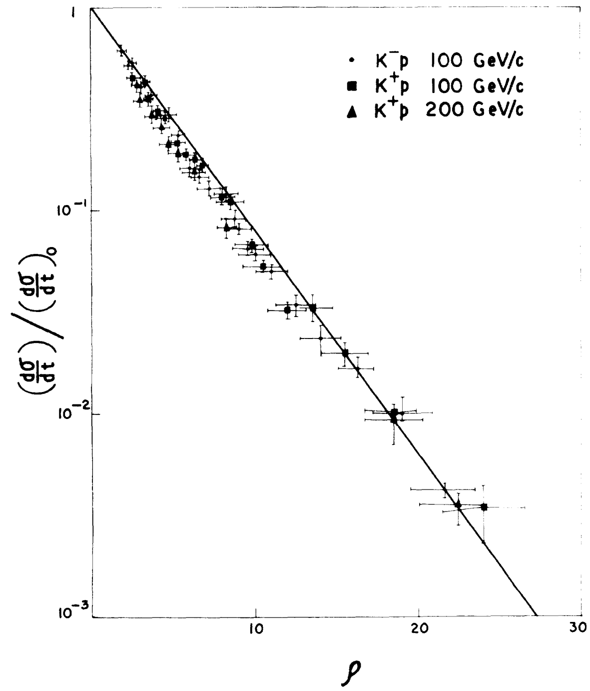

FIG. 5. K^+p data at 100 and 200 GeV/c and K^-p data at 100 GeV/c.

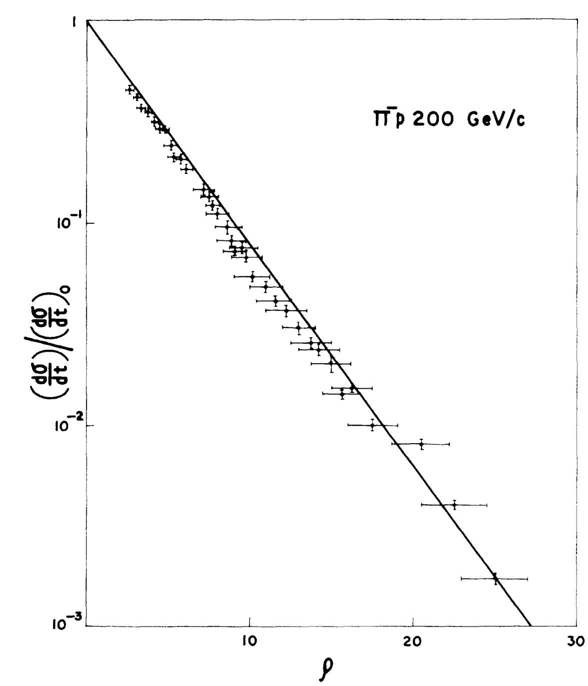

FIG. 4. $\pi^- p$ data at 200 GeV/c.

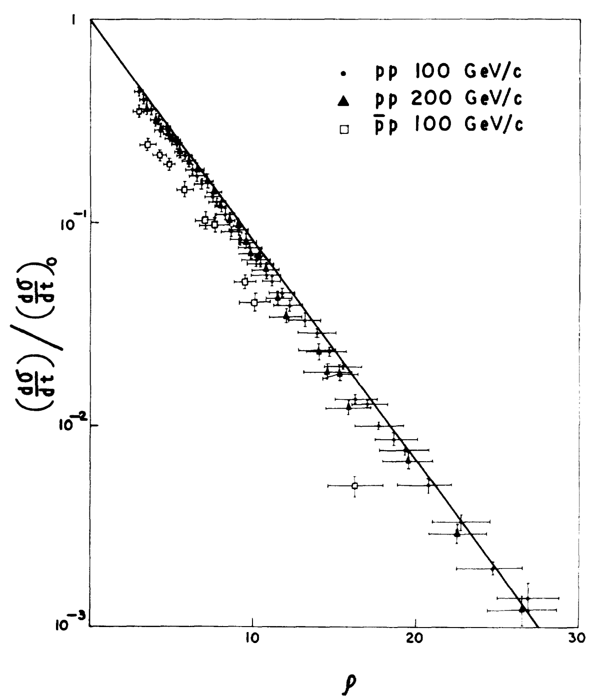

FIG. 6. pp and $\bar{p}p$ data at 100 and 200 GeV/c.

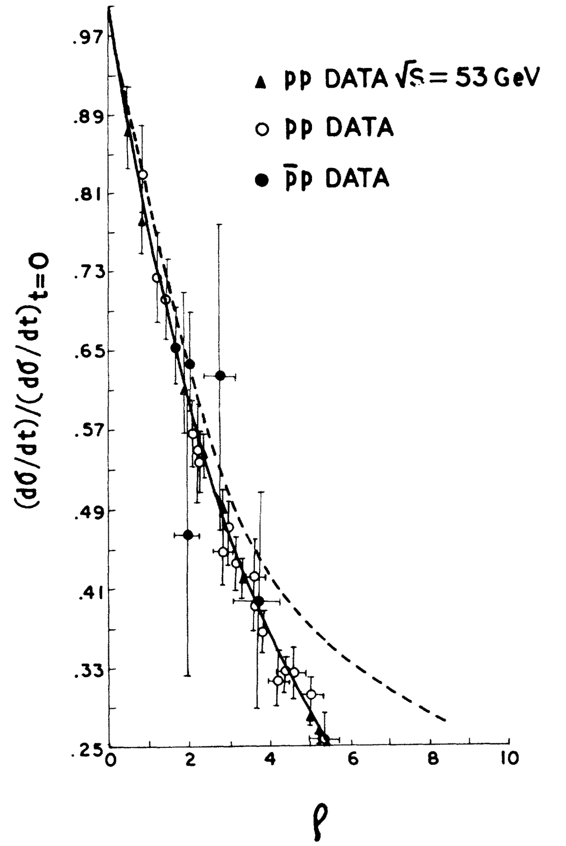

FIG. 7. Equation (16) is considered for small values of ρ for pp and $\bar{p}p$ elastic scattering. Dotted line is the unitary bound of Singh and Roy. The graph is reproduced from Ref. 7.

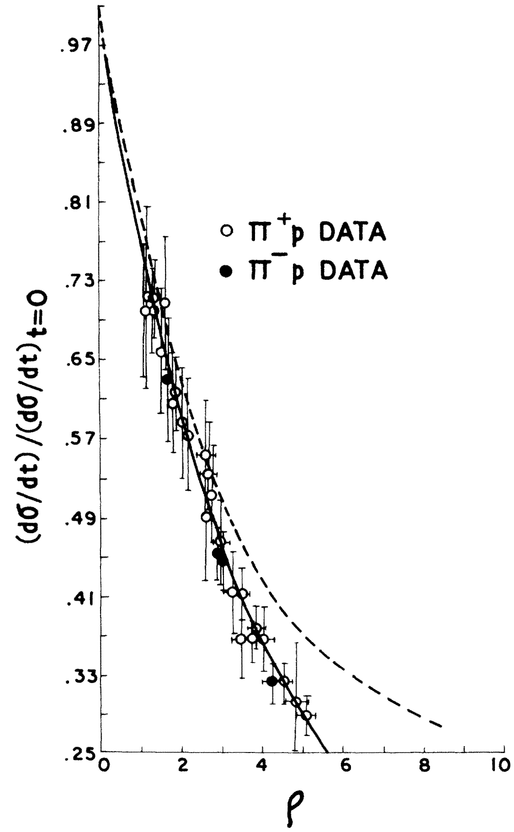

FIG. 8. Equation (16) is considered for small values of ρ for $\pi^{\pm}p$ elastic scattering. The graph is reproduced from Ref. 7.

noted that Singh and Roy[7] had introduced

$$\rho = (d\sigma/dt)_0(-4t)/\sigma_{el}$$

in connection with the consideration of unitarity bounds of the elastic amplitude in the forward direction. When the amplitude is pure imaginary, the two expressions coincide. Although the saturation of the Singh-Roy unitarity bound in the diffraction region was unexpectedly good, it now appears as an approximate saturation of Eq. (16) for small ρ with a slightly different slope (see Figs. 7 and 8).

We would like to remark that McDowell and Martin[8] made a rough estimate of the diffraction-peak width Δ as given by $\Delta^{-1} \geq \sigma_t^2/(16\pi\sigma_{el})$ and then rigorously derived the bound

$$b \geq \sigma_t^2/(18\pi\sigma_{el}) .$$

We have derived here the rough estimate above which very nearly saturates the exact bound. Also we trivially see that[9] if

$$d\sigma/dt = (d\sigma/dt)_0 \exp(bt) ,$$

then

$$b \simeq (d\sigma/dt)_0/\sigma_{el} ,$$

which is Eq. (16) above. However, this extrapolation is obviously false, e.g., for backward pp scattering, so that such a naive way of looking at things is really inadequate.

We note here that the scattering amplitude was considered for the spinless case. When we include spin, we shall have a larger number of terms including the type of contribution considered here. It appears from the agreement with experiments that the corrections to what has been derived will be small. Also the other internal quantum numbers have not been considered. It is reasonable to assume that at high energies spin flip or change in other internal quantum numbers for diffraction scattering is highly inhibited. Hence the description of the scattering in this region by a single scattering amplitude which does not change spin or unitary spin becomes reasonable. This may be behind the fact that we are able to describe diffraction scattering up to an order of 10^{-3} in a universal manner, with no free parameters, irrespective of energy or the process under investigation with a simple universal variable ρ.

In our approximation the agreement is expected to be satisfactory for

$$(d\sigma/dt) \gg (m^2/p^2)(d\sigma/dt)_0 \,. \tag{20}$$

This is so since in our identification of generators of E(2) with those of O(3) in the group contractions involved in Eqs. (1) and (2), the first one geometrical and the second one kinematic, we neglect terms such as m^2/p^2 in comparison with unity. This is the reason we consider up to $\rho \simeq 30$ in our comparison with experimental results.

We may note the similarity of the present model to the geometrical model and to the impact-parameter model. In fact this identification in many ways is expected in a heuristic manner. The particle has, in the E(2) plane which we have considered, "coordinates" $(p\,\theta/m)\cos\varphi$ and $(p\theta/m)\sin\varphi$ whose canonical conjugate "momenta" have been taken as q_a and q_b. For the impact-parameter or geometrical picture one can see that the names should be reversed, so that q_a/m and q_b/m will be the components of the impact parameter. Thus the above analysis gives a group-theoretic justification of the impact-parameter representation in Eq. (15) for forward scattering.

Also, the following results may be noted. Equation (6) taken in E(2) is equivalent in O(3) to the assumption that

$$a_l(s) = i\alpha \exp\left(-\frac{m^2\beta^2 l(l+1)}{p^2}\right). \tag{21}$$

Equation (21) is an approximation of $a_l(s)$, which has nothing to do with E(2), and thus, we would hope, we can replace Eq. (6) by (21) and consider scattering at all angles. In particular, for $\theta = 90°$, we can use[10]

$$\begin{aligned} P_l(\cos\theta) &= (-l)^{1/2}\frac{l!}{2^l[(\frac{1}{2}l)!]^2} F\left(-\frac{l}{2}, \frac{l+1}{2}, \tfrac{1}{2}; \cos^2\theta\right) \text{ when } l \text{ is even} \\ &= (-1)^{(l-1)/2}\frac{l!\cos\theta}{2^{l-1}[(\frac{1}{2}(l-1))!]^2} F\left(-\frac{l-1}{2}, \frac{l}{2}+1; \tfrac{3}{2}; \cos^2\theta\right) \text{ when } l \text{ is odd} \end{aligned} \tag{22}$$

and calculate from Eqs. (5) and (21) $d\sigma/dt$ at $\theta = 90°$. Proceeding as earlier with taking the leading terms and replacing the summation by an integral, one gets the contribution as too large and the power behavior as s^{-2}, which is too slow by several orders of magnitude. One may also use $P_l(\cos\theta) = (-1)^l P_l(-\cos\theta)$ and consider backward scattering in some cases. Again the result is bad. This illustrates that the approximation (21) in O(3) is bad but the equivalent approximation (6) in E(2) for forward scattering is quite good. In fact that (6) is the first term in the expansion in E(2) has no corresponding relevance for (21) in O(3).

Another remark is worthwhile. We note that[10]

$$P_l(\cos\theta) = F\left(l+1, -l; 1; \frac{1-\cos\theta}{2}\right) \tag{23}$$

yields the approximation (8). For large values of l, if we take the next-order approximation and retain terms of lower order in m/p, we then obtain after complicated calculations that for small t (how small we cannot specify)

$$\frac{d\sigma}{dt} = \frac{\sigma_t^{\,2}}{16\pi}\left[\exp(\tfrac{1}{2}bt)\left(1 + \frac{1-2bt-b^2t^2}{24bp^2}\right) - (24bp^2)^{-1}\right]^2. \tag{24}$$

Equation (24) yields that for large values of $-\frac{1}{2}bt$, $d\sigma/dt$ approaches a constant at fixed energy. But, since $24bp^2 \gg p^2/m^2$, the corrections in (24) are really not valid. However, an interesting qualitative feature emerges which is similar to geometrical models. It may be noted that for

$$-t_{\text{dip}} = (2/b)\ln(24bp^2)$$

$d\sigma/dt$ vanishes. For pp scattering with $\sqrt{s} = 53$ GeV/c, we have $b = 12.4$ GeV^{-2}, and thus $-t_{\text{dip}} \simeq 2$ GeV^2. This is a satisfactory *qualitative* feature when we consider that here

$$(d\sigma/dt)/(d\sigma/dt)_0 \simeq 10^{-6}\,,$$

drastically violating Eq. (20).

We thus see that a good universal approximation becomes possible for forward elastic scattering at high energies on using group contraction of O(3) to E(2), and the method is related to and gives additional understanding of allied geometrical and impact-parameter models.

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