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Test of the partial conservation of axial-vector current limit in pion-nucleon scattering

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A comparison is made of the on-shell amplitudes $\bar{F}^{(\pm)}(0,t)$, obtained using interior dispersion relations, to their off-shell current-algebra predictions in order to test the pion PCAC (partial conservation of axial-vector current) limit in pion-nucleon scattering. It is concluded that, when ν and t are used as variables, the on-shell corrections to PCAC are on the order of 10% per pion.

In recent years, the advent of accurate pion-nucleon scattering data, particularly near threshold, and of improved dispersion-theory techniques has made it reasonable to attempt to map out the subthreshold πN analytic amplitudes, as proposed by Höhler, *et al.*[1] Owing to its significance as a direct measure of chiral-symmetry breaking, the πN σ term has received particular attention from several authors, the application of numerous techniques yielding a current "world value" $\sigma_{\pi N} \simeq (65\pm5)$ MeV.[2] There are, however, other points where the values of the amplitudes are important to current-algebra theory, or to be more specific, to the pion PCAC (partial conservation of axial-vector current) hypothesis. In particular, one which has all too often been overlooked is the on-shell value of the forward background amplitude, $\bar{F}^{(+)}(\nu, t; q^2, q'^2)$, corresponding to the off-shell Adler consistency condition[3] at $q \to 0$, $q'^2 = \mu^2$,

$$\bar{F}^{(+)}(0, \mu^2; 0, \mu^2) = \bar{F}^{(+)}(0, \mu^2; \mu^2, 0) = 0\,. \tag{1}$$

It is our purpose in this note to present an interior dispersion relation (IDR) calculation of the on-shell version of the amplitude $\bar{F}^{(+)}$ as well as $\bar{F}^{(-)}$, and a discussion of their relevance as a measure of pion PCAC.

Consider first the on-shell amplitude[4]

$$F^{(+)}(\nu,t) = A^{(+)}(\nu,t) + (\nu/4m)B^{(+)}(\nu,t)\,,$$

where $\nu = (s-u)$. Since $F^{(+)}$ does not obey an unsubtracted IDR (a brief summary of IDR is given in the Appendix), we use the amplitude

$$G^{(+)}(a,t) = A^{(+)}(a,t) + \frac{t_+ - t_-}{t_+ - t} B^{(+)}(a,t)\,, \tag{2}$$

which is equal to $F^{(+)}$ for $\nu = 0$, obeys an unsubtracted IDR, and has been successfully used earlier in an IDR calculation of the πN σ term.[5] At $\nu = 0$,

$$\bar{F}^{(+)}(0,t) = \bar{G}^{(+)}(a, t_a) - \frac{g^2}{m} \frac{t_a - 2\mu^2}{2(m^2 - a)}\,, \tag{3}$$

where

$$t_a = t(\nu = 0, a)$$
$$= t_-(a)\,,$$
$$\bar{F}^{(+)}(\nu,t) = F^{(+)}(\nu,t) - \frac{g^2}{m}\left(\frac{\nu_B{}^2}{\nu_B{}^2 - \nu^2}\right),$$
$$\nu_B = t - 2\mu^2\,,$$
$$\bar{G}^{(+)}(a,t) = G^{(+)}(a,t) - G^{(+)}_{N,\mathrm{IDR}}(a,t)\,,$$
$$G^{(+)}_{N,\mathrm{IDR}}(a,t) = \frac{g^2}{4m}\left(\frac{t_+ - t_-}{t_+ - t_n}\right) \frac{(t_n - 2\mu^2)^2}{(m^2 - a)(t_- - t_n)}\,,$$

and

$$t_n = t(s = m^2, a)$$
$$= \mu^2(4m^2 - \mu^2)/(m^2 - a)\,.$$

We take $g^2/4\pi = 14.7$ ($f^2 = 0.081$), consistent with IDR determinations of the πN coupling constant,[6] but a variation in g of the order of 5% would not noticeably affect our results. Except in the neighborhood of $t_a = 2\mu^2$, where it vanishes, the second term on the right-hand side of Eq. (3) has the behavior

$$\frac{g^2}{m}\frac{t_a - 2\mu^2}{2(m^2 - a)} = \frac{g^2}{m}\mu^2 \frac{(m^2 + a)}{(m^2 - a)^2} + O(\mu^4)\,.$$

Using IDR's to determine $\bar{G}^{(+)}(a, t_a)$, as was done for $t_a = 2\mu^2$ in the determination of the σ term,[5] and making the comparably small correction for the difference in the Born terms for $F^{(+)}$ and $G^{(+)}$, the values for $\bar{F}^{(+)}(0,t)$, as shown in Fig. 1, were obtained. Figure 1 clearly demonstrates that $\bar{F}^{(+)}(0,t)$ vanishes very near $t = \mu^2$, and obeys, within the error bars, the relation $\bar{F}^{(+)}(0,0) \simeq -\bar{F}^{(+)}(0, 2\mu^2)$.

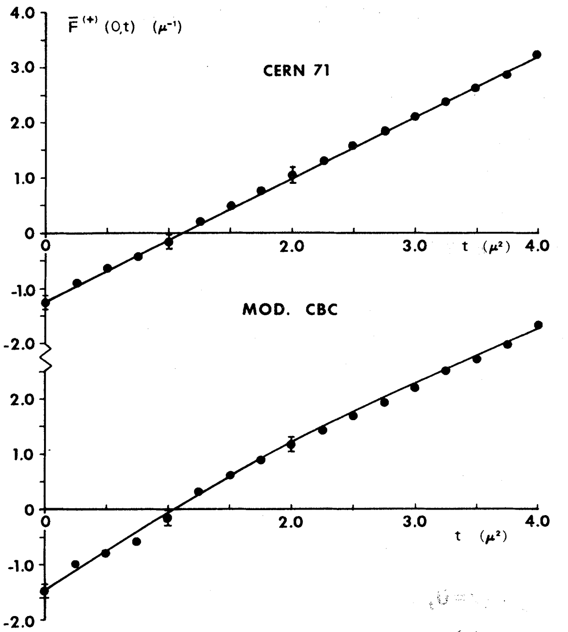

FIG. 1. $\overline{F}^{(+)}(0,t)$ in units of μ^{-1} is plotted against t for both the CERN 71 and Mod. CBC solutions (solid dots). The quadratic fits which give the values in (7) are shown as solid lines. Error bars are shown at the points representing the σ term ($t=2\mu^2$), the results in (4) ($t=\mu^2$), and $\overline{F}_{00}^{(+)}$ ($t=0$). Uncertainties in the other points are comparable.

The vanishing of $\overline{F}^{(+)}(0,t)$ near $t=\mu^2$, when combined with the off-shell Adler zero, Eq. (1), shows that the effect of taking the PCAC limit, i.e., going on the mass shell from $q^2=0$ to $q^2=\mu^2$ with ν and t fixed, is very small.

At $t=\mu^2$, we obtain

$$\overline{F}^{(+)}(0,\mu^2)=\begin{cases}(-0.16\pm0.12\mu^{-1} & \text{CERN 71}\\(-0.15\pm0.10\mu^{-1} & \text{Mod. CBC}\,,\end{cases}\qquad(4)$$

where CERN 71 and Mod. CBC correspond to two sets of partial waves used: the 1971 CERN solution of Almehed and Lovelace[7] and a modified version of the 1973 Carter, Bugg, and Carter (CBC) solution.[8] A more detailed description of these solutions as used in IDR's is given in Refs. 5 and 6.

The scale by which to compare Eq. (4) with Eq. (1) is presumably of order $1\mu^{-1}$, the value of $\mu^2\partial\overline{F}^{(+)}(0,t)/\partial t$ in the neighboring subthreshold region. This implies a PCAC correction for fixed ν and t of the order of 10% consistent, which compares favorably with the present 6% PCAC corrections to the Goldberger-Treiman relation.[9]

The phenomenological value (4) is also an independent constraint on the subthreshold on-shell expansion[1]

$$\overline{F}^{(+)}(\nu,t)=\sum_{m,n}\overline{F}_{mn}^{(+)}\nu^{2m}t^n\,.\qquad(5)$$

At $\nu=0$, $t=\mu^2$, the Höhler, Jakob, and Strauss value of $\overline{F}_{00}^{(+)}+\mu^2\overline{F}_{01}^{(+)}$ is $(-0.5\pm0.3)\mu^{-1}$,[1] while that of Nielsen and Oades is $(-0.27\pm0.10)\mu^{-1}$.[10] A semiphenomenological analysis gives[11] $-0.13\mu^{-1}$. Clearly, these numbers are consistent with (4). An additional constraint on (5) is our previously determined value of the σ term which corresponds to[12]

$$\overline{F}^{(+)}(0,2\mu^2)=\begin{cases}(1.06\pm0.13)\mu^{-1} & \text{CERN 71}\\(1.19\pm0.11)\mu^{-1} & \text{Mod. CBC}.\end{cases}\qquad(6)$$

Ignoring the term quadratic in t, we are led to a value $\overline{F}_{00}^{(+)}\simeq-1.4\mu^{-1}$, consistent with most recent determinations.

Inasmuch as the Born term contributions to (3) are relatively small over the entire physical range of the IDR path parameter a, $\overline{F}^{(+)}(0,t)$ can be calculated for a range of a values and extrapolated to $t=0$ ($a=-\infty$) to obtain a somewhat independent determination of $\overline{F}_{00}^{(+)}$. The points, as shown in Fig. 1, are best fit by quadratic forms to give the following:

	CERN 71	Mod. CBC
$\overline{F}_{00}^{(+)}(\mu^{-1})$	-1.25 ± 0.11	-1.49 ± 0.10
$\overline{F}_{01}^{(+)}(\mu^{-3})$	1.17 ± 0.10	1.46 ± 0.09
$\overline{F}_{02}^{(+)}(\mu^{-5})$	-0.02 ± 0.01	-0.07 ± 0.02.

(7)

(In Fig. 1, uncertainties are shown for the points at $t=\mu^2$ and $t=2\mu^2$, where extra pains were taken to account for systematic errors in the calculations. The uncertainties for the remaining points were taken to be comparable, except for those at $t=\frac{1}{2}\mu^2$ and $t=\frac{1}{4}\mu^2$. These last results are obtained from IDR integrations over rather limited ranges in the forward scattering direction and were weighted less than the rest. See Ref. 5 for a more thorough discussion of error estimates in IDR calculations.) The results given in (7) are generally consistent with most recent determinations, although it should be understood that they were obtained from a fit of $\overline{F}^{(+)}$ values at a single fixed value of ν (namely $\nu=0$), and that a more precise determination requires fitting over a range of t *and* ν values.[13] The fits, as given in (7), tend to displace our more precisely determined values of $\overline{F}^{(+)}(0,\mu^2)$ and $\sigma_{\pi N}$ by slight amounts.

In the above, we have taken the variables ν, t, q^2, and q'^2 as independent, with ν and t being fixed during extrapolations in q^2, as when comparing (1) with (4). An alternative set of variables is often considered[14]: $\nu, q'\cdot q$, and q^2, q'^2, where $q'\cdot q=\frac{1}{2}(q^2+q'^2-t)$. If this latter choice were correct (i.e., if this were the correct set of variables with which to test PCAC) the extrapolation from (1), with $\nu = q'\cdot q = q^2 = 0$, $q'^2 = \mu^2$ (implying $t = \mu^2$), to $\nu = q'\cdot q = 0$, $q^2 = q'^2 = \mu^2$ (implying $t = 2\mu^2$) takes us to the σ-term amplitude (6). This would mean that a small σ term is a necessary requirement for the validity of pion PCAC. The result (4), however, is roughly six times smaller in magnitude than (6), and we believe that this is significant evidence in favor of the Mandelstam variables ν and t being the correct independent analytic variables. Thus, the σ term need *not* be small for pion PCAC to be valid.

As additional evidence for this, we refer to the soft-pion theorem for both pions soft,

$$\overline{F}^{(+)}(0,0;0,0) = -\sigma_{\pi N}/f_\pi^{\,2}\,, \tag{8}$$

and invoke[15] $\sigma_{\pi N}/f_\pi^{\,2} = -\overline{F}^{(+)}(0,0;0,0) \simeq \overline{F}^{(+)}(0,2\mu^2)$ along with the numerical result that $-\overline{F}^{(+)}(0,0) \simeq \overline{F}^{(+)}(0,2\mu^2)$. This again implies that PCAC is reasonably good if ν and t are the variables to be held fixed ($\nu = t = 0$ here). On the other hand, fixing ν and $q'\cdot q$ at zero (implying $t = 0$) and bringing both pions on-shell results in a variation of t to $2\mu^2$ and brings us to $\overline{F}^{(+)}(0,2\mu^2) \simeq +1.0\mu^{-1}$. This represents a *change in sign* and a net variation of approximately $2\mu^{-1}$. $\overline{F}^{(+)}(\nu,t;q^2,q'^2)$ is thus seen to be a much smoother function of the pion masses in the PCAC limit with ν and t fixed rather than with ν and $q\cdot q'$ fixed.

An additional test of PCAC is the measurement of the on-shell analog of the Adler-Weisberger soft-pion theorem[16] ($g_A = 1.25$, $f_\pi \simeq 92$ MeV)

$$4m\nu^{-1}\overline{F}^{(-)}(0,0;0,0) = (1 - g_A^{\,2})/2f_\pi^{\,2} \simeq -0.62\mu^{-2}\,. \tag{9}$$

The on-shell value has been determined by Nielsen and Oades[10] to be $-0.52\mu^{-2}$ and by Höhler, Jakob, and Strauss[1] to be $-0.44\mu^{-2}$. We have performed a determination of this parameter by writing an IDR for the amplitude

$$H^{(-)}(a,t) = 4m\nu^{-1}F^{(-)}(\nu,t) \tag{10}$$

and proceeding as with $G^{(+)}(a,t)$ above. We have

$$4m\nu^{-1}\overline{F}^{(-)}(\nu,t) = \overline{H}^{(-)}(a,t) - g^2/(m^2 - a)\,, \tag{11}$$

where the second term on the right-hand side is the difference between the $4m\nu^{-1}F^{(-)}$ Born term (as calculated consistent with fixed-t dispersion relations) and the $H^{(-)}$ Born term, as calculated from the IDR. At $\nu = 0$, and in the limit $t_a \to 0$ ($a \to -\infty$), we have

$$4m\nu^{-1}\overline{F}^{(-)}(0,0) = \lim_{t_a \to 0}\overline{H}^{(-)}(a,t_a)\,. \tag{12}$$

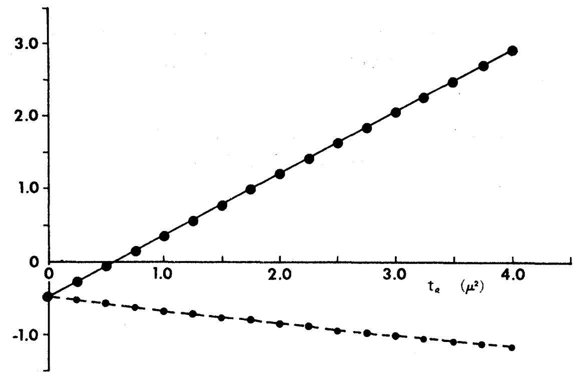

FIG. 2. The values of $\overline{H}^{(-)}(a,t_a)$ (in units of μ^{-2}) as obtained from the IDR for the CERN 71 solution (large dots). The Mod. CBC values are essentially indistinguishable from these. The points are fitted by the linear expression (solid line) $\overline{H}^{(-)}(a,t_a) = -0.49 + 0.85t_a$. Uncertainties in the points are uniformly $\simeq 0.1\mu^2$. The resulting values of $4m(s-u)^{-1}\overline{F}^{(-)}$, as calculated from Eq. (11), are also shown (dashed curve).

Extrapolation of $H^{(-)}(a,t_a)$ to $t_a = 0$ yields (cf. Fig. 2)

$$4m\nu^{-1}\overline{F}^{(-)}(0,0) = \begin{cases} (-0.49 \pm 0.02)\mu^{-2} & \text{CERN 71} \\ (-0.52 \pm 0.02)\mu^{-2} & \text{Mod. CBC}\,, \end{cases} \tag{13}$$

consistent with the previously mentioned results *and* with small pion PCAC corrections if ν and t are held fixed. However, if ν and $q\cdot q'$ were held fixed, we would need to compare Eq. (9) with $\overline{F}^{(-)}(0,2\mu^2)$, and would not have found the PCAC correction to be as small.

In summary, we have presented IDR determinations of $\overline{F}^{(\pm)}(0,t)$ which measure the validity of pion PCAC in the pion-nucleon interaction. The value of $\overline{F}^{(+)}(0,\mu^2)$ is a particularly sensitive gauge of PCAC on-shell corrections. Our results indicate an on-shell pion PCAC correction of approximately 10% per pion, when the variables ν and t are fixed, consistent with other measurements of PCAC. We also argue that the variables ν and t are the "proper" analytic variables in terms of which to discuss PCAC.

Finally, we point out that, given this proven validity of the pion PCAC extrapolation from $q^2 = 0$ to $q^2 = \mu^2$, the on-shell amplitudes $\overline{F}^{(+)}(0,\mu^2)$ and $\overline{F}^{(+)}(0,2\mu^2) = \sigma_{\pi N}/f_\pi^{\,2}$ can be used to obtain the *off-shell* spacelike-pion-momentum extrapolation of the zero-energy (Fermi sea) amplitude so prevalent in nuclear physics,[17,18]

$$\overline{F}^{(+)}(0,t;q^2,q'^2) = \sigma_{\pi N}/f_\pi^{\,2} - \frac{q^2 + q'^2 - t}{\mu^2}\overline{F}^{(+)}(0,\mu^2) + O\left(\frac{q^2}{m^2}\right)\,. \tag{14}$$

This result predicts a reasonable three-body contribution to the binding energy of nuclear matter[18] of $\frac{1}{2}$ to 1 MeV.

We wish to acknowledge valuable discussions with Dr. David C. Moir of the Los Alamos Scientific Laboratory.

APPENDIX

Interior dispersion relations (IDR) are written for reactions of the type

$$a+b\to c+d\,, \tag{A1}$$

where at least $m_a=m_c$ or $m_b=m_d$. Reactions for which both mass equalities hold are termed elastic, those for which only one holds are semielastic. We use the traditional nomenclature, calling (A1) the s-channel process,

$$a+\bar{d}\to c+\bar{b} \tag{A2}$$

the u-channel process, and

$$a+\bar{c}\to \bar{b}+d \tag{A3}$$

the t-channel process. The Mandelstam invariant quantities s, t, and u also follow conventional usage. We define the invariant

$$\nu\equiv s-u\,, \tag{A4}$$

and introduce the Kibble boundary function

$$\begin{aligned}\phi(\nu,t)&=4sp_s^{\,2}p_s'^{\,2}\sin^2\theta_s\\&=4tp_t^{\,2}p_t'^{\,2}\sin^2\theta_t\,,\end{aligned} \tag{A5}$$

where p_s and p_s' are the initial and final center-of-mass momenta and θ_s the center-of-mass scattering angle for the s-channel reaction; analogous designations hold for t-subscripted variables. Setting $\phi=0$ yields the boundary loci for the respective reactions in the ν-t plane.

For elastic and semielastic processes, the s- and u-channels are kinematically equivalent, and the invariant amplitudes are symmetric or antisymmetric in the variable ν:

$$A(s,t,u)=\pm A(u,t,s)$$

or

$$A(\nu,t)=\pm A(-\nu,t)\,.$$

We are concerned with symmetric amplitudes; antisymmetric amplitudes may be symmetrized by dividing by ν.

As a function of ν and t, an invariant amplitude presumably has only dynamical singularities. However, if one fixes a variable which is a nonanalytic function of ν and t, spurious "kinematical" singularities are introduced which must be included in any dispersion relation written in the remaining free variable. The discontinuity or residue of the amplitude at this singularity being generally unknown and unobtainable from the dynamics of the interaction, the dispersion relation so obtained has little value.

For symmetric amplitudes, a variable which can be fixed to obtain kinematical-singularity-free dispersion relations is

$$a\equiv-\phi(\nu,t)/t^2\,. \tag{A6}$$

As a function of t and a, ν is given by

$$\nu(t,a)=(\nu_t^{\,2}+4at)^{1/2}\,, \tag{A7}$$

where

$$\nu_t=-4p_tp_t'\,,$$

$$\nu_t^{\,2}=\frac{1}{t}[t-(m_a+m_c)^2][t-(m_a-m_c)^2](t-4m_b^{\,2})\,.$$

In general, fixing a introduces branch-point singularities in the t plane of the amplitude, but for symmetric amplitudes ν appears only in even powers, and the discontinuities across the kinematical cuts vanish. One may then write dispersion relations in t which contain only dynamical contributions.

By fixing the value of a one obtains an interior dispersion relation, which has the form

$$\begin{aligned}\mathrm{Re}A(t)=A_B(t)&+\mathrm{P}\frac{1}{\pi}\int_{-\infty}^{0}\frac{\mathrm{Im}A(t')}{t'-t}dt'\\&+\mathrm{P}\frac{1}{\pi}\int_{t_2}^{\infty}\frac{\mathrm{Im}A(t')}{t'-t}dt'\,.\end{aligned} \tag{A8}$$

For $a\le 0$, the first integral on the right-hand side is performed entirely within the s-channel physical region; $a=0$ corresponds to integration along the boundary, which includes contributions from backward scattering ($-\infty<t<t_0$) and forward scattering ($t_0<t<0$). t_0 is the value of t at the s-channel threshold. The second integral begins at the threshold of the t-channel unitary cut, $t=t_2$, includes an unphysical-region contribution in which appear most of the lower-mass t-channel resonances, and finishes with a contribution from the physical t-channel reaction. It is this integral which is called the *discrepancy function*, and which is extrapolated from the s-channel physical region in order to determine $\mathrm{Re}A(t)$ below threshold. A_B is a Born term.

For *elastic scattering*

$$\cos\theta_s = \frac{a+s(t,a)}{a-s(t,a)}, \quad \text{(A9)}$$

where

$$2s(t,a) = \Sigma - t + \nu(t,a),$$

$$\Sigma = 2m_a^2 + 2m_b^2.$$

The laboratory angle for elastic scattering is fixed within the s-channel physical region:

$$\cos\theta_{1ab} = -\frac{a+(m_b^2 - m_a^2)}{[a^2 - a\Sigma + (m_b^2 - m_a^2)^2]^{1/2}}. \quad \text{(A10)}$$

Further details can be found in Moir *et al.*, Ref. 2, and in Hite *et al.*, Ref. 19.

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[13]A more complete IDR determination of expansion coefficients for πN amplitudes is currently under way.

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