# $J = 0$ $N\bar{N} \to \pi\pi$ amplitude in the unphysical region $t > 4\mu^2$ *

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The magnitude of the $J = 0$ $N\bar{N} \to \pi\pi$ amplitude is evaluated in the pseudophysical region above the $\pi\pi$ threshold. The relevance of this quantity to the dispersion theory of nucleon-nucleon scattering is discussed and the 425-MeV $l = 5,6,7$ $NN$ phase parameters calculated from the dispersion theory are given.

## I. INTRODUCTION

The formalism for evaluating the two-pion-exchange part of the nucleon-nucleon interaction has been known since the early 1960's, when Amati, Leader, and Vitale[1] published the details of a dispersion-theoretic approach based on the assumption of Mandelstam analyticity. With this approach one expresses the $NN$ amplitude as an integration over the two-pion branch cut in the variable $t$.[2] From unitarity in the $N\bar{N} \to N\bar{N}$ channel one writes the absorptive parts entering into the dispersion integrals as products of $N\bar{N} \to \pi\pi$ amplitudes evaluated in the kinematical region above the $\pi\pi$ threshold.

Until recently calculations based on this formalism have relied on physical models for the $N\bar{N} \to \pi\pi$ amplitudes. However, the availability of accurate $\pi N$ phase shifts and a knowledge of the low-$l$ $\pi\pi$ phase shifts has made it possible to evaluate the first few $N\bar{N} \to \pi\pi$ partial-wave amplitudes using certain assumptions about their analytic structures. These $N\bar{N} \to \pi\pi$ amplitudes then allow one to calculate the intermediate-range $NN$ interaction in a way which is fully relativistic, is consistent with crossing and analyticity, and includes correct $\pi\pi$ and $\pi N$ substructures.

The $NN$ phase parameters can be calculated with this dispersion theory and one expects that for $l$ values larger than some energy-dependent minimum value the parameters will be dominated by the one- and two-pion exchanges. It may be advantageous to incorporate the intermediate-$l$ phase parameters calculated in this way into phase-shift analyses, a procedure analogous to the usual practice in such analyses of approximating the high-$l$ partial-wave amplitudes by one-pion exchange. This procedure might be especially useful at higher energies, where the number of parameters which differ significantly from their one-pion-exchange values becomes large.

The effect of the two-pion-exchange interaction on the intermediate-$l$ phase parameters at 425 MeV can be seen in Table I. The $J = 0$ $N\bar{N} \to \pi\pi$ amplitude used in this calculation is evaluated in this paper (solid curve in Fig. 1, to be discussed below), while the $J = 1, 2$ values are from Ref. 3. The error limits correspond to an uncertainty in the square of the $J = 0$ amplitude, which is 50% larger than the error bars in Fig. 1. The $NN$ calculation requires adding the $J = 0, 1, 2$ contributions to the fourth-order amplitudes (the box diagrams, proportional to $g^4$) with the $J = 0, 1, 2$ partial waves removed. This complication arises because the fourth-order terms do not have a convergent $N\bar{N} \to N\bar{N}$-channel partial-wave expansion in the kinematical region of interest. Further details have been given previously.[3] One sees that the corrections to the one-pion-exchange values are particularly large for the triplet odd $J = l + 1$ cases where the correction is 80% for $l = 7$.

Calculations with this formalism have already been used to provide the intermediate-range part of an otherwise phenomenological potential.[4] The derivation of a potential from the dispersion-theoretic (on-shell) amplitude must include assumptions about the nonlocality of the interaction.[5-8]

These applications of the $NN$ dispersion formalism require that the input is reliable and that the uncertainties are understood. Presently, the least reliable input is the magnitude of the $J = 0$ $N\bar{N} \to \pi\pi$ amplitude $f_+^0$. This amplitude also plays a part in reactions other than nucleon-nucleon scattering, an example of which is the $\Lambda N$ interaction where two-pion exchange gives the longest-range force.[9] In this work we show how the accurate low-energy $\pi N$ phase shifts now available make a sufficiently accurate determination of $|f_+^0|$ possible.

## II. PROCEDURE FOR EVALUATING $|f_+^0|$

The backward non-spin-flip $\pi N$ amplitude $F^{(+)}$ is written in terms of the more familiar amplitudes $A^{(+)}$ and $B^{(+)}$ as[10,11]

$$F^{(+)}(t) = \frac{1}{m}\left[A^{(+)}(k, \cos\theta = -1) + \frac{m\omega}{E} B^{(+)}(k, \cos\theta = -1)\right], \quad (1)$$

where $t$ is the negative of the c.m. $\pi N$ momentum

TABLE I. $NN$ nuclear bar phase parameters in degrees at 425 MeV lab energy. The one-pion-exchange values are in parentheses.

| $l$ | Singlet | $J=l-1$ | $J=l$ | $J=l+1$ | $\epsilon(J=l+1)$ |
|---|---|---|---|---|---|
| 5 | −2.64 ±0.07<br>(−2.11) | 1.04 ±0.06<br>(0.820) | −1.09 ±0.07<br>(−1.82) | 1.10 ±0.08<br>(0.372) | −0.765 ±0.000<br>(−0.793) |
| 6 | 0.784 ±0.022<br>(0.484) | −1.61 ±0.01<br>(−1.29) | 3.17 ±0.03<br>(3.25) | −0.601 ±0.025<br>(−0.626) | 1.50 ±0.00<br>(1.51) |
| 7 | −1.05 ±0.01<br>(−0.991) | 0.297 ±0.007<br>(0.250) | −0.600 ±0.008<br>(−0.700) | 0.234 ±0.009<br>(0.130) | −0.341 ±0.000<br>(−0.344) |

transfer squared and the square of the $N\bar{N} \to \pi\pi$ c.m. energy, $\omega$ and $E$ are the pion and nucleon energies in the $\pi N$ c.m. frame, $k$ is the $\pi N$ c.m. momentum, and $m$ is the nucleon mass. We use units such that the charged-pion mass is unity.[12] For $\cos\theta = -1$ we may write $k^2 = -\frac{1}{4}t$ and

$$\frac{\omega}{E} = \left(\frac{4-t}{4m^2-t}\right)^{1/2}.$$

The $N\bar{N} \to \pi\pi$-channel partial-wave expansion was given by Frazer and Fulco.[13] Since backward $\pi N$ scattering implies also backward $N\bar{N} \to \pi\pi$, the first two terms of the expansion for $F^{(+)}$ are

$$F^{(+)}(t) = \frac{16\pi}{m(4m^2-t)} f_+^0(t) - \frac{5\pi(t-4)}{m} f_+^2(t) + \cdots . \tag{2}$$

The unitarity condition on $f_+^0$ applied at $4 < t < 16$, where only the $\pi\pi$ intermediate state contributes, gives the phase relation[13]

$$f_+^0(t) = \pm |f_+^0(t)| e^{i\delta_0^0(t)}, \tag{3}$$

where $\delta_0^0$ is the $I=J=0$ $\pi\pi$ phase shift. Actually Eq. (3) is correct for all values of $t$ such that the $I=J=0$ $\pi\pi$ state scatters elastically. This appears to be true experimentally for $t \leq 50$.[14] The sign is known to be positive from extrapolations of $A^{(+)}$ toward $t=4$ by fixed-$t$ dispersion relations[15,16] [$A^{(+)}$ is purely $s$ wave at $t=4$]. The positive sign is also indicated by the current-algebra prediction for $A^{(+)}$ at the Adler point.[17]

We define the function $D(t)$ by[18,13]

$$D(t) = \exp\left[\frac{-t}{\pi}\int_4^\infty \frac{\delta_0^0(t')}{t'(t'-t-i\epsilon)}dt'\right]. \tag{4}$$

The singularity structure of the product $D(t)\tilde{F}^{(+)}(t)$ consists of the $\pi N$ physical cut at $t<0$, the nucleon pole at $t = 4 - 1/m^2$, and the $\pi\pi$ pseudophysical cut at $t>4$. There is no contribution from $f_+^0$ to the discontinuity along the $t>4$ branch cut for values of $t$ such that Eq. (3) is valid.

The unknown distant singularities can be isolated into a "discrepancy function" $\Delta(t)$ by writing[19]

$$\mathrm{Re}[D(t)\tilde{F}^{(+)}(t)] = \Delta(t) - \frac{1}{\pi}\int_{L_1}^0 \frac{D(t')\,\mathrm{Im}F^{(+)}(t')dt'}{t'-t} + \frac{t}{\pi}\int_4^{L_2} \frac{\mathrm{Im}[D(t')\tilde{F}^{(+)}(t')]dt'}{t'(t'-t)}, \tag{5}$$

where

$$\tilde{F}^{(+)}(t) = F^{(+)}(t) - F_B^{(+)}(t),$$

and the nucleon-pole (Born) contribution is

$$F_B^{(+)}(t) = -g^2\frac{t-4}{m^2(t-a)},$$

$$a = 4 - \frac{1}{m^2}, \quad \frac{g^2}{4\pi} = 14.28.$$

Note that $F_B^{(+)}$ contains both the $s$- and $u$-channel poles. We will take $L_2 < -L_1$ so that the discrepancy $\Delta(t)$ is an analytic function for $|t| < L_2$.

Estimates of the isobar contributions and the

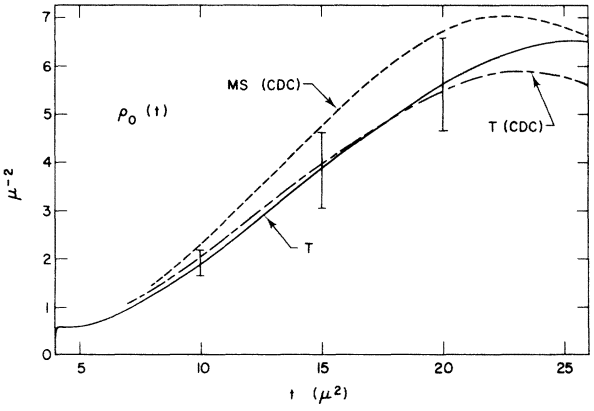


FIG. 1. The spectral function $\rho_0$. The low-energy $\delta_0^0$ was from Ref. 18 for curves $T$ and $T$ (CDC) and from Ref. 32 for $MS$ (CDC). The optimal expansion was used for $T$ (CDC) and $MS$ (CDC). See the text for the meaning of the error bars.

small size of the $J \geq 4$ $\pi\pi$ phase shifts indicate that the $J \geq 4$ $N\bar{N} \to \pi\pi$ partial-wave amplitudes in the pseudophysical region are almost totally dominated by the nucleon poles, at least for values of $t$ of interest here ($t \lesssim 45$). This point is discussed in the Appendix. For $t > 4$, $\tilde{F}^{(+)}(t)$ can, therefore, be written in terms of the $J = 0, 2$ partial-wave amplitudes:

$$\mathrm{Re}[D(t)\tilde{F}^{(+)}(t)] = \frac{16\pi |D(t)|}{m(4m^2 - t)}[|f_+^0(t)| - \cos\delta_0^0 f_{+B}^0(t)] - \frac{5\pi(t-4)|D(t)|}{m}[\cos\delta_0^0 \mathrm{Re}\tilde{f}_+^2(t) - \sin\delta_0^0 \mathrm{Im} f_+^{\,2}(t)], \tag{6}$$

$$\mathrm{Im}[D(t)\tilde{F}^{(+)}(t)] = \frac{-5\pi(t-4)|D(t)|}{m}[\cos\delta_0^0 \mathrm{Im} f_+^{\,2}(t) - \sin\delta_0^0 \mathrm{Re}\tilde{f}_+^{\,2}(t)] + \frac{16\pi \sin\delta_0^0 |D(t)|}{m(4m^2 - t)} f_{+B}^0(t), \tag{7}$$

where $\tilde{f}_+^J = f_+^J - f_{+B}^J$. The appearance of a $J = 0$ term in Eq. (7) is caused by the subtraction of the nucleon-pole terms from $F^{(+)}$. The nucleon-pole contributions to $f_+^0$ and $f_+^{\,2}$ are

$$f_{+B}^0(t) = \frac{mg^2}{4\pi}[h \arctan h^{-1} - 1], \tag{8}$$

$$f_{+B}^{\,2}(t) = \frac{2mg^2}{\pi} \frac{h}{(t-4)(4m^2 - t)} \times [(3h^2 + 1)\arctan h^{-1} - 3h], \tag{9}$$

where

$$h = \frac{t-2}{[(t-4)(4m^2 - t)]^{1/2}}.$$

The $N\bar{N} \to \pi\pi$-channel partial-wave expansion of $\tilde{F}^{(+)}$ converges within an ellipse whose size is limited by the $s$- and $u$-channel ($\pi N$) threshold branch points. For $4 < t < 4m^2$ one finds singularities in the cosine of the scattering angle

$$\cos\theta_t = \frac{s-u}{[(t-4m^2)(t-4)]^{1/2}} \tag{10}$$

at $\cos\theta_t = \pm x_c$, where

$$x_c = \frac{i(t+4m)}{[(4m^2 - t)(t-4)]^{1/2}}.$$

The smallest convergence ellipse occurs at $t = 4m$ with a semimajor axis of 1.35, so that for $\cos\theta_t = -1$, as in Eq. (2), convergence is expected throughout the pseudophysical region.

In the $\pi N$ physical region, $t < 0$, $F^{(+)}$ can be evaluated from its partial-wave expansion:

$$F^{(+)}(t) = \frac{4\pi(E+\omega)}{mE} \sum_{l=0}^{\infty} (-1)^l [(l+1) f_{l+}^{(+)}(k) + l f_{l-}^{(+)}(k)],$$

$$f_{l\pm}^{(+)} = \tfrac{1}{3}(f_{l\pm}^{(1/2)} + 2 f_{l\pm}^{(3/2)}),$$

$$f_{l\pm}^{(T)} = \frac{1}{2ik}[\eta_{l\pm}^{(T)} \exp(2i\delta_{l\pm}^{(T)}) - 1],$$

where the superscript $T$ denotes the $\pi N$ isospin $\frac{1}{2}$ or $\frac{3}{2}$.

## III. RESULTS

The low-$t$ phase shift $\delta_0^0$ used in this work is the solution recently given by Tryon[20] based on twice-subtracted dispersion relations for the $\pi\pi$ amplitudes. At low dipion energy the solution is well represented by a simple function of one parameter which is determined by fitting the $K_{e4}$ decay results for $\delta_0^0$. The resulting scattering length is $a_0 = 0.26 \pm 0.08\mu^{-1}$. Above $t = 20$ we have used the Berkeley[14] solution obtained by an energy-dependent fit to data primarily on $\pi N \to \pi\pi\Delta$. The $J = 2$ $N\bar{N} \to \pi\pi$ amplitude $f_+^{\,2}$ was evaluated in Ref. 3 using a partial-wave dispersion relation involving the $I = 0$ $d$-wave $\pi\pi$ phase shift.

To perform the dispersion integrals we have used three sets of $\pi N$ phase shifts:

(a) Almehed and Lovelace[21] $-165 < t < -21$,
(b) Carter, Bugg, and Carter[22] $-19 < t < -4.4$,
(c) Nielsen and Oades[23] $0 < t < -2$.

To evaluate the left-hand side of Eq. (5) and hence $\Delta(t)$, where accuracy is especially important, we have used only the CBC phase shifts[22,24] for the final results.

The discrepancies were calculated at the 10 points of the CBC analysis using $L_1 = -165$ and $L_2 = 45$, then extrapolated to $t > 4$ via a Legendre expansion in $t$. The discrepancies and a quadratic fit are shown in Fig. 2.

By equating the right-hand sides of Eq. (5) and Eq. (6) we can write $|f_+^0(t)| = \alpha(t)[\Delta(t) + B(t) + I(t)]$, where $\alpha(t) = m(4m^2 - t)/16\pi|D(t)|$, $B(t)$ is from the $f_{+B}^0$ and $f_+^{\,2}$ contributions in Eq. (6), and $I(t)$ denotes the integrals in Eq. (5). The quantities $\Delta$, $B$, and $I$ are displayed in Fig. 3. One sees that the largest part of $|f_+^0|$ comes from the extrapolated discrepancy.

At the $\pi\pi$ threshold we obtained $f_+^0(4) = 113\mu$ and $A^{(+)}(s, t=4) = 4\pi f_+^0(4)/(m^2 - \mu^2) = 32\mu^{-1}$. This value is consistent with the current-algebra value at the Adler point[17] and with the value and derivatives at $t = 0$. For example, using the value $(25.9\mu^{-1})$ and the slope $(1.16\mu^{-3})$ of $A^{(+)}(s = u, t)$ at $t = 0$ from Ref. 3, one finds $A^{(+)}(s, t = 4) = 31\mu^{-1}$. The same value

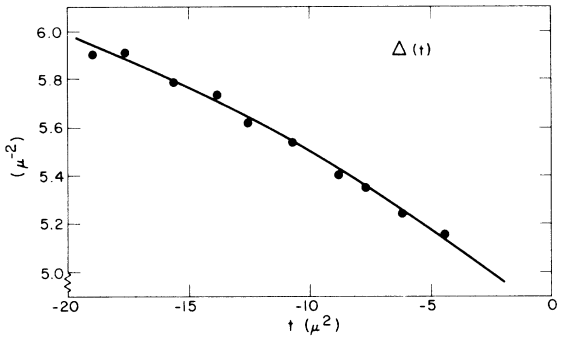


FIG. 2. The discrepancies and fit which give curve $T$ of Fig. 1.

is found from Ref. 25. We note, however, that using the value and derivatives at $t = 0$ to extrapolate to the $\pi\pi$ threshold is of questionable accuracy since one is extrapolating to the limit of the domain of convergence. The procedure used here does not suffer this fault.

The results for $t > 4$ are given by the solid curve in Fig. 1. The quantity shown there is $\rho_0$ defined as

$$\rho_0(t) = \frac{8\pi}{(4m^2 - t)^2}\left(\frac{t-4}{t}\right)^{1/2} |f_+^0(t)|^2 .$$

This quantity is the $J = 0$ spectral function required in the $NN$ dispersion theory. Note that $\rho_0(4) = 0$, and that the peculiar shape near threshold is caused by a rapid change in the nucleon pole terms, which vanish at threshold.

The results of the extrapolation may be expected to depend on the function used to represent $\Delta(t)$. The conformal mapping of Cutkosky, Deo,[26] and Ciulli[27] (CDC) is designed to maximize the convergence of a polynomial expansion by introducing the locations of the branch cuts explicitly into the expansion variable. The result of the mapping is to take the entire cut plane into an ellipse whose size is determined by the locations of the branch points. The results of performing the extrapolation of $\Delta(t)$ via a quadratic expansion in the CDC variable are shown in Fig. 1.[28] These results differ from those of the expansion in $t$ by less than 10% for all $t < 24$.

Ciulli *et al.*[29] have emphasized the need for information other than analyticity to stabilize extrapolations against experimental errors. Stabilization is achieved here by restricting the order of the polynomial approximation to $\Delta(t)$. A quadratic approximation gives a good fit to the experimental discrepancy values without being highly sensitive to the data noise.

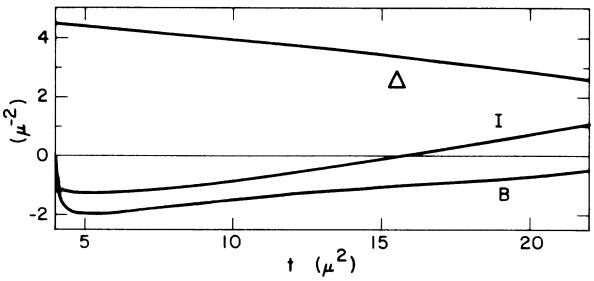


FIG. 3. The contributions to $|f_+^0(t)|$.

## IV. ESTIMATION OF ERROR LIMITS

Although no error matrices are available for the CBC phase-shift analysis, we have obtained information on the correlations in the phase shifts by private communication.[30] We have approximated the correlated errors by decreasing the $P_{33}$, $S_{31}$, $P_{31}$, $S_{11}$, $P_{13}$, and $P_{11}$ phase shifts by 1.5, 15, 15, 15, 10, and 6 percent, respectively. The results for $\rho_0$ are changed by about ±10% except at the smallest values of $t$, where the change is smaller. We note that the correlations are important since a 1.5% change in just the $P_{33}$ phase shift gives twice these error limits.

As an indication of the stability of the extrapolation to changes in the distribution of discrepancies used in the fit, we have defined a second measure of error as follows. Define each of 10 subsets of discrepancies by removing one of the 10 values from the complete set. Perform the extrapolation with each subset to obtain a total of 10 values of $\rho_0(t)$ for every $t$. Then define the quantity

$$\epsilon^2(t) = \tfrac{1}{9}\sum_{k=1}^{10}[\rho_0^{(k)}(t) - \bar{\rho}_0(t)]^2,$$

where $\rho_0^{(k)}$ is obtained from the $k$th discrepancy subset and $\bar{\rho}_0$ is the mean. We found that $\bar{\rho}_0$ differs from the result using all 10 discrepancies by less than 1% at all values of $t$ in Fig. 1. The quantity $\epsilon(t)$ is shown as error bars in Fig. 1. The ratio $\epsilon(t)/\bar{\rho}_0(t)$ increases with $t$ and reaches 20% at $t = 20$.

We have also performed the analysis using the later Nielsen-Oades[25] $\pi N$ phase shifts rather than the CBC phase shifts. The Nielsen-Oades phase shifts differ from the CBC phase shifts in that the former were constrained by fixed-$t$ dispersion relations. The results for $\rho_0$ are almost unchanged.

The fact that the right-hand cut in the procedure used here involves the $J = 2$ $N\bar{N} \to \pi\pi$ amplitude in an unphysical region does not introduce a large uncertainty. The lack of structure in the low-energy $I = 0$, $J = 2$ $\pi\pi$ partial wave causes $\tilde{f}_+^{\,2}$ to vary slowly so that it can be accurately evaluated by the procedure used in Ref. 3. The $J = 2$ amplitude contri-

butes only a small part of the right-hand side of Eq. (6); however, it does cancel much of the $J=0$ contribution to Eq. (7). Nevertheless, we found that a 10% reduction in $\tilde{f}_+^2(t)$ for all $t>4$ causes only an 8% reduction in $\rho_0(20)$.

The largest source of uncertainty is the phase shift $\delta_0^0$, especially its value at low $\pi\pi$ energy. One can find experimental evidence that the scattering length may be negative[31] or as large as[32] $0.6\mu^{-1}$, but most recent work is consistent with Weinberg's prediction[33] of $a_0=0.2\mu^{-1}$. To judge the sensitivity of our results to $\delta_0^0$, we have performed the analysis with the phase shift obtained from the current-algebra model of Morgan and Shaw[34] rather than Tryon's[20] values. The low-energy behavior of the $s$-wave amplitudes can be compared by writing $\mathrm{Ref}_0^0=a_0+b_0q^2+\cdots$; Morgan and Shaw have $a_0=0.21\mu^{-1}$, $b_0=0.25\mu^{-3}$, whereas Tryon has $a_0=0.26\mu^{-1}$, $b_0=0.14\mu^{-3}$. The results for $\rho_0$ using the Morgan-Shaw $\delta_0^0$ and the CDC conformal mapping are shown in Fig. 1. The results without the CDC mapping differ by less than 10% for $t<25$.

## V. DISCUSSION

Dispersion relations for the backward $\pi N$ amplitudes have had extensive use in the past.[10,11,35,36] Much of this work has been directed toward determining the $\pi\pi$ phase shifts, especially $\delta_0^0$, via Eq. (3). The attempts to determine $\delta_0^0$ have had only limited success, primarily because they required an extrapolation to the $\pi\pi$ branch cut and because the backward amplitude in the $\pi N$ physical region is only slightly sensitive to the shape of $\delta_0^0(t)$. On the other hand, small changes in the near-threshold $\pi N$ amplitude can produce unacceptably large changes in $\delta_0^0$.[11]

Since it appears that $\delta_0^0$ may be more reliably determined by other means we have used it to make an "interior to interior" extrapolation possible, avoiding the extrapolation to a branch cut. This technique has been used by Elvekjaer[37] and by Nielsen and Oades[38] in conjunction with partial-wave dispersion relations. However, in their case the discontinuity on the left-hand cut was known only to $t\simeq -26$, and was determined by another extrapolation from the $\pi N$ physical region. In the procedure used here the $\pi N$ amplitude required on the left-hand cut is always in the physical region, and the left-hand cut integral can be evaluated to large values of $-t$.

Twice-subtracted dispersion relations have recently been used to obtain $f_+^0$.[3,7] The result of Epstein and McKellar[7] differs from ours primarily in the region around $t=10$, where their $\rho_0$ is somewhat larger than the value found here.

We have found that $\rho_0$ could be determined below $t=20$ to within about $\pm 20\%$ if the $\pi\pi$ phase shift $\delta_0^0$ were known without uncertainty. Variations in the low energy $\delta_0^0$ near the current-algebra predictions are expected to leave $\rho_0$ within or near these limits. These limits could probably be improved when accurate Coulomb-corrected $\pi N$ phase shifts very near threshold become available (the lowest energy in the CBC analysis is $E_\pi=88.5$ MeV).

## ACKNOWLEDGMENTS

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## APPENDIX

We noted in Sec. II that the point $\cos\theta_t=-1$ is within the domain of convergence of the $N\bar{N}\to\pi\pi$-channel partial-wave expansion above the $\pi\pi$ threshold, provided the nucleon poles are removed. Thus, we were able to truncate the expansion to write Eq. (6) and Eq. (7). We wish to estimate the accuracy of this approximation.

We begin with the fixed-$t$ dispersion relations:

$$A^{(+)}(x,t)=\frac{1}{\pi}\int_{x_0}^{\infty}dx'\,\sigma_A(x',t)\left(\frac{1}{x'-x}+\frac{1}{x'+x}\right),$$

$$\tilde{B}^{(+)}(x,t)=\frac{1}{\pi}\int_{x_0}^{\infty}dx'\,\sigma_B(x',t)\left(\frac{1}{x'-x}-\frac{1}{x'+x}\right),$$

where $x$ is related to the Mandelstam variables by $x=\frac{1}{2}(s-u)$ and $x_0=\frac{1}{2}(2m+t)$. These relations have been written in unsubtracted form although the relation for $A^{(+)}$ may in fact require one subtraction for $t<4$.[39-41] If a Mandelstam representation is valid, then these equations may be extended to the region $t>4$, although the number of subtractions required may change. Subtractions would modify Eq. (11) and Eq. (12) given below, but Eq. (13) would be unchanged if no more than two subtractions are required for $A^{(+)}$ and one subtraction for $\tilde{B}^{(+)}$.[41]

Using Eq. (10) with $\cos\theta_t=-1$, we find the following expression for $\tilde{F}^{(+)}(t)$:

$$\tilde{F}^{(+)}(t)=\frac{1}{\pi m p_- q}\int_{x_0}^{\infty}dx'\left(h(x',t)\sigma_A(x',t)-\frac{mq}{p_-}\sigma_B(x',t)\right)\times\frac{1}{h^2(x',t)+1},\tag{11}$$

where

$$q=(t/4-1)^{1/2},\quad p_-=(m^2-t/4)^{1/2},$$

$$h(x',t)=x'/2p_-q\,.$$

The following expressions for $\tilde{f}_+^0$ and $\tilde{f}_+^2$ are obtained by using the projection formulas given by Frazer and Fulco[13]:

$$\tilde{f}_+^0(t) = \frac{1}{4\pi^2}\int dx' \left[ \frac{p_-}{q}\sigma_A \arctan h^{-1} + m\sigma_B(h \arctan h^{-1} - 1)\right] \tag{12a}$$

$$\tilde{f}_+^2(t) = \frac{1}{8\pi^2 p_- q^3}\int dx' \left(\sigma_A + \frac{mq}{p_-}h\sigma_B\right) \times \left[(3h^2+1)\arctan h^{-1} - 3h\right] . \tag{12b}$$

The integration limits and the arguments of $h$ and $\sigma_{A,B}$ remain as in Eq. (11).

We define $H(t)$ to be the difference between the amplitude of Eq. (11) and the sum of the first two partial waves as given by Eq. (2) and Eq. (12). The large-$x'$ contributions to $H$ are suppressed as one can see by writing

$$H(t) = \frac{1}{\pi}\int dx' \mathcal{K}(x', t),$$

and expanding $\mathcal{K}$ in powers of $h^{-1}$. We find

$$\mathcal{K}(x', t) = \frac{1}{mp_- q}\frac{8}{35}h^{-5}\left(\sigma_A + \frac{mq}{p_-}h\sigma_B\right) + \cdots . \tag{13}$$

The actual suppression depends on the behavior of $\sigma_{A,B}(x', t)$.

To obtain an estimate of the numerical size of $H(t)$ we have included the lowest-mass isobar, the $\Delta(1236)$, in the narrow-width approximation. In this approximation the $\sigma_{A,B}$ have the form

$$\sigma_{A,B}(x', t) = \pi G_{A,B}(t)\,\delta(s' - m_\Delta^2), \tag{14}$$

with

$$s' = x' + m^2 + 1 - \tfrac{1}{2}t .$$

We have used the residues $G_{A,B}$ derived by Epstein and McKellar,[7] which in our notation are $G_A/4\pi = 36.0 + 1.6t$, $G_B/4\pi = -12.2 + 0.1t$.

The third column of Table II contains the numerical value of $H(t)$ computed from Eqs. (11), (2), (12), and (14). If $H$ were included in Eq. (6) it would appear multiplied by just $\cos\delta_0^0$. For comparison, therefore, the value of $\mathrm{Re}(e^{-i\delta_0^0}\tilde{F}^{(+)})$ computed from Eq. (5) is also given.

TABLE II. Estimates of the $J \geq 4$ contributions to $F^{(+)}$ and $\rho_0$.

| $t$ | $\mathrm{Re}(e^{-i\delta_0^0}\tilde{F}^{(+)})$ | $H_\Delta$ | $R_\Delta$ |
|---|---|---|---|
| 5 | 4.5 | $4.0\times10^{-4}$ | 0.0012 |
| 10 | 4.5 | 0.011 | −0.0020 |
| 20 | 4.4 | 0.050 | −0.0042 |

Including only the first two partial waves in Eq. (6) affects the definition and extrapolation of the discrepancy through the second integral of Eq. (5). The contribution of $H(t)$ to this integral is

$$Y(t) \equiv \frac{-t}{\pi}\int_4^{L_2}\frac{|D(t')|\sin\delta_0^0(t')H(t')}{t'(t'-t)}\,dt' .$$

The best[42] orthogonal polynomial expansion in $t$ over the region where the experimental discrepancies exist, $a < t < b$ (we have $a = -19$, $b = -4$), is the Legendre one:

$$Y(t) = \sum_{n=0}^{\infty} c_n P_n(z(t)),$$

$$c_n = \frac{-(2n+1)}{\pi}\int_4^{L_2}|D(t')|\sin\delta_0^0(t')H(t') \times \left[\frac{2Q_n(z(t'))}{b-a} - \frac{\delta_{n,0}}{t'}\right]dt',$$

$$z(t) = \frac{2t-a-b}{b-a} .$$

The error in $\rho_0$ caused by illegally continuing this series to $t \geq 4$ as a part of the quadratic approximation to $\Delta(t)$ is given approximately by the ratio

$$R(t) = \frac{\alpha(t)[\,Y(t) - c_0 - c_1 z(t) - c_2 P_2(z(t))]}{|f_+^0(t)|} .$$

The fourth column of Table II contains this quantity with only the $\Delta(1236)$ included in $H(t)$. An actual fit to $Y(t)$, $a < t < b$, confirms these estimates. The $|f_+^0|$ used in this ratio and the numbers in the second column of Table II correspond to curve $T$ of Fig. 1.

The $P_{11}(1470)$ and $D_{13}(1520)$ isobar terms have also been computed; their effect is a small fraction of that of the $\Delta(1236)$. The numerical values in Table II suggest that the error incurred by neglecting all but the first two partial waves of $\tilde{F}^{(+)}$ is at most a few percent.

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