

## Analysis of radial scaling in single-particle inclusive reactions

F. E. Taylor

*Northern Illinois University, DeKalb, Illinois 60115*

D. C. Carey, J. R. Johnson,\* R. Kammerud, D. J. Ritchie, A. Roberts,  
J. R. Sauer,† R. Shafer, D. Theriot, and J. K. Walker

*Fermi National Accelerator Laboratory, Batavia, Illinois 60510†*

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An analysis of an extensive sample of the world's data has been performed to test the hypothesis of radial scaling. We have studied the inclusive reactions  $p + p \rightarrow (\pi^{\pm,0} \text{ or } K^{\pm} \text{ or } p \text{ or } \bar{p}) + \text{anything}$  to determine the behavior of the invariant cross section as a function of  $p_{\perp}$ ,  $x_R = E^*/E_{\text{max}}^*$ , the radial scaling variable, and  $s$ . The data cover a range in  $p_{\perp}$  from 0.25 to  $\sim 6.0$  GeV/c and a range in  $\sqrt{s}$  from 3.0 to 63 GeV. For small  $x_R$  and all available  $p_{\perp}$  the single-particle inclusive cross sections for the reactions studied scale to a good approximation for all  $\sqrt{s}$ , even down to the kinematic threshold. For large  $x_R$ , the single-particle inclusive cross sections for increasing  $\sqrt{s}$  show a rapid approach to the scaling limit from above. In these cases the scaling limit is always approached by  $\sqrt{s} \approx 10$  GeV. Thus, data for all particles to a good approximation exhibit radial scaling at all available  $p_{\perp}$  and  $x_R$  over the CERN ISR energy range. A comparison of radial scaling with Feynman scaling is given. It is shown that in the Feynman case the cross sections for small  $x_{\parallel}$  ( $x_{\parallel} = p_{\parallel}^*/p_{\text{max}}^*$ ) approach their scaling limit from below, and that the approach to the scaling limit is slower than is exhibited for the case of small  $x_R$ . The systematic differences among the inclusive cross sections of various particles are discussed in the range of  $\sqrt{s}$  where radial scaling has been shown to be valid. In particular, the  $p_{\perp}$  and  $x_R$  distributions of  $E d\sigma/dp^3$  are examined.

### I. INTRODUCTION

In a separate paper we presented data on inclusive  $\pi^0$  production from  $pp$  collisions at the Fermi National Accelerator Laboratory.<sup>1</sup> Here we present an analysis and interpretation of these  $\pi^0$  data. In addition, we examine other single-particle inclusive cross sections in order to search for systematics in their behavior.

To study the single-particle inclusive experiments, a set of variables should be chosen such that the single-particle inclusive cross section displays the simplest behavior with respect to energy,  $p_{\perp}$ , the center-of-momentum (c.m.) frame angle, etc. It is natural to choose a parameterization of the inclusive cross section which is meaningful in the exclusive limit.<sup>2</sup> We therefore define the scaling variable

$$x_R = \left( \frac{E^*}{E_{\text{max}}^*} \right), \quad (1)$$

where  $E^*$  = the energy of the detected particle in the center-of-momentum frame, and  $E_{\text{max}}^*$  = the maximum energy kinematically available to the detected particle in the c.m. frame. The range of  $x_R$  is  $0 < x_R \leq 1$  for all  $p_{\perp}$ , and the case  $x_R = 1$  corresponds to the exclusive limit. Since this variable is independent of the center-of-momentum angle and depends on only the radial distance from the kinematic boundary, we have called it the "radial" scaling variable.

Using this variable  $x_R$ , the proton-proton single-

particle inclusive cross section can be expressed as a function of three variables,  $s$ ,  $p_{\perp}$ , and  $x_R$ ,

$$E \frac{d\sigma}{dp^3} = f(s, p_{\perp}, x_R), \quad (2)$$

where  $s$  is the square of the total c.m. energy. In this work we study whether the invariant cross section at sufficiently high energy scales, that is, becomes independent of  $s$ . It is shown that this scaling is reached at a lower  $s$  than obtained with the use of the Feynman variable  $x_{\parallel}$ .

In an analysis of a single- $\pi^0$  inclusive experiment in  $p$ - $p$  collisions,<sup>3</sup> it has been shown that for sufficiently high  $\sqrt{s}$  ( $\sqrt{s} \approx 10$  GeV) there is radial scaling

$$E \frac{d\sigma}{dp^3} \approx f(p_{\perp}, x_R). \quad (3)$$

This scaling is observed for  $0.3 \text{ GeV}/c \leq p_{\perp} \leq 3.0 \text{ GeV}/c$ ,  $10 \text{ GeV} \leq \sqrt{s} \leq 27 \text{ GeV}$ , and for center-of-mass angle  $20^\circ \leq \theta^* \leq 150^\circ$ . It is therefore interesting to test this new form of scaling over as wide a kinematic range as possible for other particles produced in  $p$ - $p$  collisions. We shall discuss the reactions

$$p + p \rightarrow (\pi^{\pm,0} \text{ or } K^{\pm} \text{ or } p \text{ or } \bar{p}) + \text{anything}.$$

In Sec. II, the variable  $x_R$  is discussed and it is compared with the Feynman scaling variable  $x_{\parallel}$ . Section III is a discussion of the method of data analysis. A comparison of radial scaling with Feynman scaling is given in Sec. IV. Section V

is a presentation of the various systematic differences among single-particle inclusive cross sections for various particles in  $p$ - $p$  collisions in the radial scaling region. A summary is given in Sec. VI.

## II. THE $x_R$ VARIABLE

In an analysis of single- $\pi^0$  inclusive production,<sup>3</sup> it has been shown that the  $s$  dependence and the laboratory angular dependence of the invariant cross section could be succinctly described by writing the invariant cross section in terms of the transverse momentum  $p_\perp$  and the variable  $x_R$ . This variable may be written in Lorentz-invariant form as

$$x_R = \frac{E^*}{E_{\max}^*} = \frac{1 - M_X^2/s + M_c^2/s}{1 - \bar{M}_X^2/s + M_c^2/s}, \quad (4)$$

where  $M_X$  = the invariant mass of the unspecified particle(s) (undetected),  $\bar{M}_X$  = the minimum possible  $M_X$  value,  $M_c$  = the rest mass of the detected particle, and  $s$  = the square of the total energy in the center-of-momentum frame. The variable  $x_R$  has been used by several authors and was probably first used by Kinoshita and Noda<sup>4</sup> in 1971, although apparently it had been discussed by Feynman<sup>5</sup> in 1969.

The maximum energy ( $E_{\max}^*$ ) kinematically available to the detected particle  $c$  in the inclusive reaction

$$p + p \rightarrow c + \text{anything}$$

is determined by  $s$ , the square of the total energy in the c.m. frame, by the mass  $M_c$  of the detected particle  $c$ , and by  $\bar{M}_X$ , the minimum mass of the undetected particle system consistent with quantum number conservation (charge, strangeness, baryon number, etc.). For a single-particle inclusive reaction this minimum value of  $M_X$  corresponds to the exclusive limit. A compilation of the minimum value of  $M_X$  and the exclusive limit reactions is given in Table I. We can express  $E_{\max}^*$  as

$$E_{\max}^* = \frac{s - \bar{M}_X^2 + M_c^2}{2\sqrt{s}}. \quad (5)$$

Since  $x_R$  is a ratio of total energies, the value of  $x_R$  at the particle  $c$  production threshold is 1, and for fixed, finite  $E^*$ ,  $x_R = 0$  only at infinite  $s$ .

The radial scaling variable  $x_R$  is distinguished from the Feynman<sup>6</sup> variable  $x_\parallel = p_\parallel^*/p_{\max}^* \cong 2p_\parallel^*/\sqrt{s}$  in that the detected particle's total energy is scaled by its maximum value, rather than its longitudinal momentum by the maximum momentum. The use

TABLE I. For a given inclusive reaction, the corresponding exclusive reaction and the minimum mass  $\bar{M}_X$  are tabulated. The mass  $\bar{M}_X$  is determined by the mass of the undetected particle system in the exclusive limit reaction.

| Inclusive reaction                            | Exclusive reaction                        | $\bar{M}_X$             |
|---|---|-------------------------|
| $p + p \rightarrow \pi^+ + \text{anything}$   | $p + p \rightarrow p + n + \pi^+$         | 1.88 GeV/c <sup>2</sup> |
| $p + p \rightarrow \pi^- + \text{anything}$   | $p + p \rightarrow p + p + \pi^+ + \pi^-$ | 2.02 GeV/c <sup>2</sup> |
| $p + p \rightarrow \pi^0 + \text{anything}$   | $p + p \rightarrow p + p + \pi^0$         | 1.88 GeV/c <sup>2</sup> |
| $p + p \rightarrow K^+ + \text{anything}$     | $p + p \rightarrow \Lambda^0 + p + K^+$   | 2.05 GeV/c <sup>2</sup> |
| $p + p \rightarrow K^- + \text{anything}$     | $p + p \rightarrow p + p + K^+ + K^-$     | 2.37 GeV/c <sup>2</sup> |
| $p + p \rightarrow \bar{p} + \text{anything}$ | $p + p \rightarrow p + \bar{p}$           | 0.94 GeV/c <sup>2</sup> |
| $p + p \rightarrow \bar{p} + \text{anything}$ | $p + p \rightarrow p + p + p + \bar{p}$   | 2.81 GeV/c <sup>2</sup> |

of only the longitudinal momentum component means that the variable  $x_\parallel$  is not related to the location of the kinematic boundary at finite  $p_\perp$ . One might, therefore, expect the invariant cross section to have an  $s$  dependence for fixed  $p_\perp$ , owing entirely to kinematic effects. In particular, one would expect  $E d\sigma/dp^3$  for fixed  $p_\perp$  and  $x_\parallel \cong 0$  to rise with increasing  $\sqrt{s}$  as the point in the  $x_\parallel$ - $x_\perp$  plane at which the measurement is being made moves further from the kinematic boundary. Therefore, in the region of the center-of-mass angle  $\theta^* \cong \pi/2$ ,  $x_\parallel$  is clearly not the best variable for studying scaling and the systematics of single-particle inclusive production. Other frequently used variables such as rapidity,

$$y^* = \frac{1}{2} \ln \left( \frac{E^* + p_\parallel^*}{E^* - p_\parallel^*} \right), \quad (6)$$

and  $x_\perp$ ,

$$x_\perp = \frac{2p_\perp}{\sqrt{s}}, \quad (7)$$

also are not related to the location of the kinematic boundary, and suffer the same defects. On the other hand, the expression of the invariant cross section in terms of the variables  $x_R$  and  $p_\perp$  allows the  $s$  dependence of the cross section to be studied at a fixed distance from the kinematic boundary.

It is instructive to contrast radial scaling with Feynman scaling by examining the methods by which single-particle inclusive measurements are made. In both cases, consider the invariant cross section at a fixed  $p_\perp$  and compare, for example,  $x_\parallel = 0.3$  to  $x_R = 0.3$ . Figure 1 shows the curves as a function of  $s$  along which these measurements are made in both cases. This plot is made in the plane of  $x_\perp = 2p_\perp/\sqrt{s}$  and  $x_\parallel = 2p_\parallel^*/\sqrt{s}$ . In the limit of  $s \rightarrow \infty$  we see that radial scaling and Feynman scaling are identical hypotheses:

$$E \frac{d\sigma}{dp^3} \cong f(p_\perp, x_R) = f(p_\perp, x_\parallel). \quad (8)$$

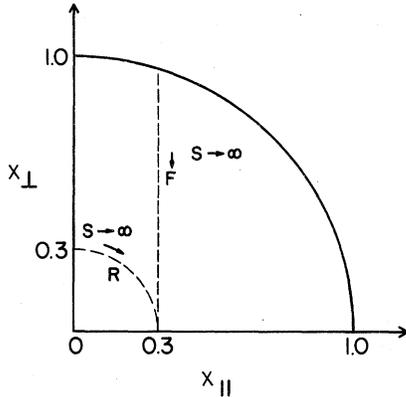


FIG. 1. The plot shows the lines in the center-of-momentum frame along which measurements are made to study scaling in the case of fixed Feynman ( $F$ ) variable  $x_{\parallel}$  and fixed radial ( $R$ ) variable  $x_R$ .

However, the finite- $s$  behavior is quite different in the two cases. In the case of Feynman scaling the point at which the measurement is made moves away from the kinematic boundary. A large  $s$ -dependent increase in the cross section due to increasing phase space is thereby introduced. On the other hand, measurements at fixed  $x_R$  require that the fractional distance to the kinematic boundary remain constant. In this way, it appears that the  $s$  dependence of the dynamics may be more directly probed.

Another property of the variable  $x_R$  which distinguishes it from the Feynman variable  $x_{\parallel}$  is its totally different  $s$ -threshold behavior. The  $s$  threshold for a given value of  $p_{\perp}$  is defined as the minimum value of  $s$  which can have the specified value of  $x_R$  or  $x_{\parallel}$ . For the  $x_R$  variable, this threshold is given by

$$s_T = 2T + \bar{M}_X^2 - M_c^2 + 2[T^2 + T(\bar{M}_X^2 - M_c^2)]^{1/2}, \quad (9)$$

where

$$T = \frac{p_{\perp}^2 + M_c^2}{x_R^2}.$$

Hence for the limit  $x_R = 0$ , the  $s$  threshold  $s_T$  is  $\infty$ , for a fixed value of  $p_{\perp}$ . The limit  $x_R = 1$  (the exclusive limit) corresponds to a finite  $s_T$  for finite  $p_{\perp}$ . Comparing the  $s$ -threshold values at these two extreme values of  $x_R = 0$  and  $x_R = 1$  with the corresponding  $s$  thresholds for the same extremes for  $x_{\parallel}$ , there are very great differences. The  $s$  threshold for the Feynman variable  $x_{\parallel}$  is determined by the kinematic boundary, so letting  $p_{\text{max}}^*$  be the maximum possible momentum of the particle in the c.m. frame, then,

$$p_{\text{max}}^* = \frac{p_{\perp}}{(1 - x_{\parallel}^2)^{1/2}} \approx \frac{\sqrt{s_T}}{2}. \quad (10)$$

Hence, in the limit  $x_{\parallel} = 0$  and  $p_{\perp}$  fixed,

$$\sqrt{s_T} \approx 2p_{\perp}. \quad (11)$$

The limit  $x_{\parallel} = 1$  corresponds to  $\sqrt{s} = \infty$  for fixed  $p_{\perp}$ . Thus for a given  $x_{\parallel}$  or  $x_R$  and fixed  $p_{\perp}$  the threshold energy in the two cases is quite different.

We wish to compare the  $s$  dependence of the invariant cross section for fixed  $x_R$ ,  $p_{\perp}$  with the invariant cross section for the same numerical value of  $x_{\parallel}$  at the same value of  $p_{\perp}$ . In this way the two cross sections will approach the same asymptotic limit. We are interested in this approach to the asymptotic limit. There is clearly some point at which the thresholds occur at the same energy  $\sqrt{s_T}$ . This happens for (using high-energy approximations)

$$\frac{2p_{\perp}}{x_R} = \frac{2p_{\perp}}{(1 - x_{\parallel}^2)^{1/2}}, \quad (12)$$

i.e.,  $x = x_R = x_{\parallel} = 1/\sqrt{2}$ . Hence for  $x_R = x_{\parallel} < 1/\sqrt{2}$ , the threshold for the radial variable  $x_R$  lies at a higher  $\sqrt{s}$  than the threshold for the Feynman variable  $x_{\parallel}$ , whereas the opposite is true for  $x_R = x_{\parallel} > 1/\sqrt{2}$ .

Finally, it should be emphasized that  $E d\sigma/dp^3$  is always finite at the  $s$  threshold for fixed  $x_R$  and  $p_{\perp}$ , but is zero (excluding  $p$ - $p$  elastic scattering) at the  $s$  threshold for fixed  $x_{\parallel}$  and  $p_{\perp}$ . This means that the cross section for fixed  $x_{\parallel}$ ,  $p_{\perp}$  must rise over some range in  $\sqrt{s}$ , owing to purely kinematic effects. This  $s$  dependence seems to be a major cause for the observed fixed  $x_{\parallel}$ ,  $p_{\perp}$  behavior of inclusive cross sections.

### III. THE DATA ANALYSIS

#### A. The object of the data analysis

The object of the data analysis is to convert the data from each experiment into a table of invariant cross sections as a function of  $\sqrt{s}$  for given values of  $p_{\perp}$  and  $x_R$  or  $p_{\perp}$  and  $x_{\parallel}$ . We divided the range of  $x_R$  into 10 bins of 0.1 units, ranging from 0.0 to 1.0, and the  $p_{\perp}$  range from 0.125 to 10.125 GeV/c was divided into 40 bins of 0.25 GeV/c. The variable  $\sqrt{s}$  was not binned, and therefore each value of  $\sqrt{s}$  of a given experiment provided a unique entry into the compilation.

Since a given datum generally did not fall at exactly the middle of the  $p_{\perp}$ ,  $x_R$ , or  $x_{\parallel}$  bin, a small adjustment was performed to move it to the center of the bin. This procedure is described in more detail later. Only statistical errors were used to denote the experimental uncertainty of each data point. The experiment-to-experiment systematic errors were estimated from the consis-

TABLE II. Listed are the references used in this data compilation. The values of  $\sqrt{s}$  and the particle type for each reference are tabulated.

| Reference  | $\sqrt{s}$ (GeV)   | Particle                             |
|--|--|--------------------------------------|
| J. T. Reed <i>et al.</i> , Phys. Rev. <u>168</u> , 1495 (1968)   | 2.83<br>2.98   | $K^+$                                |
| W. J. Hogan <i>et al.</i> , Phys. Rev. <u>166</u> , 1472 (1968)  | 2.88<br>2.99<br>3.03   | $K^+$                                |
| A. C. Melissinos <i>et al.</i> , Phys. Rev. <u>128</u> , 2373 (1962)   | 2.98   | $\pi^+, \pi^-$                       |
| E. R. Gellert, thesis, Report No. LBL-749, 1972 (unpublished)  | 3.78   | $\pi^+, \pi^-$                       |
| V. Blobel <i>et al.</i> , DESY Report No. 73/76, 1973 (unpublished).   | 4.93<br>6.84   | $\pi^+, \pi^-$                       |
| C. W. Akerlof <i>et al.</i> , Phys. Rev. D <u>3</u> , 645 (1971)   | 5.02   | $\pi^+, \pi^-, K^+, K^-, p, \bar{p}$ |
| D. Dekkers <i>et al.</i> , Phys. Rev. <u>137</u> , B962 (1965)   | 6.09<br>6.72   | $\pi^+, \pi^-, K^+, K^-, p, \bar{p}$ |
| A. N. Diddens <i>et al.</i> , Nuovo Cimento <u>31</u> , 961 (1964)   | 6.12<br>6.84   | $\pi^+, \pi^-, K^+, K^-, p, \bar{p}$ |
| J. V. Allaby <i>et al.</i> , Report No. CERN 70-12, 1970 (unpublished)   | 6.15   | $\pi^+, \pi^-, K^+, K^-, p, \bar{p}$ |
| W. H. Sims <i>et al.</i> , Nucl. Phys. <u>B41</u> , 317 (1972)   | 7.43   | $\pi^+, \pi^-$                       |
| D. C. Carey <i>et al.</i> , Phys. Rev. Lett. <u>33</u> , 327 (1974); preceding paper, Phys. Rev. D <u>14</u> , 1196 (1976).<br>(The values of $\sqrt{s}$ from this reference were combined in this compilation to diminish the size of the data sample.) | 10.2<br>11.1<br>12.3<br>13.4<br>14.6<br>15.7<br>16.9<br>18.2<br>19.5<br>20.7<br>21.8<br>23.1<br>24.6<br>25.7<br>27.0 | $\pi^0$                              |
| J. W. Cronin <i>et al.</i> , Phys. Rev. D <u>11</u> , 3105 (1975)  | 19.4<br>23.8<br>27.4   | $\pi^+, \pi^-, K^+, K^-, p, \bar{p}$ |
| P. Capiluppi <i>et al.</i> , Nucl. Phys. <u>B79</u> , 189 (1974)   | 23.3<br>30.6<br>44.6<br>53.0<br>62.7   | $\pi^+, \pi^-, K^+, K^-, p, \bar{p}$ |
| B. Alper <i>et al.</i> , Nucl. Phys. <u>B87</u> , 19 (1975)  | 23.4<br>30.6<br>44.6<br>52.8<br>63.0   | $\pi^+, \pi^-, K^+, K^-, p, \bar{p}$ |
| M. G. Albrow <i>et al.</i> , Nucl. Phys. <u>B73</u> , 40 (1974)  | 31.0<br>45.0<br>53.2<br>62.6   | $\pi^+, K^+, p$                      |
| M. G. Albrow <i>et al.</i> , Nucl. Phys. <u>B56</u> , 333 (1973)   | 23.6<br>30.8<br>45.0<br>53.2<br>62.8   | $\pi^-, K^-, \bar{p}$                |

TABLE II (Continued)

| Reference  | $\sqrt{s}$ GeV                       | Particle                   |
|--|--------------------------------------|----------------------------|
| M. G. Albrow <i>et al.</i> , Nucl. Phys. <u>B54</u> , 6 (1973)   | 31.0                                 | $p$                        |
| M. Banner <i>et al.</i> , Phys. Lett. <u>41B</u> , 547 (1972)  | 23.2<br>30.4<br>44.4<br>52.7         | $\pi^+, \pi^-, p, \bar{p}$ |
| F. W. Büsser <i>et al.</i> , Phys. Lett. <u>46B</u> , 471 (1973), and F. W. Büsser <i>et al.</i> , Phys. Lett. <u>55B</u> , 232 (1975) | 23.5<br>30.6<br>44.8<br>52.7<br>62.4 | $\pi^0$                    |
| K. Eggert <i>et al.</i> , paper submitted to the Palermo International Conference on HEP, Palermo, Italy, 1975 (unpublished)           | 23.6<br>30.8<br>45.1<br>53.2<br>62.9 | $\pi^0$                    |

tency of the data set to be  $\approx \pm 15\%$ , but in many cases they could not be reliably determined, and were therefore not included.

#### B. Criteria for choice of data

The requirement for including data in the compilation was the existence of a published table of cross sections for the production process  $p + p \rightarrow c + \text{anything}$ ,  $c = \pi^+, \pi^0, K^+, p$ , or  $\bar{p}$ . In only a few instances were fits to the data used to generate values of  $E d\sigma/dp^3$ . These exceptions were made when there were no other data in the same kinematic range. A list of the data used is given in Table II.

#### C. The finite-binning corrections

Each set of data was binned in 0.25-GeV/ $c$  units of  $p_\perp$  and in 0.1 units of  $x_R$  or  $x_\parallel$ , and it was found that the variation of  $E d\sigma/dp^3$  even for these small bins was sizable. Therefore, it was necessary to adjust the data to the center of each bin, both in  $p_\perp$  and in  $x_R$  or  $x_\parallel$ . The value of the invariant cross section for a given experiment entered into the compilation for a fixed  $p_\perp$  and  $x_R$  was computed by an expression of the form

$$E \frac{d\sigma}{dp^3}(p_\perp^0, x_R^0, s) = \left\langle E \frac{d\sigma}{dp^3}(\hat{p}_\perp, \hat{x}_R, s) \right\rangle g(\hat{p}_\perp, p_\perp^0) \times f(\hat{x}_R, x_R^0), \quad (13)$$

where  $p_\perp^0, x_R^0$  are the central values of the  $p_\perp$  and  $x_R$  bins;  $\langle E(d\sigma/dp^3)(\hat{p}_\perp, \hat{x}_R, s) \rangle$  is the cross section averaged over the  $p_\perp, x_R$  bins weighted by its statistical error. The functions  $f(\hat{x}_R, x_R^0)$  and  $g(\hat{p}_\perp, p_\perp^0)$  are the finite-binning corrections in  $x_R$  and  $p_\perp$ , respectively. They shift the data average from the

statistical mean values  $\hat{x}_R$  and  $\hat{p}_\perp$  to the centers of the  $x_R$  and  $p_\perp$  bins (to  $x_R^0, p_\perp^0$ ). The functions  $f$  and  $g$  depend on the particle type, and  $g(\hat{p}_\perp, p_\perp^0)$  was also allowed to depend on  $x_R$ . The explicit functional forms of  $f(\hat{x}_R, x_R^0)$  and  $g(\hat{p}_\perp, p_\perp^0)$  were determined by performing a rough fit to the binned, uncorrected data. Since these corrections are typically  $\approx 30\%$ , a rough determination of these functional forms was adequate to describe the data over one bin width.

The form adapted for the  $f(\hat{x}_R, x_R^0)$  correcting function is given by

$$f(\hat{x}_R, x_R^0) = \frac{(1 - x_R^0)^n}{(1 - \hat{x}_R)^n}, \quad (14)$$

where the exponent  $n$  is a function of the detected particle type.

The function  $g(\hat{p}_\perp, p_\perp^0)$  used is given by

$$g(\hat{p}_\perp, p_\perp^0) = \frac{(\hat{p}_\perp^2 + m^2)^q}{(p_\perp^{02} + m^2)^q}, \quad (15)$$

where the power  $q$  and the parameter  $m^2$  showed a slight dependence on  $x_R$  but was roughly independent of particle species.

Although the explicit determination of the various parameters  $n, m^2, q$  was approximate, the resulting error in these corrections was small. It is estimated that the error in the parameter  $n$  is  $\pm 1$ , the error in  $q$  is  $\pm 1$ , and the error in  $m^2$  is  $\pm 0.1 \text{ GeV}^2$ , giving rise to an error in the  $x_R$ -correcting function of

$$\frac{\Delta f}{f} \leq \pm 5\%,$$

and in the  $p_\perp$ -correcting function of

$$\frac{\Delta g}{g} \approx \pm 6\%.$$

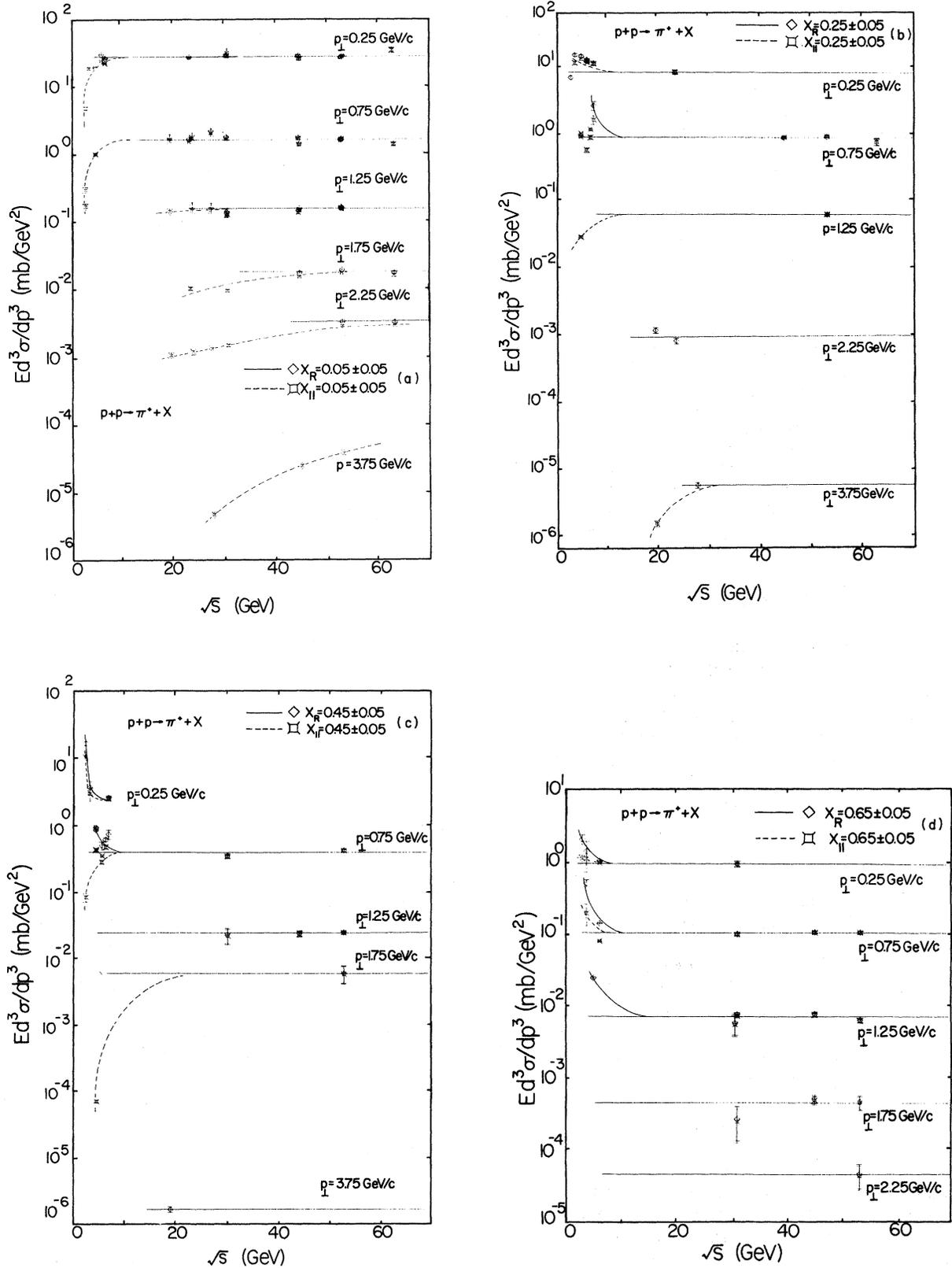


FIG. 2. (Continued on following page)

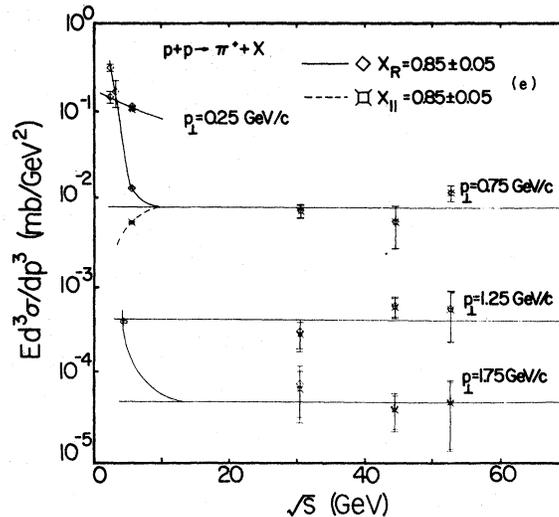


FIG. 2. The  $\pi^+$  invariant cross section  $E d\sigma/dp^3$  for constant  $x_R$  or  $x_{\parallel}$  and for various constant values of  $p_{\perp}$  versus the center-of-momentum energy  $\sqrt{s}$ . The dotted line is to guide the eye through points of constant  $x_{\parallel}$  and  $p_{\perp}$ . The smooth line is the average value of  $E d\sigma/dp^3$  for constant  $x_R$  and  $p_{\perp}$  for  $\sqrt{s} \gtrsim 10$  GeV, and extends from the  $\sqrt{s}$  threshold [calculated using Eq. (9), taking into account the finite bin widths in  $x_R$  and  $p_{\perp}$ ] to  $\sqrt{s} = 70$  GeV. The cases where there was only one high- $p_{\perp}$  data point were not plotted. (a)  $x_R$  or  $x_{\parallel} = 0.05 \pm 0.05$ , (b)  $x_R$  or  $x_{\parallel} = 0.25 \pm 0.05$ , (c)  $x_R$  or  $x_{\parallel} = 0.45 \pm 0.05$ , (d)  $x_R$  or  $x_{\parallel} = 0.65 \pm 0.05$ , (e)  $x_R$  or  $x_{\parallel} = 0.85 \pm 0.05$ .

#### D. The rebinning in $x_{\parallel}$

The final stage of the data analysis was to compute the invariant cross section for fixed  $\sqrt{s}$ ,  $p_{\perp}$ , with  $x_{\parallel}$  replacing  $x_R$ . This was performed on the compiled data for fixed  $x_R$ ,  $p_{\perp}$ , by calculating for each table entry the corresponding value of  $x_{\parallel}$  and using a small correction to shift the  $x_{\parallel}$  binned data to the middle of the  $x_{\parallel}$  bin. This small correction ( $\lesssim 30\%$ ) was performed by using the  $x_R$  dependence of  $E d\sigma/dp^3$  to shift to the appropriate  $x_R$  value corresponding to the middle of the  $x_{\parallel}$  bin.

#### IV. COMPARISON OF RADIAL SCALING WITH FEYNMAN SCALING

The invariant cross sections for  $\pi^{\pm,0}$ ,  $K^{\pm}$ ,  $p$ , and  $\bar{p}$  are presented in Figs. 2 through 8 for constant  $p_{\perp}$ ,  $x_R$  and constant  $p_{\perp}$ ,  $x_{\parallel}$  versus  $\sqrt{s}$ . These graphs cover roughly 25% of the data compilation. The qualitative features of this comparison between the radial scaling hypothesis and the Feynman scaling hypothesis for these single-particle inclusive cross sections are given below.

(1) In the low- $x$  region ( $x \leq 0.2$ ) radial scaling is good to within the estimated  $\pm 15\%$ , experiment-to-experiment systematic errors, from the  $s$  threshold to the highest energy available at the CERN ISR. On the other hand, above the corresponding  $x_{\parallel}$  energy threshold, there is a very large energy dependence of the cross sections.

In this low- $x_{\parallel}$  region, the approach to the Feynman scaling limit is from below, and in the case of large  $p_{\perp}$  ( $\sim 3.0$  GeV/c) the Feynman scaling limit is not obtained even in the CERN ISR energy range.<sup>7,8</sup> If one restricts the comparison of these two types of scaling to the  $\sqrt{s}$  region above the radial variable  $s$  threshold, one observes less difference. In the same range of  $\sqrt{s}$ , the  $s$  dependence of  $E d\sigma/dp^3$  for fixed  $x_{\parallel}$  is slightly greater than that for fixed  $x_R$ , and the cross section appears to rise to its asymptotic limit at large  $\sqrt{s}$ . Therefore, in the low- $x$  ( $x_R$  or  $x_{\parallel}$ ) region, the primary breakdown in Feynman scaling is below the radial variable  $s$  threshold.

(2) For  $x_R > 0.20$  there is a rapid approach to a radial scaling after  $\sqrt{s}$  passes above the  $x_R$  threshold. Furthermore, for  $\sqrt{s} \gtrsim 10$  GeV radial scaling appears to be achieved for all  $p_{\perp}$ .<sup>9</sup> The deviation from radial scaling at low  $\sqrt{s}$  is greater for larger values of  $x_R$ . In all cases the radial scaling limit is approached from above for increasing  $\sqrt{s}$ .

For  $x_{\parallel} > 0.2$ , the approach to the Feynman scaling limit is also from above. This is in contrast to the situation at low  $x_{\parallel}$ , where the approach to the scaling limit is from below. Since for a given point on the Peyrou plot  $x_{\parallel} \leq x_R$ , large  $x_{\parallel}$  implies  $x_{\parallel} \approx x_R$ , and therefore in this limit  $x_R$  and  $x_{\parallel}$  scaling are the same. Because of this transition from an approach to scaling from below at small  $x_{\parallel}$  to an approach from above at larger  $x_{\parallel}$ , there are

values of  $x_{\parallel}$  and  $p_{\perp}$  where Feynman scaling is good at very low  $\sqrt{s}$  ( $\sim 5.0$  GeV). For example, Feynman scaling for  $\pi^+$ ,  $x_{\parallel}=0.25$  and  $p_{\perp}=0.75$  GeV/c is good to within experimental systematic errors from  $\sqrt{s}=5$  GeV up to  $\sqrt{s}=53$  GeV. It appears that a kinematic threshold effect is superimposed on a dynamically induced decreasing cross section. Thus the  $s$  dependences of the cross sections, for fixed  $x_{\parallel}$ ,  $p_{\perp}$  and for fixed  $x_R$ ,  $p_{\perp}$ , as they fall to their scaling limit, are differ-

ent. In the former case, kinematic boundary effects distort the behavior.

A discussion of the various inclusive cross sections will now be given. Referring to Figs. 2 through 8 we see the following:

(a)  $\pi^+$  (Figs. 2 and 3). For  $x_R=0.05$ , radial scaling is good down to the radial scaling threshold, as is evident from Figs. 2(a) and 3(a). Feynman scaling for  $x_{\parallel}=0.05$  is evidently violated even for small  $p_{\perp}$ , and the invariant cross section for fixed

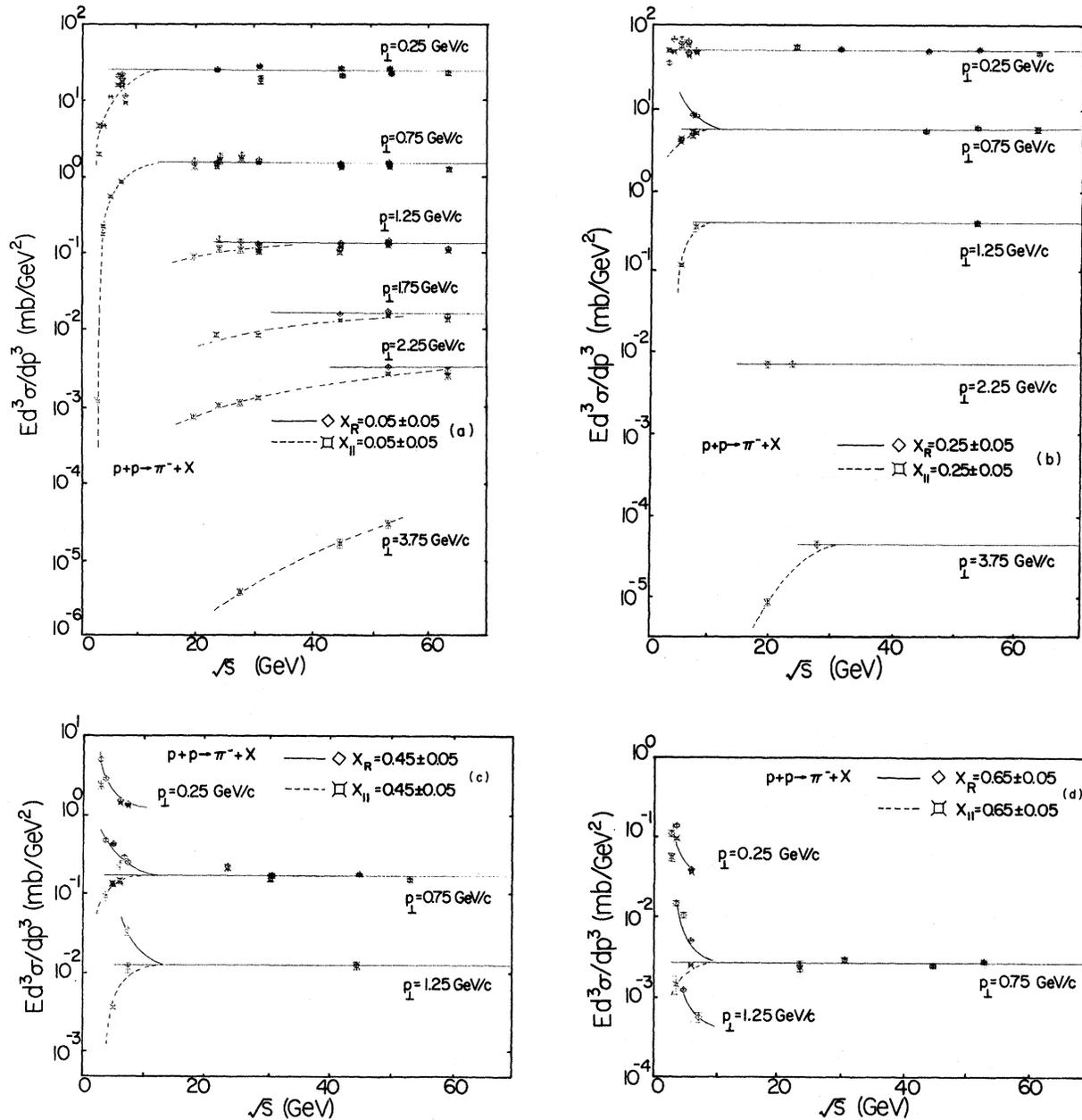


FIG. 3. The  $\pi^-$  invariant cross section  $Ed\sigma/dp^3$  for constant  $x_R$  or  $x_{\parallel}$  and constant values of  $p_{\perp}$  versus  $\sqrt{s}$ . (a)  $x_R$  or  $x_{\parallel}=0.05\pm 0.05$ , (b)  $x_R$  or  $x_{\parallel}=0.25\pm 0.05$ , (c)  $x_R$  or  $x_{\parallel}=0.45\pm 0.05$ , (d)  $x_R$  or  $x_{\parallel}=0.65\pm 0.05$ .

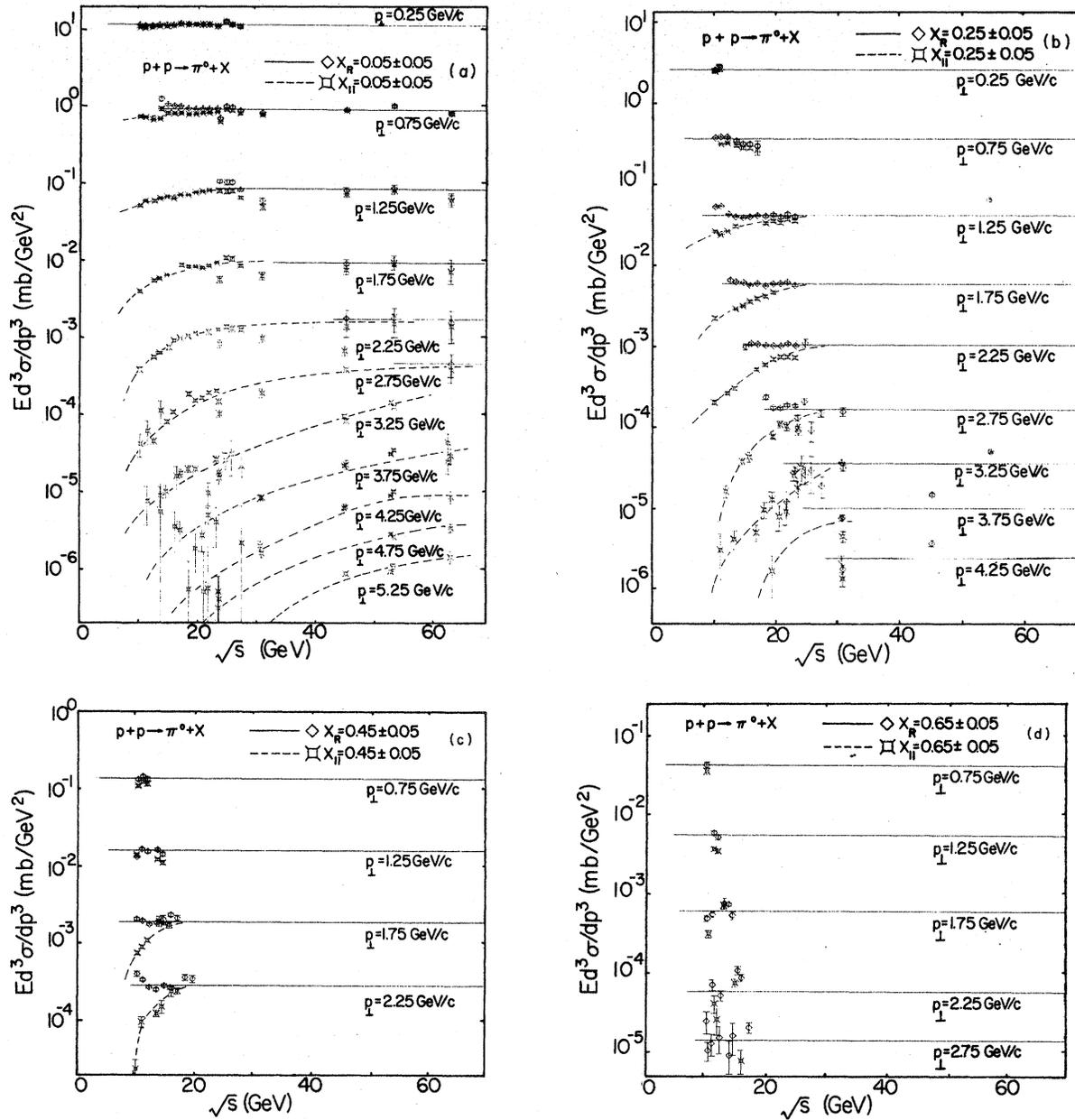


FIG. 4. The  $\pi^0$  invariant cross section  $Ed\sigma/dp^3$  for constant  $x_R$  or  $x_{||}$  and constant values of  $p_{\perp}$  versus  $\sqrt{s}$ . (a)  $x_R$  or  $x_{||}=0.05\pm 0.05$ , (b)  $x_R$  or  $x_{||}=0.25\pm 0.05$ , (c)  $x_R$  or  $x_{||}=0.45\pm 0.05$ , (d)  $x_R$  or  $x_{||}=0.65\pm 0.05$ .

$x_{||}$  and  $p_{\perp}$  is rising with increasing  $\sqrt{s}$ . For  $x_R \geq 0.20$  the radial scaling limit is always approached from above for increasing  $\sqrt{s}$ , and by roughly  $\sqrt{s} \approx 10$  GeV the scaling limit has been attained for all  $p_{\perp}$ . The approach to the Feynman scaling limit for  $x_{||} > 0.2$  shows a turnover. For small  $p_{\perp} (\leq 1$  GeV/c) the approach to the scaling limit is from above with increasing  $\sqrt{s}$ , but for larger  $p_{\perp} (\geq 1.25$  GeV/c) the approach to the scaling limit is from below. In all cases radial scaling appears to be

good for  $\sqrt{s} \geq 10$  GeV, whereas Feynman scaling may be violated by a factor of 10 from  $\sqrt{s} \sim 27$  to 53 GeV.

(b)  $\pi^0$  (Fig. 4). For  $x_R = 0.05 \pm 0.05$ , there is good evidence for radial scaling down to the radial scaling threshold. Feynman scaling for  $x_{||}$  in the same range is again violated and the invariant cross section rises from below for increasing  $\sqrt{s}$ . For  $x_R = 0.25 \pm 0.05$  radial scaling is good down to  $\sqrt{s} \approx 10$  GeV for all  $p_{\perp}$ . For  $x_{||} = 0.25 \pm 0.05$  Feyn-

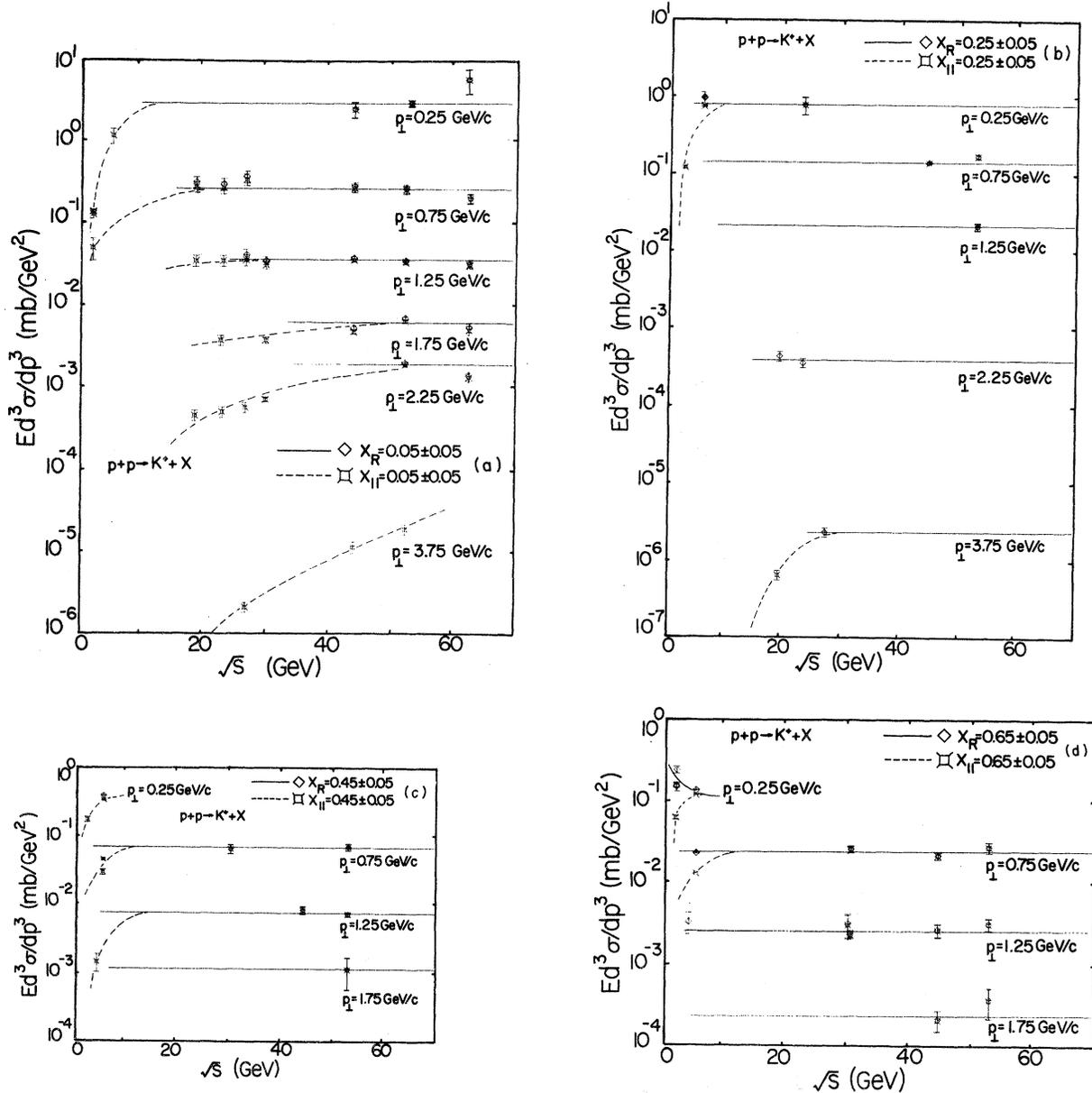


FIG. 5. The  $K^+$  invariant cross section  $Ed\sigma/dp^3$  for constant  $x_R$  or  $x_{\parallel}$  and constant values of  $p_{\perp}$  versus  $\sqrt{s}$ . (a)  $x_R$  or  $x_{\parallel} = 0.05 \pm 0.05$ , (b)  $x_R$  or  $x_{\parallel} = 0.25 \pm 0.05$ , (c)  $x_R$  or  $x_{\parallel} = 0.45 \pm 0.05$ , (d)  $x_R$  or  $x_{\parallel} = 0.65 \pm 0.05$ .

man scaling over the same  $\sqrt{s}$  region is violated by approximately a factor of 2 for  $p_{\perp} = 1.25$  GeV/c and by a factor of 4 for  $p_{\perp} = 2.25$  GeV/c. Feynman scaling appears to hold for  $p_{\perp} \leq 0.75$  GeV/c. For  $x_R \geq 0.35$  radial scaling is good to within experimental errors for  $\sqrt{s} > 10$  GeV, whereas for  $x_{\parallel} \geq 0.35$  Feynman scaling is obeyed for only low  $p_{\perp}$ .

(c)  $K^+$  (Figs. 5 and 6). The  $K^+$  data show the same qualitative features as the  $\pi^{\pm,0}$  data. However, there is a difference between  $K^+$  and  $K^-$ . The  $K^+$  data for fixed  $p_{\perp}$ ,  $x_{\parallel}$  appear to have some-

what less  $s$  dependence than the corresponding  $K^-$  data. A comparison of the two scaling hypotheses in the same range of  $\sqrt{s}$  for  $K^+$  shows a systematic  $s$  dependence for fixed  $p_{\perp}$ ,  $x_{\parallel}$  (especially for  $K^-$ ), which is absent in the data for fixed  $x_R$ .

(d)  $p$  (Fig. 7). The data at  $x_R = 0.05 \pm 0.05$  exhibit good radial scaling down to the radial scaling  $s$  threshold, but for large  $p_{\perp} \approx 3.75$  GeV/c the Feynman scaling hypothesis for  $x_{\parallel} = 0.05$  is violated by almost an order of magnitude over the ISR energy range, and appears to be approaching

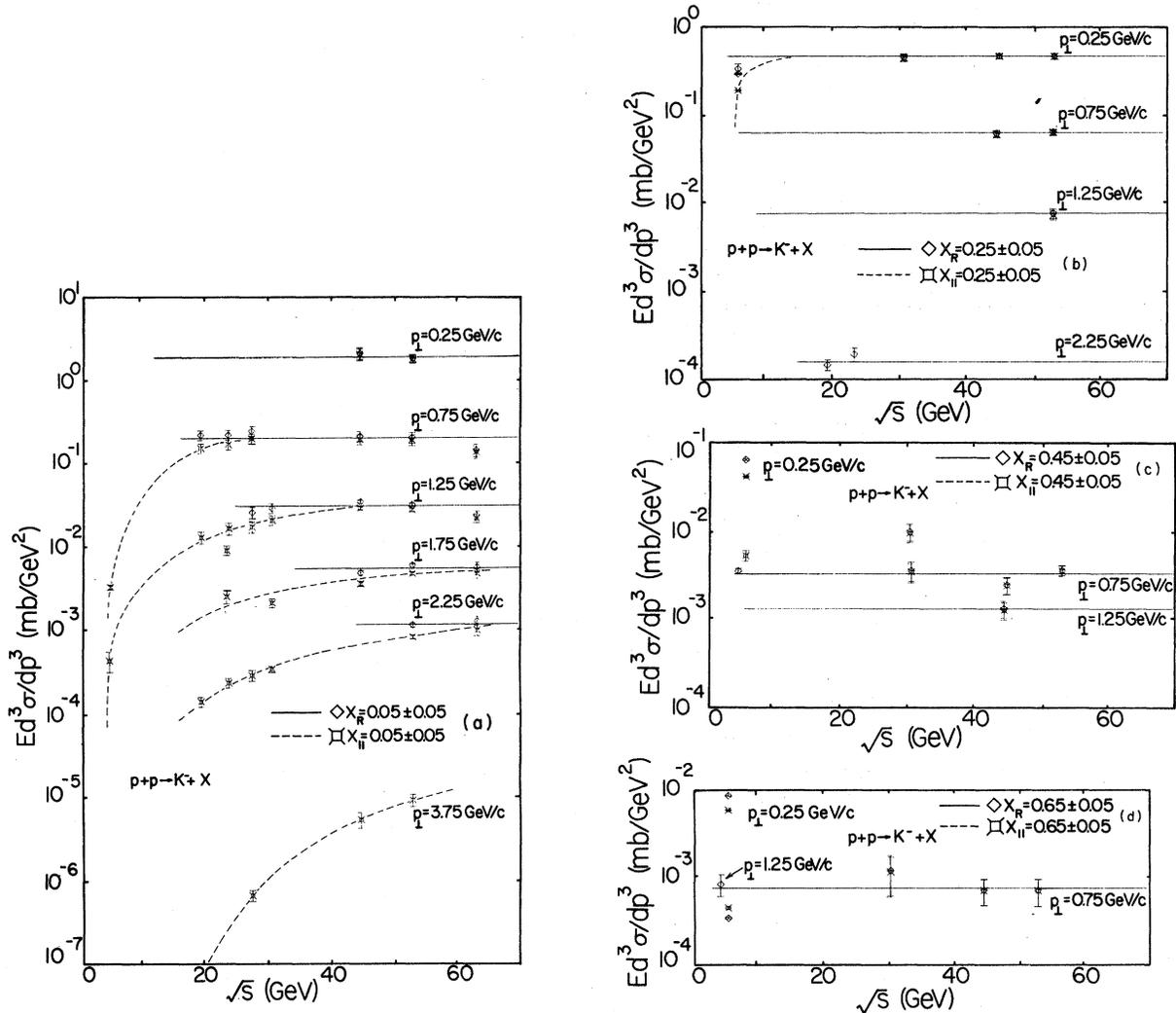


FIG. 6. The  $K^-$  invariant cross section  $Ed\sigma/dp^3$  for constant  $x_R$  or  $x_{||}$  and constant values of  $p_{\perp}$  versus  $\sqrt{s}$ . (a)  $x_R$  or  $x_{||} = 0.05 \pm 0.05$ , (b)  $x_R$  or  $x_{||} = 0.25 \pm 0.05$ , (c)  $x_R$  or  $x_{||} = 0.45 \pm 0.05$ , (d)  $x_R$  or  $x_{||} = 0.65 \pm 0.05$ .

the scaling limit from below. For small  $p_{\perp}$  ( $\leq 0.75$  GeV/c) Feynman scaling is approached from above. In general, the violations of Feynman scaling are less severe for protons than for any other particle. At low  $p_{\perp}$ , high  $x_R$  ( $\approx 0.65$ ) the radial scaling limit is approached from above, and as for the other particles, this approach to radial scaling is more pronounced for large rather than small  $x_R$ .

(e)  $\bar{p}$  (Fig. 8). Radial scaling for small  $x_R \leq 0.2$  appears to be satisfied for antiprotons down to the radial scaling  $s$  threshold. For larger  $x_R$  there are indications that the radial scaling limit is approached from above, although the data are incomplete. Feynman scaling is badly violated for antiprotons even at very high energies. For ex-

ample, at  $p_{\perp} = 1.25$  GeV/c,  $x_{||} = 0.05$ , the antiproton invariant cross section rises by roughly a factor of 2 for the  $\sqrt{s}$  range from 30 to 60 GeV.

In conclusion, for single-particle inclusive reactions use of the radial scaling variable  $x_R$  leads to an earlier scaling of the invariant cross sections than use of the  $x_{||}$  variable. Unlike the Feynman scaling limit, the radial scaling limit is always approached from above for increasing  $s$  and is reached by  $\sqrt{s} \approx 10$  GeV. This is consistent with the  $s$  dependence of the total proton-proton cross section at low  $\sqrt{s}$  ( $\sim 5$  to 10 GeV). The Feynman scaling limit is approached either from below, from above, or is exact depending on the dominance of phase-space effects, dynamic effects, or the fortuitous cancellation of these two effects. At

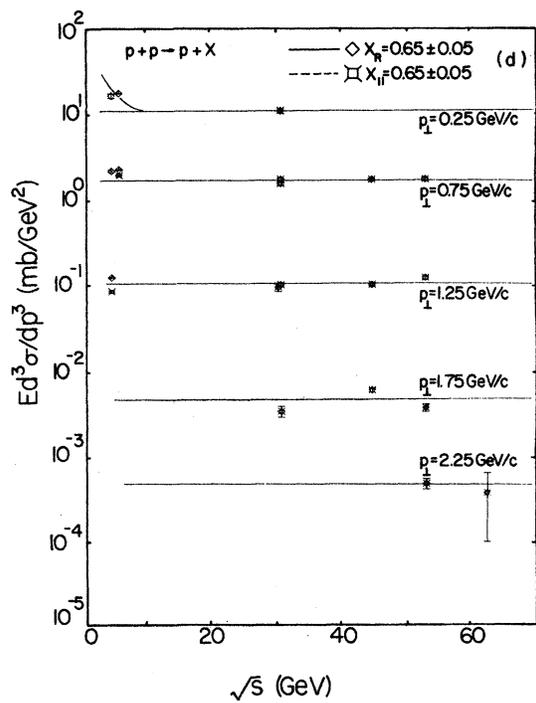
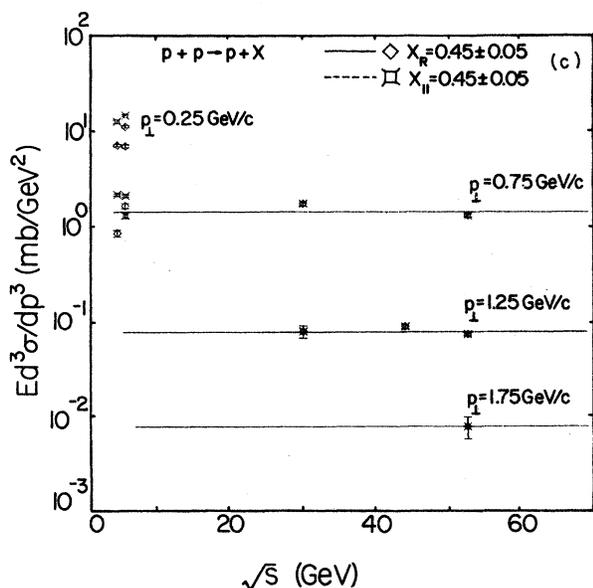
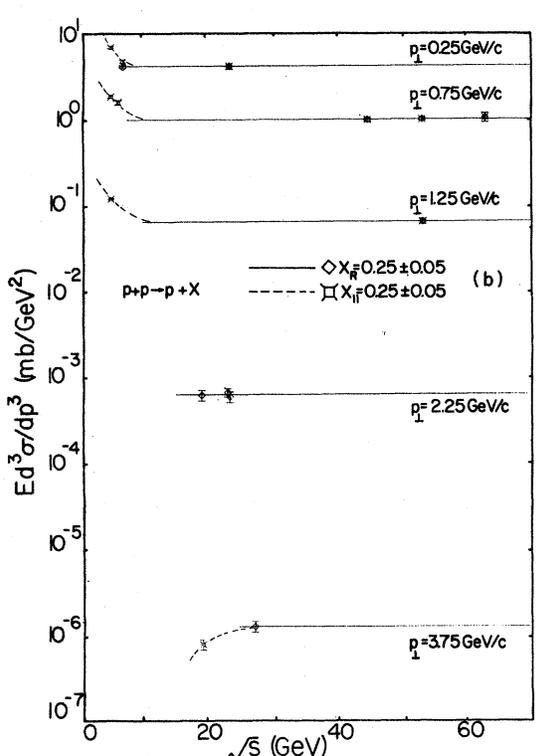
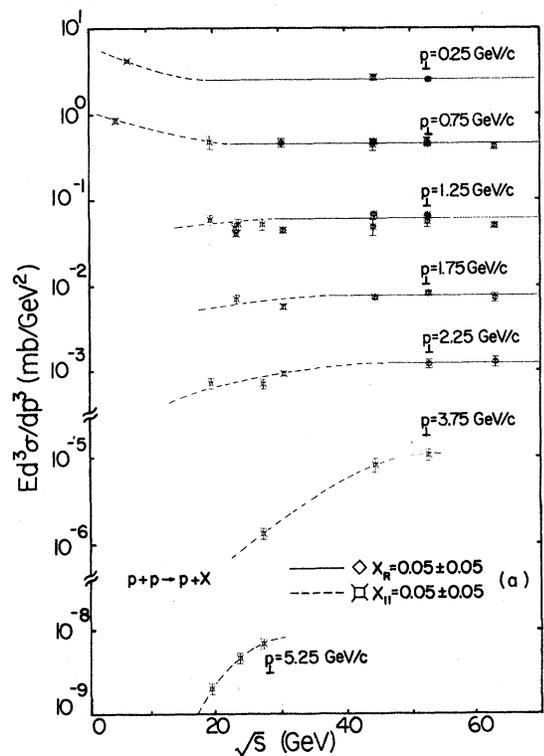


FIG. 7. (Continued on following page)

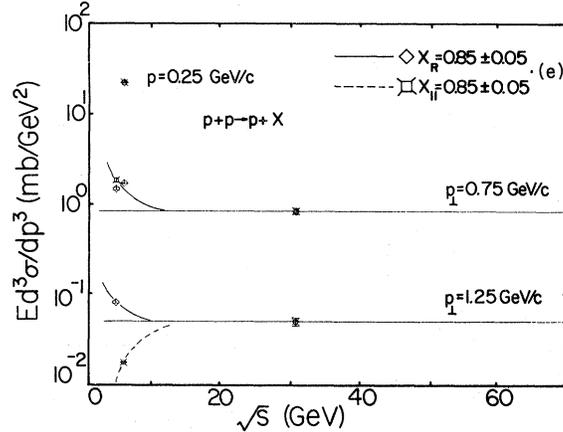


FIG. 7. The proton invariant cross section  $E d\sigma/dp^3$  for constant  $x_R$  or  $x_{||}$  and constant values of  $p_{\perp}$  versus  $\sqrt{s}$ . (a)  $x_R$  or  $x_{||}=0.05\pm 0.05$ , (b)  $x_R$  or  $x_{||}=0.25\pm 0.05$ , (c)  $x_R$  or  $x_{||}=0.45\pm 0.05$ , (d)  $x_R$  or  $x_{||}=0.65\pm 0.05$ , (e)  $x_R$  or  $x_{||}=0.85\pm 0.05$ .

small  $x_{||}$  ( $\sim 0.05$  to  $0.20$ ) there are large violations of Feynman scaling owing to large changes in the phase-space suppression, which for large  $p_{\perp}$  (especially for  $\bar{p}$  and  $K^-$  data) remain even at ISR energies.

#### V. INCLUSIVE CROSS SECTIONS IN THE RADIAL SCALING REGION

Since all single-particle inclusive cross sections for fixed  $x_R$  and  $p_{\perp}$  appear to scale for energies  $\sqrt{s} \gtrsim 10$  GeV, there are sufficient data to examine the systematic differences between particle species in this radial scaling region.

To illustrate the kinematic range in the radial scaling region ( $\sqrt{s} \gtrsim 10$  GeV) covered by the compilation, normalized Peyrou plots for each particle type are presented in Fig. 9. In these plots,  $x_{||} = 2p_{||}^*/\sqrt{s}$  is plotted against  $x_{\perp} = 2p_{\perp}/\sqrt{s}$  for a given  $x_R$  value of a given particle. It is seen that only in the case of  $\pi^0$  mesons [Fig. 9(c)] is a wide range in  $x_{||}, x_{\perp}$  for a given  $x_R$  covered.<sup>1</sup> For the other particles, most of the data for large  $x_R$  are concentrated either in the forward direction or near  $90^\circ$ .

It is of particular interest to compare the dependence of  $E d\sigma/dp^3$  on  $p_{\perp}$  for constant  $x_R$  and on  $x_R$  for constant  $p_{\perp}$ . We make this comparison for each detected particle in the single-particle inclusive reactions:  $p + p \rightarrow (\pi^{\pm,0} \text{ or } K^{\pm} \text{ or } p \text{ or } \bar{p}) + \text{anything}$ .

(a) The  $p_{\perp}$  dependence of  $E d\sigma/dp^3$  for constant  $x_R$  is displayed in Fig. 10 for each of the single-particle inclusive reactions listed above. These graphs were generated by projecting on the  $p_{\perp}$  axis all of the data in the particle compilation for a given constant value of  $x_R$ . A separate point is plotted for each  $\sqrt{s}$  value in the table above 10

GeV. Referring to these figures, it is noticed that the invariant cross section for  $p_{\perp} \gtrsim 1.5$  GeV/c for all particles becomes steeper for increasing  $p_{\perp}$  as  $x_R$  increases. Furthermore, the slope of  $E d\sigma/dp^3$  versus  $p_{\perp}$  at low  $p_{\perp}$  decreases for increasing  $x_R$ . This  $p_{\perp}$  dependence of  $E d\sigma/dp^3$  can be parameterized by the following universal form, applicable to all particles and at all  $x_R$ :

$$E \frac{d\sigma}{dp^3} \Big|_{x_R = \text{constant}} \approx \frac{A}{(1 + p_{\perp}^2/m^2)^q} \quad (16)$$

A minimum- $\chi^2$  fit was performed on the data to determine the values of the parameters  $A$  (mb/GeV<sup>2</sup>),  $m^2$  (GeV<sup>2</sup>/c<sup>2</sup>), and  $q$  for each particle in each slice in  $x_R$ . These parameters are presented in Table III. It is evident that for low  $x_R$  and small  $p_{\perp}$  the largest invariant cross section is for  $\pi^+$ . Then  $\pi^-$ ,  $\pi^0$ ,  $K^+$ ,  $K^-$ ,  $p$ , and  $\bar{p}$  respectively follow in order of decreasing magnitude.<sup>10</sup> At large  $x_R$ , the proton inclusive cross section dominates because of leading-particle effects.

The fit parameters  $m^2$  and  $q$  for mesons display an interesting similarity. The evident general trend of the parameter  $m^2$  is to grow from approximately  $0.3$  GeV<sup>2</sup>/c<sup>2</sup> at  $x_R \sim 0.05$  to roughly  $1.5$  GeV<sup>2</sup>/c<sup>2</sup> at  $x_R \approx 0.55$ . This growth in  $m^2$  with increasing  $x_R$  is an expression of the flattening at low  $p_{\perp}$  of  $E d\sigma/dp^3$  with increasing  $x_R$ . The fit parameter  $q$  shows the general trend of growing for increasing  $x_R$  from approximately 3 at small  $x_R \approx 0.05$  to roughly 6 at  $x_R \approx 0.55$ . The  $m^2$  and  $q$  parameters from the fits to the  $p$  and  $\bar{p}$  invariant cross section show this same behavior with increasing  $x_R$ . However,  $m^2$  for protons and antiprotons grows from  $\sim 1.0$  (GeV/c)<sup>2</sup> at  $x_R \approx 0.05$  to  $\sim 2.0$  (GeV/c)<sup>2</sup> at  $x_R \approx 0.55$  and is therefore generally larger than

the value of  $m^2$  for mesons. The parameter  $q$  for protons and antiprotons is also different from mesons:  $q$  increases from roughly 4 at  $x_R \approx 0.05$  to 8 at  $x_R \approx 0.55$  and is therefore systematically larger than the corresponding values for mesons.

A check was made to ensure that this general behavior of the parameters  $m^2$  and  $q$  is not a consequence of relative normalization errors among various experiments. A fit to only the  $\pi^0$  data of Carey *et al.*<sup>1</sup> showed the same general behavior.

To verify that the trend is not due to an increasing  $p_{\perp}$  range with increasing  $x_R$ , additional fits were made with cuts in the  $p_{\perp}$  range to constrain it between  $0.5 \leq p_{\perp} \leq 2.25$  GeV/c for  $\pi^+$  and for  $p_{\perp} \leq 2.25$  GeV/c for  $\pi^0$ . The calculated parameters  $A$ ,  $m^2$ , and  $q$  for this test were found to be consistent with the values from the unconstrained fits.

It is interesting to note that in the limit as  $x_R \rightarrow 0$  (or equivalently,  $p_{\perp}$  fixed as  $\sqrt{s} \rightarrow \infty$ ) the extrapolated form of the invariant cross section for

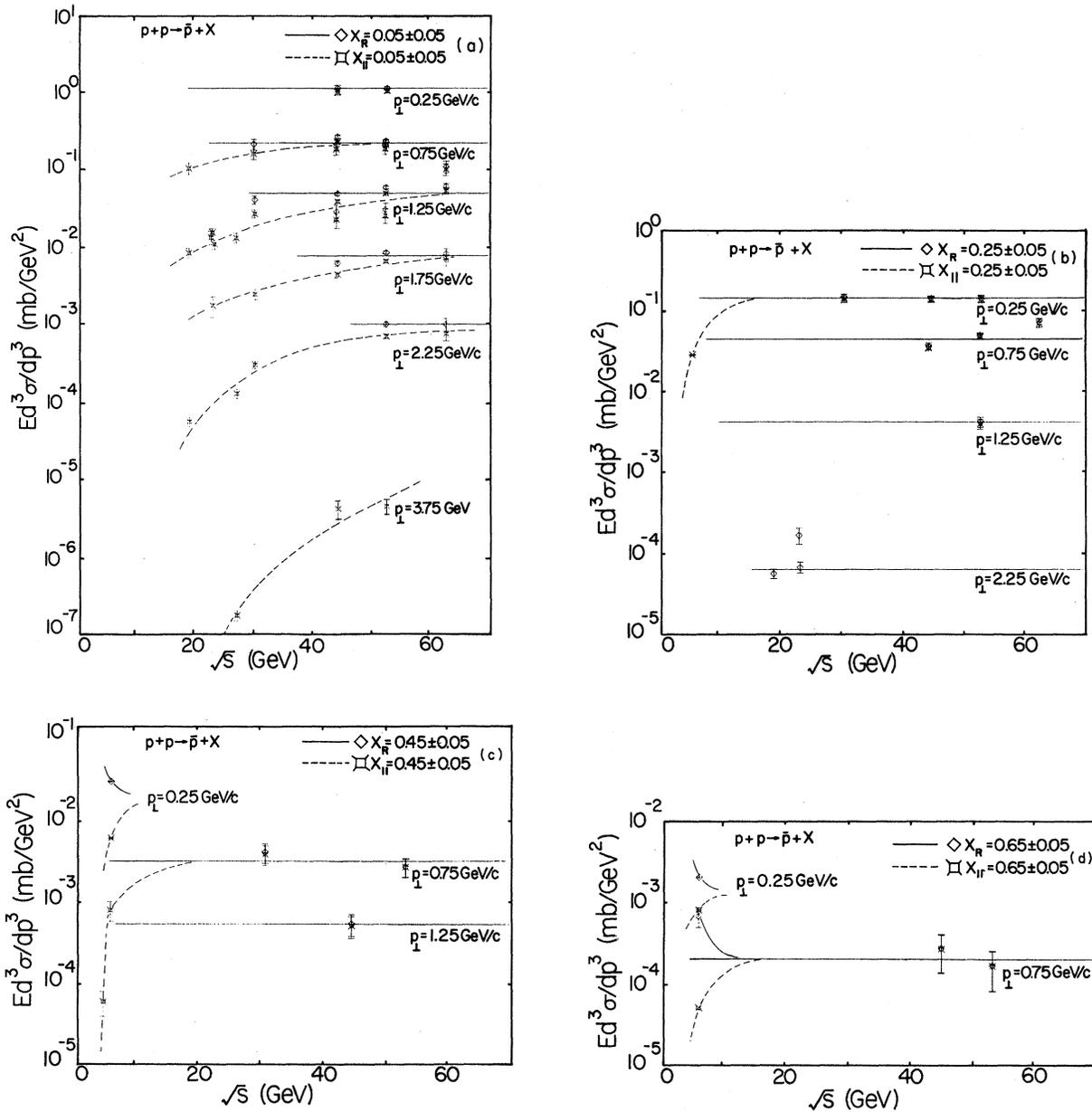


FIG. 8. The antiproton invariant cross section  $Ed\sigma/dp^3$  for constant  $x_R$  or  $x_{\parallel}$  and constant values of  $p_{\perp}$  versus  $\sqrt{s}$ . (a)  $x_R$  or  $x_{\parallel} = 0.05 \pm 0.05$ , (b)  $x_R$  or  $x_{\parallel} = 0.25 \pm 0.05$ , (c)  $x_R$  or  $x_{\parallel} = 0.45 \pm 0.05$ , (d)  $x_R$  or  $x_{\parallel} = 0.65 \pm 0.05$ .

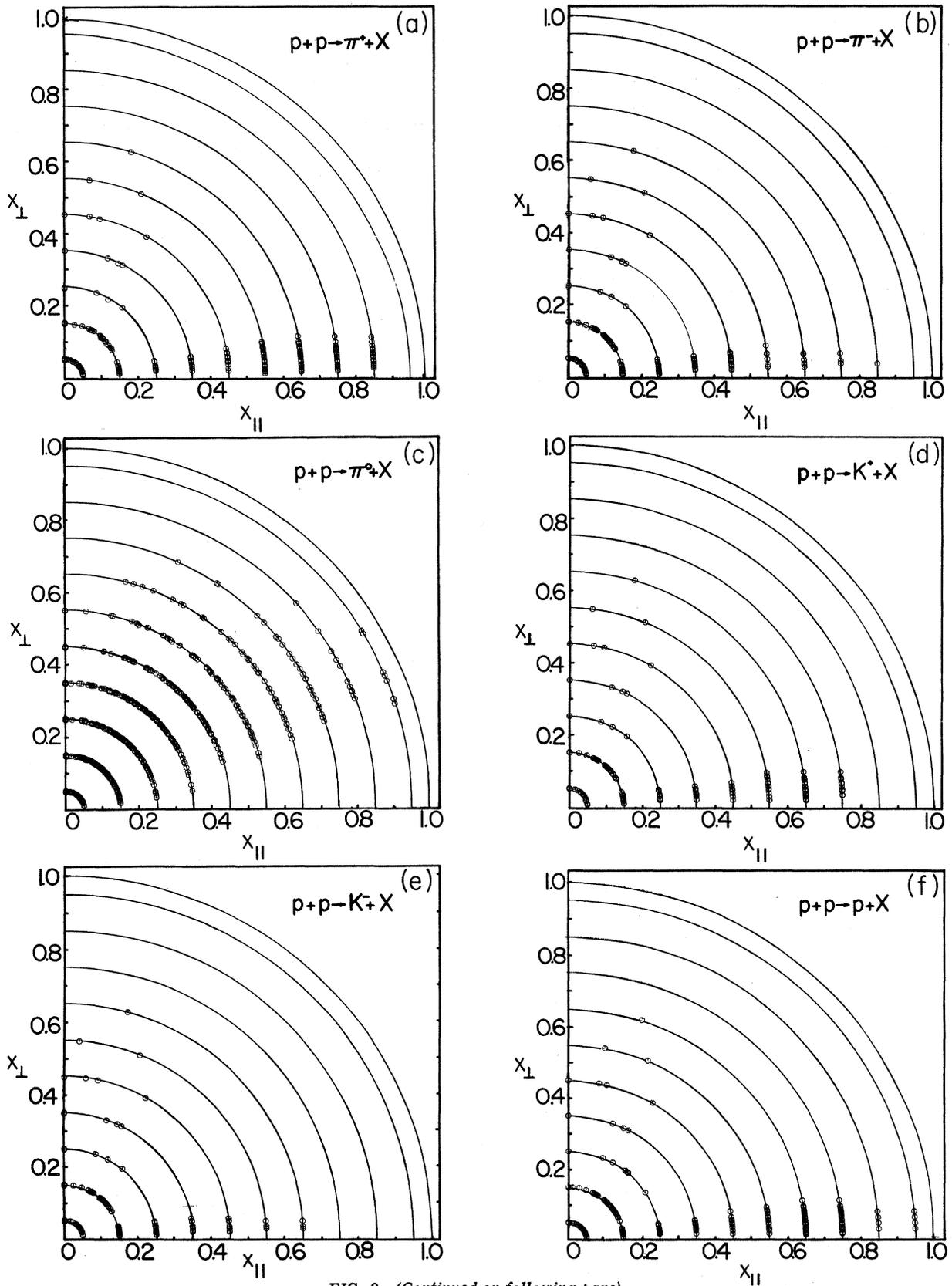


FIG. 9. (Continued on following page)

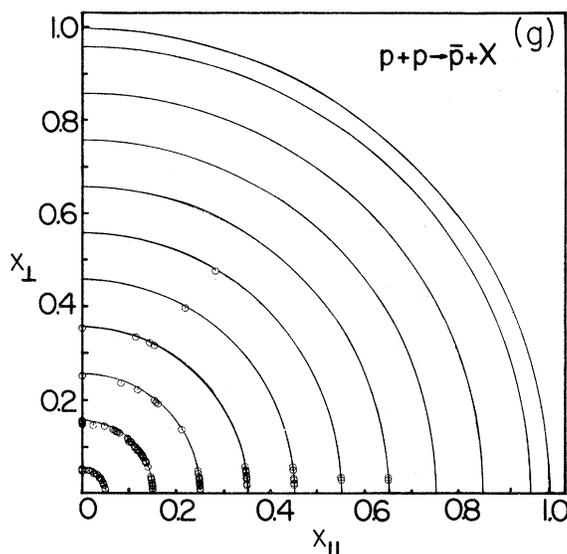


FIG. 9. The normalized Peyrou plots showing the kinematic region covered for each particle species in the data compilation. The forward-backward symmetry of  $p$ - $p$  collisions was used to map  $-|x_{||}|$  to  $+|x_{||}|$ . The quarter circles are lines of various constant values  $x_R$ . (a) Normalized Peyrou plot for  $\pi^+$ , (b) for  $\pi^-$ , (c) for  $\pi^0$ , (d) for  $K^+$ , (e) for  $K^-$ , (f) for  $p$ , (g) for  $\bar{p}$ .

mesons is consistent with

$$\lim_{x_R \rightarrow 0} \left( E \frac{d\sigma}{dp^3} \right) \simeq \frac{\tilde{A}}{p_{\perp}^{5+1}}. \quad (17)$$

This observed  $p_{\perp}$  dependence seems to exclude the constituent-interchange model,<sup>11</sup> which predicts a  $1/p_{\perp}^8$  behavior, and is closer to the  $1/p_{\perp}^4$  dependence postulated by Berman, Bjorken, and Kogut.<sup>12</sup> However, protons and antiprotons appear to be more consistent with the form

$$\lim_{x_R \rightarrow 0} \left( E \frac{d\sigma}{dp^3} \right) \simeq \frac{A}{(1+p_{\perp}^2/1.0)^4}, \quad (18)$$

giving a  $p_{\perp}$  dependence of  $\approx 1/p_{\perp}^8$  at large  $p_{\perp}$ .

(b) The  $x_R$  dependence of  $E d\sigma/dp^3$  for constant  $p_{\perp}$  various particles in the radial scaling region of  $\sqrt{s} \gtrsim 10$  GeV is shown in Fig. 11. It is seen that the cross sections for all particles with the exception of protons at low  $p_{\perp}$  fall as  $x_R \rightarrow 1$  for all  $p_{\perp}$ . To parameterize this  $x_R$  dependence (for all particles except protons), we performed a minimum- $\chi^2$  fit of the invariant cross section to the theoretically motivated form<sup>4,11</sup>

$$E \frac{d\sigma}{dp^3} \Big|_{\text{fixed } p_{\perp}} \simeq B(1-x_R)^n, \quad (19)$$

where  $B$  and  $n$  are free parameters. Reasonable fits were obtained in all cases. The resulting values of  $B$  (mb/GeV<sup>2</sup>) and  $n$  are tabulated in Table

IV. The particles listed, in rough order of increasing  $n$ , are  $p, K^+, \pi^+, \pi^-, \pi^0, K^-, \bar{p}$ . To check that these fits were independent of the  $x_R$  interval, a cut in  $x_R$  was made for  $\pi^0$  data constraining  $0.15 \leq x_R \leq 0.35$ , and a minimum- $\chi^2$  fit was again performed. Within errors, the fit parameters were the same.

The power of  $n$  for a fixed  $p_{\perp}$ , in the preceding parameterization of  $E d\sigma/dp^3|_{p_{\perp}}$ , reflects, perhaps, the quantum-number conservation requirements (charge, baryon number, strangeness, etc.) in the production of particle  $c$ .<sup>11</sup> These requirements may be calculated from the exclusive limit of the invariant cross section, since the undetected particles in the inclusive experiment must contain the same quantum numbers as the exclusive limit.<sup>2</sup> It appears that the more a given single-particle inclusive reaction is forbidden (i.e., more quantum numbers to balance in the production of particle  $c$ ), the larger is the value of the exponent  $n$ .

## VI. SUMMARY

We have parameterized the single-particle inclusive production cross sections in terms of  $s$ ,  $p_{\perp}$ , and the radial scaling variable  $x_R$ ,

$$E \frac{d\sigma}{dp^3} = f(s, p_{\perp}, x_R),$$

where  $x_R = E^*/E_{\text{max}}^*$ .

We have found that above  $\sqrt{s} \approx 10$  GeV, all of

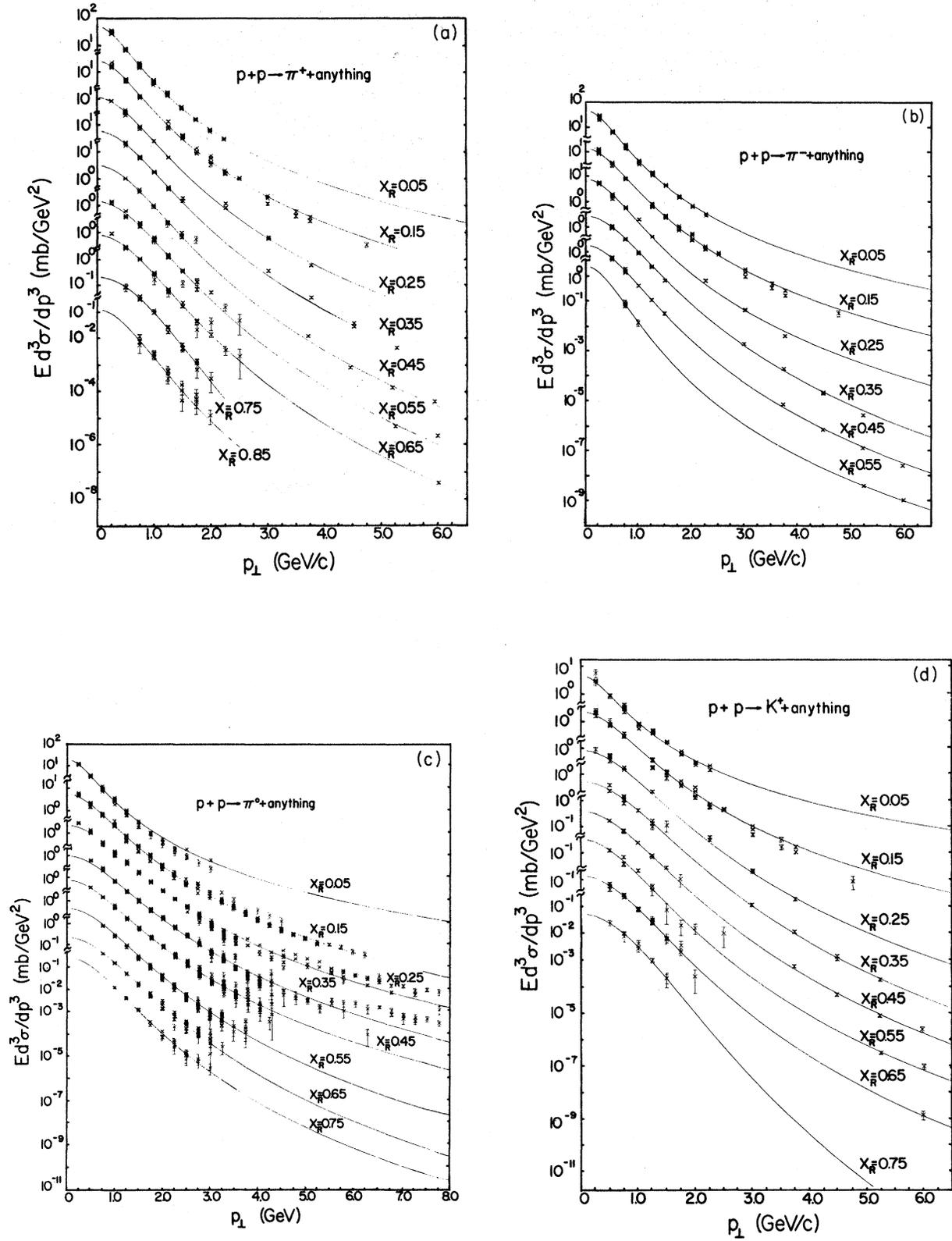


FIG. 10. (Continued on following page)

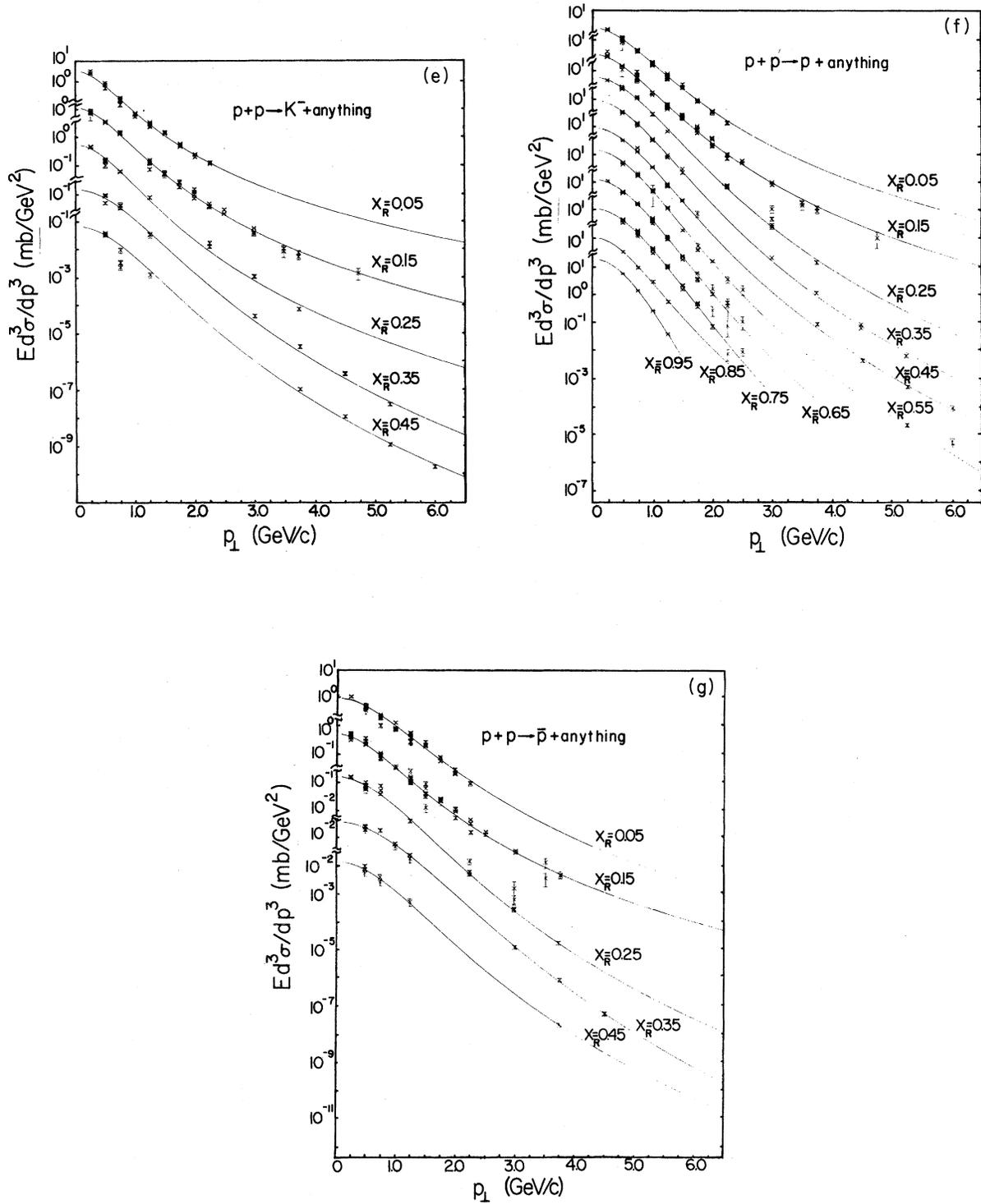


FIG. 10. The transverse-momentum dependence of the invariant cross section  $Ed\sigma/dp^3$  for various constant values of  $x_R$  in the radial scaling region:  $\sqrt{s} \approx 10$  GeV. The solid lines are the function  $A/(1+p_{\perp}^2/m^2)^q$ , where the values of  $A$ ,  $m^2$ ,  $q$  are given in Table III. (a)  $\pi^+$  invariant cross section, (b)  $\pi^-$  invariant cross section, (c)  $\pi^0$  invariant cross section, (d)  $K^+$  invariant cross section, (e)  $K^-$  invariant cross section, (f)  $p$  invariant cross section, (g)  $\bar{p}$  invariant cross section.

TABLE III. The fit parameters  $A$  (mb/GeV<sup>2</sup>),  $m^2$  (GeV<sup>2</sup>/c<sup>2</sup>), and  $q$  defined in Eq. (16) in the text for various inclusive reactions in the radial scaling region  $\sqrt{s} \gtrsim 10$  GeV are tabulated. These parameters describe the  $p_{\perp}$  dependence of the single-particle inclusive cross sections for constant  $x_R$ .

| $x_R$ |       | $\pi^+$     | $\pi^-$     | $\pi^0$      | $K^+$        | $K^-$      | $p$         | $\bar{p}$   |
|-------|-------|-------------|-------------|--------------|--------------|------------|-------------|-------------|
| 0.05  | $A$   | 51.460±2.90 | 45.220±2.10 | 19.210±0.060 | 4.820±1.630  | 2.180±0.43 | 2.95±0.220  | 0.960±0.080 |
|       | $m^2$ | 0.298±0.02  | 0.348±0.02  | 0.358±0.001  | 0.285±0.070  | 0.538±0.09 | 1.00±0.095  | 1.680±0.200 |
|       | $q$   | 3.390±0.07  | 3.450±0.06  | 3.230±0.005  | 2.717±0.140  | 3.210±0.16 | 4.25±0.190  | 4.840±0.300 |
| 0.15  | $A$   | 25.150±2.24 | 14.170±0.50 | 5.330±0.030  | 2.139±0.320  | 1.080±0.09 | 3.62±0.130  | 0.547±0.030 |
|       | $m^2$ | 0.523±0.04  | 0.610±0.03  | 0.908±0.001  | 0.836±0.110  | 0.757±0.07 | 1.13±0.040  | 1.150±0.070 |
|       | $q$   | 4.010±0.09  | 4.080±0.07  | 4.357±0.004  | 4.020±0.180  | 3.890±0.12 | 4.85±0.070  | 4.490±0.110 |
| 0.25  | $A$   | 10.660±1.15 | 7.910±0.29  | 2.070±0.006  | 0.791±0.150  | 0.535±0.07 | 5.84±0.300  | 0.165±0.006 |
|       | $m^2$ | 0.873±0.09  | 0.726±0.05  | 1.332±0.001  | 1.620±0.360  | 0.990±0.18 | 1.88±0.120  | 2.550±0.040 |
|       | $q$   | 5.060±0.17  | 4.650±0.15  | 4.953±0.002  | 5.610±0.460  | 4.780±0.31 | 7.06±0.200  | 7.180±0.060 |
| 0.35  | $A$   | 5.970±0.53  | 2.740±0.09  | 0.993±0.005  | 0.536±0.080  | 0.135±0.02 | 8.45±0.430  | 0.038±0.002 |
|       | $m^2$ | 1.198±0.07  | 1.320±0.06  | 1.569±0.002  | 2.070±0.260  | 2.280±0.36 | 1.91±0.070  | 4.290±0.060 |
|       | $q$   | 5.840±0.10  | 5.740±0.09  | 5.231±0.004  | 6.350±0.260  | 6.750±0.37 | 7.59±0.110  | 9.040±0.070 |
| 0.45  | $A$   | 3.100±0.36  | 1.790±0.11  | 0.770±0.020  | 0.365±0.033  | 0.070±0.01 | 9.18±0.550  | 0.015±0.004 |
|       | $m^2$ | 1.330±0.09  | 1.109±0.07  | 1.410±0.040  | 1.895±0.120  | 2.250±0.33 | 1.91±0.070  | 2.500±1.270 |
|       | $q$   | 6.070±0.12  | 5.640±0.10  | 5.200±0.020  | 6.340±0.130  | 6.890±0.32 | 7.81±0.110  | 7.080±1.510 |
| 0.55  | $A$   | 1.427±0.14  | 2.490±0.38  | 0.420±0.040  | 0.319±0.070  |            | 13.05±0.400 |             |
|       | $m^2$ | 1.460±0.11  | 0.590±0.05  | 1.730±0.120  | 1.630±0.280  |            | 2.02±0.010  |             |
|       | $q$   | 6.390±0.19  | 5.220±0.07  | 5.860±0.170  | 6.390±0.310  |            | 8.48±0.040  |             |
| 0.65  | $A$   | 0.821±0.05  |             | 0.210±0.020  | 0.134±0.011  |            | 12.11±0.160 |             |
|       | $m^2$ | 1.520±0.07  |             | 2.110±0.030  | 2.025±0.160  |            | 2.28±0.010  |             |
|       | $q$   | 6.680±0.14  |             | 6.529±0.040  | 7.070±0.290  |            | 9.24±0.030  |             |
| 0.75  | $A$   | 0.210±0.03  |             | 0.279±0.100  | 0.052±0.012  |            | 10.47±0.170 |             |
|       | $m^2$ | 3.010±0.94  |             | 1.600±0.500  | 3.390±1.250  |            | 2.55±0.014  |             |
|       | $q$   | 10.700±2.39 |             | 6.200±0.700  | 10.900±2.900 |            | 10.13±0.040 |             |
| 0.85  | $A$   | 0.370±0.37  |             |              |              |            | 10.20±0.600 |             |
|       | $m^2$ | 0.330±0.21  |             |              |              |            | 1.24±0.030  |             |
|       | $q$   | 3.740±0.44  |             |              |              |            | 6.47±0.130  |             |
| 0.95  | $A$   |             |             |              |              |            | 18.30±0.600 |             |
|       | $m^2$ |             |             |              |              |            | 2.59±0.030  |             |
|       | $q$   |             |             |              |              |            | 13.10±0.100 |             |

the cross-section data are consistent with radial scaling for all  $p_{\perp}$ :

$$E \frac{d\sigma}{dp^3} = f(p_{\perp}, x_R).$$

Below  $\sqrt{s} = 10$  GeV and for  $x_R \gtrsim 0.2$ , there is a rapid approach to the scaling limit from above. For  $x_R \lesssim 0.20$ , there is good radial scaling down to the threshold.

Feynman scaling is achieved at a larger value of  $\sqrt{s}$ , where the suppression due to the presence

of the kinematic boundary is sufficiently small. In the cases of large  $p_{\perp}$ , (or even small  $p_{\perp}$  for  $\bar{p}$  and  $K^-$ ), this  $s$  dependence is still present even in the ISR energy range.

Parameterizing the single-particle inclusive cross sections in terms of  $p_{\perp}$ ,  $x_R$ , and  $\sqrt{s}$ , we find in the radial scaling region  $\sqrt{s} \gtrsim 10$  GeV that the shape of the  $p_{\perp}$  dependence of  $E d\sigma/dp^3$  for all particles for constant  $x_R$  as a function of  $x_R$  shows the same general behavior. From the minimum- $\chi^2$  fits to the invariant cross section to the form

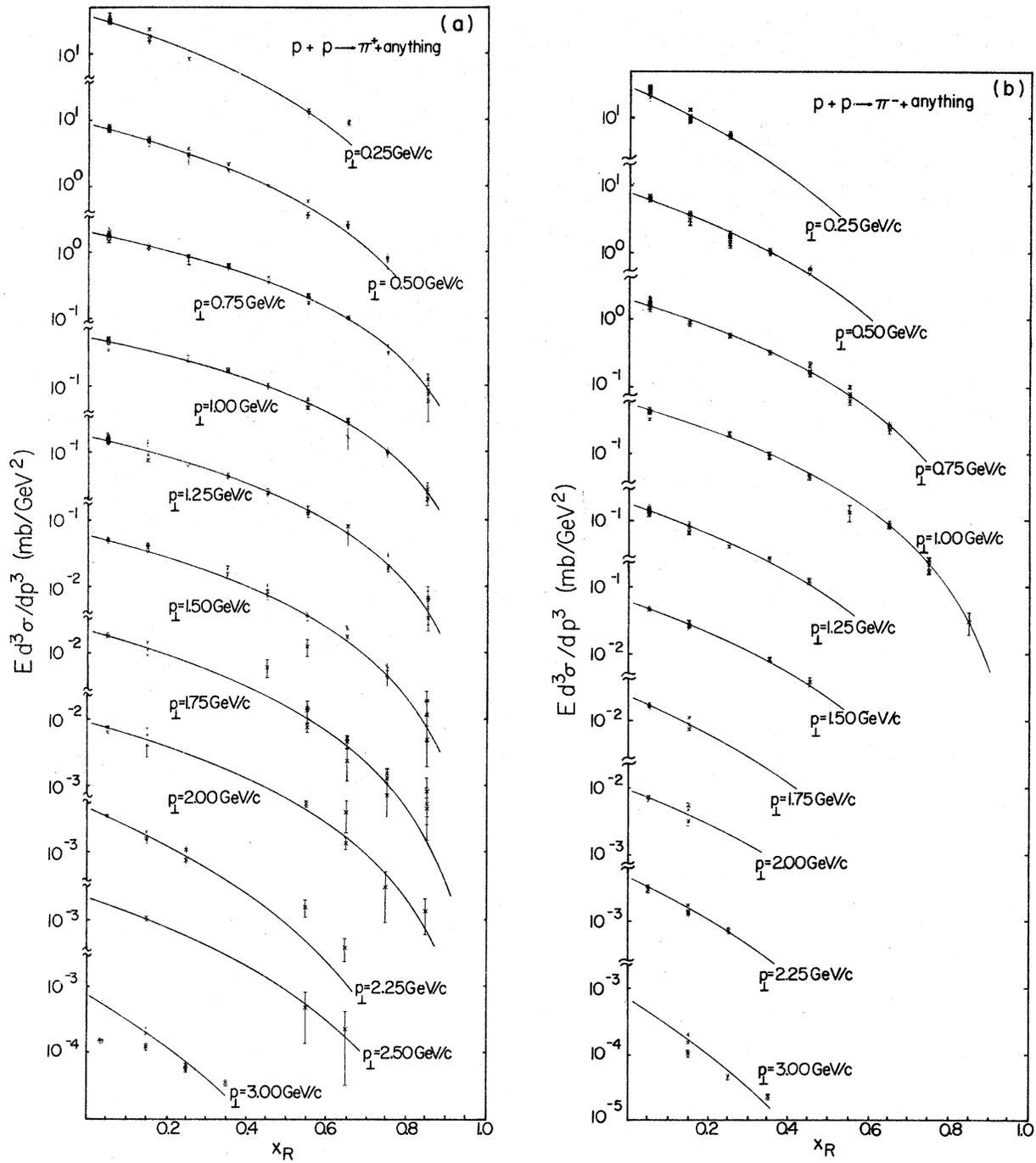


FIG. 11. (Continued on following page)

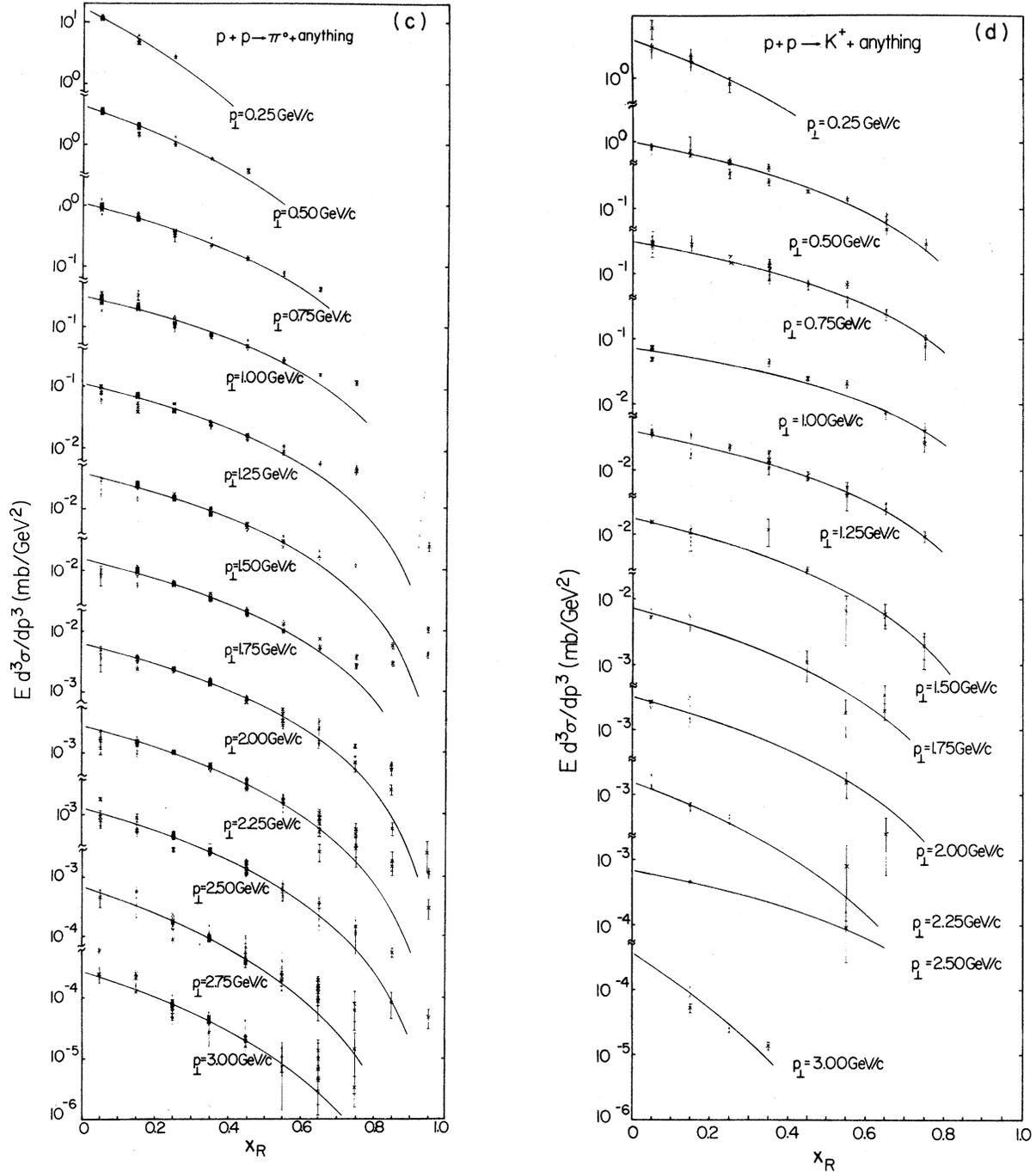


FIG. 11. (Continued on following page)

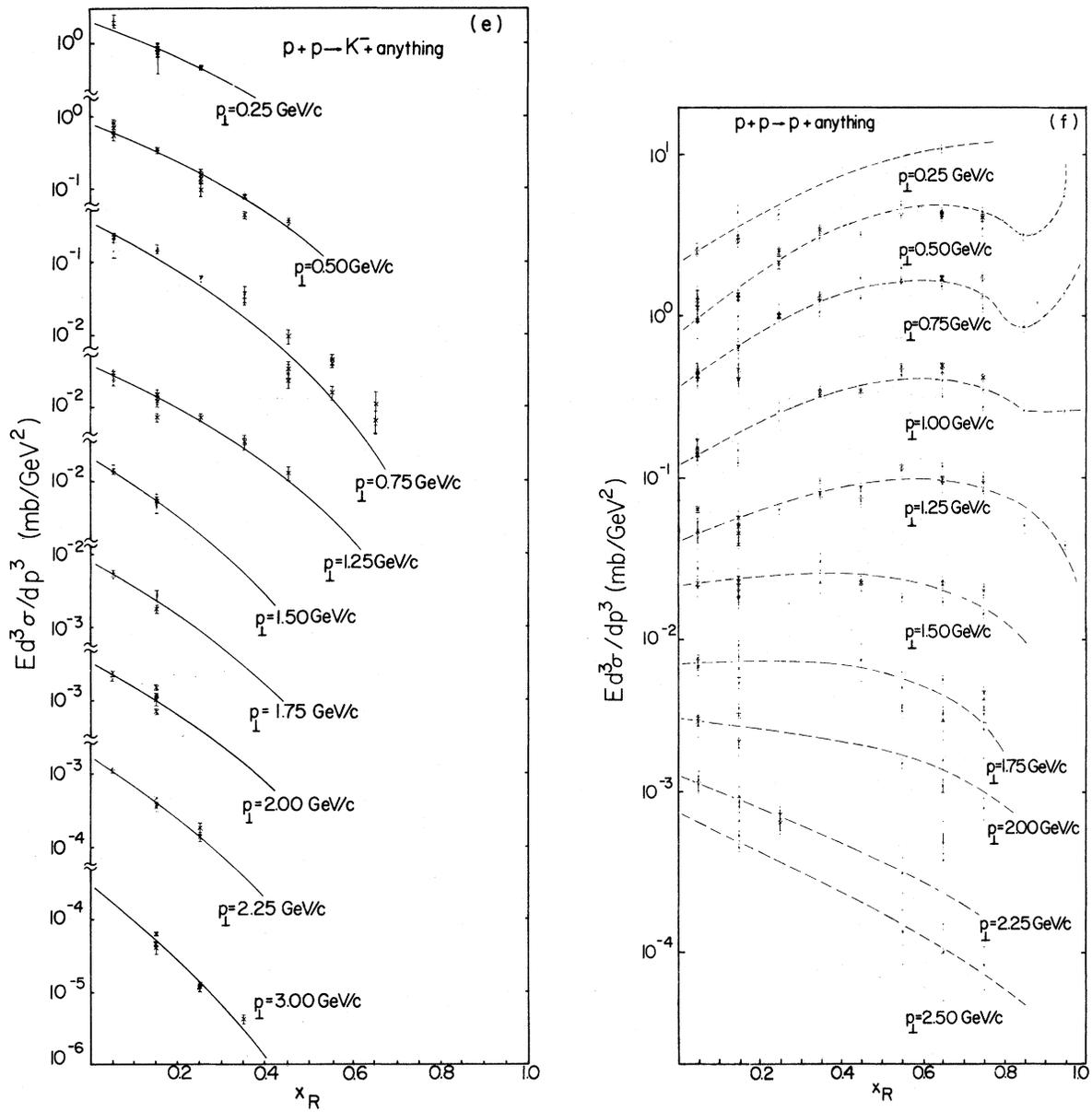


FIG. 11. (Continued on following page)

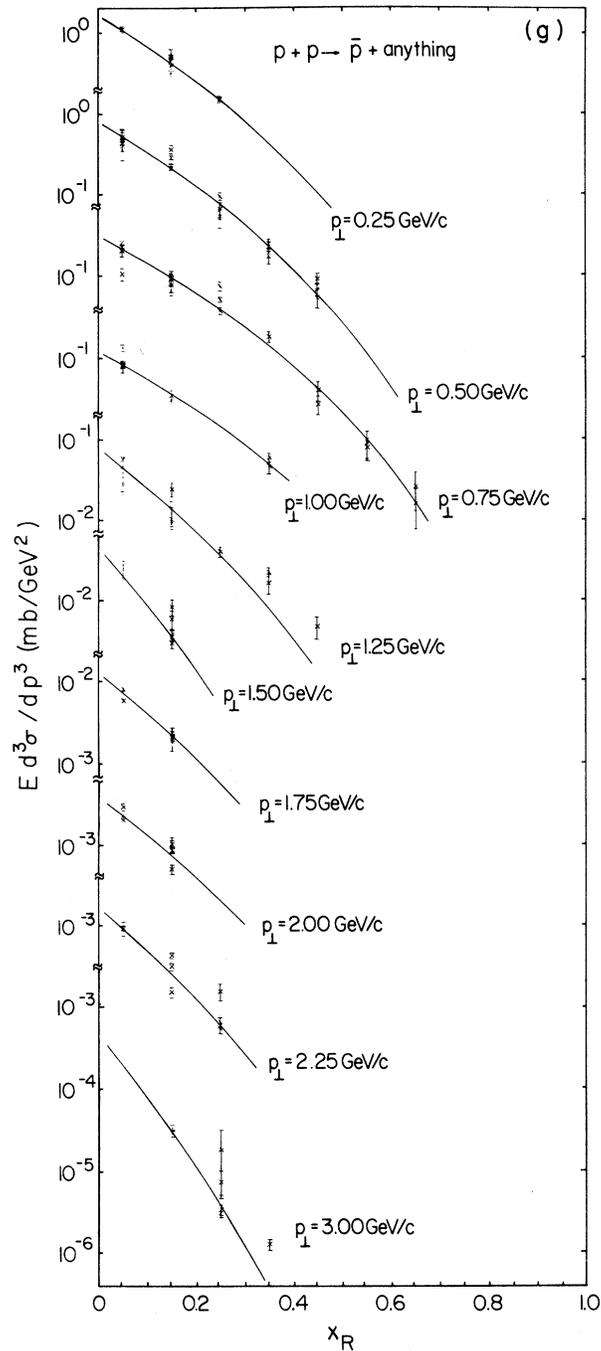


FIG. 11. The  $x_R$  dependence of the invariant cross section  $E d\sigma/dp^3$  for various constant values of  $p_\perp$  in the radial scaling region:  $\sqrt{s} \gtrsim 10$  GeV. The solid lines are the function  $B(1-x_R)^n$ , where  $B$  and  $n$  are determined from the minimum- $\chi^2$  fit to  $E d\sigma/dp^3|_{\langle p_\perp \rangle}$ , and are given in Table IV. (a)  $\pi^+$  invariant cross section, (b)  $\pi^-$  invariant cross section, (c)  $\pi^0$  invariant cross section, (d)  $K^+$  invariant cross section, (e)  $K^-$  invariant cross section, (f)  $p$  invariant cross section (the dotted line us to guide the eye, since no fit of the form  $B(1-x_R)^n$  was performed), (g)  $\bar{p}$  invariant cross section.

TABLE IV. The fit parameters  $B$  (mb/GeV<sup>2</sup>) and  $n$  defined in Eq. (19) in the text for various inclusive reactions in the radial scaling region  $\sqrt{s} \gtrsim 10$  GeV are presented. These parameters describe the  $x_R$  dependence of the single-particle inclusive cross sections for constant  $p_{\perp}$ .

| $p_{\perp}$ (GeV/ $c$ ) | $\pi^+$   | $\pi^-$                                   | $\pi^0$                                   | $K^+$                                     | $K^-$                                  | $\bar{p}$                              |
|-------------------------|---|---|---|---|--|--|
| 0.25                    | $B$ 36.3±1.4<br>$n$ 4.2±0.2                       | 28.7±2.8<br>6.0±0.5                       | 16.6±0.4<br>7.0±0.3                       | 4.0±0.5<br>4.9±0.9                        | 2.0±0.2<br>5.1±0.4                     | 1.7±0.2<br>8.5±0.5                     |
| 0.5                     | $B$ 8.72±0.28<br>$n$ 3.52±0.08                    | 7.7±0.4<br>4.6±0.1                        | 4.29±0.07<br>4.60±0.10                    | 1.06±0.06<br>2.90±0.10                    | (8.1±0.9)×10 <sup>-1</sup><br>5.6±0.3  | (8.0±0.7)×10 <sup>-1</sup><br>8.1±0.3  |
| 0.75                    | $B$ 2.01±0.04<br>$n$ 2.87±0.04                    | 1.93±0.05<br>4.07±0.04                    | 1.09±0.02<br>3.50±0.10                    | (3.1±0.1)×10 <sup>-1</sup><br>2.5±0.1     | (3.5±0.5)×10 <sup>-1</sup><br>7.0±0.4  | (3.2±0.2)×10 <sup>-1</sup><br>7.2±0.4  |
| 1.0                     | $B$ (5.32±0.14)×10 <sup>-1</sup><br>$n$ 2.82±0.06 | (5.28±0.01)×10 <sup>-1</sup><br>3.94±0.07 | (3.25±0.08)×10 <sup>-1</sup><br>3.20±0.10 | (7.3±0.6)×10 <sup>-2</sup><br>2.1±0.2     | (3.8±0.3)×10 <sup>-2</sup><br>6.1±0.7  | (1.3±0.7)×10 <sup>-1</sup><br>7.4±0.5  |
| 1.25                    | $B$ (1.79±0.05)×10 <sup>-1</sup><br>$n$ 3.24±0.07 | (1.76±0.07)×10 <sup>-1</sup><br>4.60±0.30 | (1.14±0.04)×10 <sup>-1</sup><br>3.30±0.09 | (3.9±0.2)×10 <sup>-2</sup><br>2.7±0.2     | (3.8±0.3)×10 <sup>-2</sup><br>6.1±0.7  | (7.9±0.9)×10 <sup>-2</sup><br>10.5±1.0 |
| 1.50                    | $B$ (6.00±0.2)×10 <sup>-2</sup><br>$n$ 3.51±0.09  | (6.03±0.02)×10 <sup>-2</sup><br>4.90±0.20 | (3.89±0.13)×10 <sup>-2</sup><br>3.24±0.09 | (1.80±0.06)×10 <sup>-2</sup><br>3.20±0.10 | (2.1±0.7)×10 <sup>-2</sup><br>8.5±0.4  | (4.7±0.7)×10 <sup>-2</sup><br>15.7±1.3 |
| 1.75                    | $B$ (2.33±0.09)×10 <sup>-2</sup><br>$n$ 3.90±0.20 | (2.3±0.02)×10 <sup>-2</sup><br>5.8±1.10   | (1.51±0.05)×10 <sup>-2</sup><br>3.33±0.09 | (7.6±0.6)×10 <sup>-3</sup><br>3.7±0.5     | (7.8±1.0)×10 <sup>-3</sup><br>7.6±1.2  | (1.3±0.2)×10 <sup>-2</sup><br>10.8±1.2 |
| 2.0                     | $B$ (9.3±0.7)×10 <sup>-3</sup><br>$n$ 3.7±0.4     | (9.3±1.5)×10 <sup>-3</sup><br>5.2±1.3     | (6.28±0.17)×10 <sup>-3</sup><br>3.38±0.07 | (3.2±0.4)×10 <sup>-3</sup><br>3.7±0.8     | (3.3±0.6)×10 <sup>-3</sup><br>7.4±1.6  | (3.9±0.8)×10 <sup>-3</sup><br>10.1±1.7 |
| 2.25                    | $B$ (4.8±0.4)×10 <sup>-3</sup><br>$n$ 5.7±0.5     | (4.7±0.4)×10 <sup>-3</sup><br>6.6±0.6     | (2.88±0.1)×10 <sup>-3</sup><br>3.70±0.1   | (1.6±0.2)×10 <sup>-3</sup><br>5.1±0.7     | (1.7±0.1)×10 <sup>-3</sup><br>8.8±0.6  | (1.7±0.4)×10 <sup>-3</sup><br>11.5±1.5 |
| 2.5                     | $B$ (2.2±0.3)×10 <sup>-3</sup><br>$n$ 4.6±0.7     | (1.29±0.06)×10 <sup>-3</sup><br>3.90±0.10 | (1.29±0.06)×10 <sup>-3</sup><br>3.90±0.10 | (6.9±0.9)×10 <sup>-4</sup><br>2.6±0.8     |  |  |
| 2.75                    | $B$   |   | (6.84±0.6)×10 <sup>-4</sup><br>4.60±0.2   |   |  |  |
| 3.0                     | $B$ (7.7±2.4)×10 <sup>-4</sup><br>$n$ 8.2±1.6     | (7.2±2.5)×10 <sup>-4</sup><br>8.8±1.8     | (2.72±0.2)×10 <sup>-4</sup><br>4.40±0.2   | (3.7±1.6)×10 <sup>-4</sup><br>8.8±2.2     | (3.0±0.9)×10 <sup>-4</sup><br>10.7±1.4 | (4.8±0.2)×10 <sup>-4</sup><br>16.7±2.0 |

$$E \frac{d\sigma}{dp^3} \Big|_{x_R \text{ fixed}} = \frac{A}{(1 + p_{\perp}^2/m^2)^q},$$

we find that

(1) the parameter  $m^2$  grows with increasing  $x_R$  for all particles and is generally larger for protons and antiprotons than for mesons,

(2) the parameter  $q$  grows with increasing  $x_R$  for all particles,

(3) the parameter  $A$ , which reflects the overall magnitude of the invariant cross section, decreases for increasing  $x_R$  for  $\pi^+$ ,  $\pi^-$ ,  $\pi^0$ ,  $K^+$ ,  $K^-$ ,  $\bar{p}$  (for protons,  $A$  increases with increasing  $x_R$ ),

(4) in the limit of  $x_R \rightarrow 0$ , the invariant cross section for meson production is consistent with

$$\lim_{x_R \rightarrow 0} \left( E \frac{d\sigma}{dp^3} \right) \propto \frac{1}{p_{\perp}^5},$$

and for protons and antiprotons (although the errors in the  $\bar{p}$  cross section are large) with the form

$$\lim_{x_R \rightarrow 0} \left( E \frac{d\sigma}{dp^3} \right) \propto \frac{1}{p_{\perp}^6}.$$

The shape as a function of  $x_R$  of the single-particle inclusive cross section for constant  $p_{\perp}$  shows a very strong dependence on the species of the

detected particle. In particular, the more forbidden the production of the detected particle is, the steeper the slope of  $E d\sigma/dp^3 \Big|_{p_{\perp} \text{ fixed}}$  versus  $x_R$ . Parameterizing the invariant cross section for fixed  $p_{\perp}$  in the form

$$E \frac{d\sigma}{dp^3} \Big|_{p_{\perp} \text{ fixed}} = B(1 - x_R)^n,$$

we find that the parameter  $n$  for the various particles is given by the following increasing order:  $p, K^+, \pi^+, \pi^0, \pi^-, K^-, \bar{p}$ . Only protons at low  $p_{\perp}$  show an increasing  $E d\sigma/dp^3$  versus increasing  $x_R$  for  $p_{\perp}$  fixed.

In conclusion, the most important result of this investigation is that all data for hadron production in proton-proton collisions exhibit radial scaling at all  $p_{\perp}$  for  $\sqrt{s} \gtrsim 10$  GeV.

Expressing single-particle inclusive cross sections in terms of  $p_{\perp}$  and  $x_R$  allows a simple and systematic behavior to be revealed for particle production.

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- \*Present address: Physics Department, Northeastern University, Boston, Massachusetts 02115.  
 †Present address: Physics Department, Indiana University, Bloomington, Indiana 47401.  
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<sup>1</sup>D. C. Carey *et al.*, preceding paper, Phys. Rev. D **14**, 1196 (1976).  
<sup>2</sup>J. D. Bjorken and J. Kogut, Phys. Rev. D **8**, 1341 (1973).  
<sup>3</sup>D. C. Carey *et al.*, Phys. Rev. Lett. **33**, 327 (1974); **33**, 330 (1974); Report No. Fermilab-PUB-75/20-EXP (unpublished).  
<sup>4</sup>K. Kinoshita and H. Noda, Prog. Theor. Phys. **46**, 1639 (1971); **49**, 896 (1973); Memoirs of the Faculty of Science, Kyushu Univ. **B4**, No. 5 (1973); Prog. Theor. Phys. **50**, 915 (1973). A preliminary analysis of the proton-proton single-particle inclusive data using the radial scaling variable was performed by E. Yen, Phys. Rev. D **10**, 836 (1974).  
<sup>5</sup>As far as we are aware, the variable  $x_R$  was first suggested by R. P. Feynman, 1969 (private communication from N. Byers).  
<sup>6</sup>R. P. Feynman, Phys. Rev. Lett. **23**, 1415 (1969); J. Benecke, T. T. Chou, C. N. Yang, and E. Yen, Phys. Rev. **188**, 2159 (1969).  
<sup>7</sup>T. Ferbel, Phys. Rev. D **8**, 2321 (1973).  
<sup>8</sup>A. M. Rossi *et al.*, Nucl. Phys. **B84**, 269 (1975).  
<sup>9</sup>There is some evidence in unpublished data for a small rise in the single-particle inclusive cross section for constant  $p_{\perp}$ ,  $x_R$  with increasing  $\sqrt{s}$  over the ISR energy

range. This could be related to the rise in the total  $pp$  cross section. It is therefore of interest to obtain high-precision data in this energy range in order to study the onset of this radial scaling breakdown. It should be noted that the approach to a radial scaling limit is from above at low energies, so that if the single-particle cross sections at the ISR really begin to rise, it corresponds to a minimum having occurred in the cross sections—reminiscent of the total-cross-section behavior. The corresponding behavior of the single-particle cross sections for fixed  $x_{\parallel}$  is much more complicated, and any subsequent rise over the ISR energy range produces a complex energy dependence arising from both kinematical and dynamical effects. In this case the cross sections approach their limit from below for  $x_{\parallel} < 0.2$ , and any subsequent rise over the ISR range tends to be seen as a retarded approach to an ultimate scaling limit. For  $x_{\parallel} > 0.2$ , the cross sections at low energy first rise rapidly from threshold, then turn around and approach their limits from above. Any subsequent rise over the ISR energy range then produces a very complex energy dependence.  
<sup>10</sup>Typical  $\pi^0$  cross sections measured at Fermilab and ISR appear to be typically half of the average of the  $\pi^+$ ,  $\pi^-$  production cross section. Of course, there is no reason in principle why this may not be the case. However, there are severe difficulties in absolute normalization of  $\pi^0$  cross sections. For this reason, quoted systematic errors in absolute normalization of  $\pi^0$  cross sections are large enough that no definitive conclusion can be made about the difference of  $\pi^0$  and

$(\pi^+ + \pi^-)/2$  cross sections.

- <sup>11</sup>Stanley J. Brodsky and Glennys R. Farrar, *Phys. Rev. D* 11, 3109 (1975); R. Blankenbecler and S. J. Brodsky, *ibid.* 10, 2973 (1974); J. Gunion, *ibid.* 10, 242 (1974); J. Gunion, in *Particles and Fields—1974*, proceedings

of the Williamsburg meeting of the Division of Particles and Fields of the American Physical Society, edited by C. E. Carlson (A. I. P., New York, 1975).

<sup>12</sup>S. M. Berman, J. D. Bjorken, and J. B. Kogut, *Phys. Rev. D* 4, 3388 (1971).