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Isospin decompositions of three-pion scattering amplitudes and their partial-wave analyses*

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The explicit connection is made between the partial-wave-analyzed forms of the isospin-decomposed three-pion amplitudes generated by permutation-symmetry methods and by Cartesian-tensor methods. In the minimal K-matrix model, the partial-wave-analyzed amplitudes generated by the former method lead to fewer coupled amplitudes per J^π,I channel than those generated by the latter method. This is due to the fact that kernels of the permutation-symmetry amplitude integral equations contain only one subsystem isospin while the kernels of the Cartesian-tensor amplitude integral equations contain all possible subsystem isospins. However, the Cartesian-tensor method has the virtues of easy determination of the number of independent three-pion scattering amplitudes prior to partial-wave analysis and gives a clearer insight into the structure of three-pion-to-three-pion scattering.

Recently, there have appeared two isospin decompositions of the three-pion-to-three-pion scattering amplitude which were applied to the minimal K-matrix model[1] for three-to-three scattering. In the first method,[2] by constructing states which transform according to the irreducible representations of the three-particle permutation group, one obtains the scattering amplitudes $[\chi]^I_{\nu n}$ with the $0 \le I \le 3$ systems requiring one, nine, four, and one such amplitudes, respectively. The set of coupled integral equations satisfied by the $[\chi]^I_{\nu n}$ is given by Eq. (6.7) of Ref. 2(b).[3]

In the second method,[4] by employing a Cartesian-tensor analysis of the three-pion-to-three-pion amplitude, one obtains seemingly simpler results. For the $I=0$, 3 systems, again only one amplitude ($\mathcal{T}_0, \mathcal{T}_3$, respectively) is required. For the $I=1$ system, one begins again with nine amplitudes $\overline{T}_{ij}$ with $i,j=1,2,3$. However, it is shown that these satisfy the conditions

$$\overline{T}_{ij} = \overline{T}_{i1} e(j)$$

and (1)

$$\overline{T}_{ij} = o(1)\overline{T}_{2j}\,,$$

where $e(j)$ and $o(1)$ represent the permutations on the initial- or final-state kinematic variables $(123) \to (jki)$ and $(123) \to (132)$, respectively. Thus, the $I=1$ amplitudes may be reduced down to a set of two, $\overline{T}_{11}$ and $\overline{T}_{21}$, which satisfy the coupled integral equations of Eq. (5.11) of Ref. 4. Similarly, for $I=2$, one begins with the six amplitudes $\overline{F}_i$ and $\overline{G}_i$ with $i=1,2,3$. Again these are shown to satisfy

$$\sum_i \overline{F}_i = 0 = \sum_i \overline{G}_i$$

and (2)

$$\overline{G}_i = o(1)\overline{F}_i\,,$$

reducing the system down to the two amplitudes $\overline{F}_1$ and $\overline{F}_2$ which satisfy Eq. (5.13) of Ref. 4.[5]

It is of interest to obtain the explicit connection between the results of these two approaches both to understand the numbers of amplitudes needed in describing the $I=1,2$ systems and to determine whether either approach possesses specific advantages over the other in the performing of numerical calculations. To accomplish this, we will employ for the most part the notation of Ref. 4. As to the connection between Refs. 2 and 4, we define the operators $[X]^I_{\nu n}$ by

$$\begin{aligned}
&[X]^0_{10} \equiv 3\mathcal{T}_0\,,\\
&[X]^1_{i-1,n} \equiv \sum_{j=1}^{3} \mathfrak{z}_{ij} a_{jn}\,,\\
&[X]^2_{11} \equiv -\tfrac{3}{4}\sqrt{3}(\overline{F}_1 + 2\overline{F}_2 - \overline{G}_1 - 2\overline{G}_2)\,,\\
&[X]^2_{12} \equiv \tfrac{9}{4}(\overline{F}_1 - \overline{G}_1)\,,\\
&[X]^2_{21} \equiv -\tfrac{9}{4}(\overline{F}_1 + \overline{G}_1)\,,\\
&[X]^2_{22} \equiv -\tfrac{3}{4}\sqrt{3}(\overline{F}_1 + 2\overline{F}_2 + \overline{G}_1 + 2\overline{G}_2)\,,
\end{aligned} \tag{3}$$

and

$$[X]^3_{20} \equiv 3\mathcal{T}_3\,,$$

where in the $I=1$ case

$$\begin{aligned}
&a_{j0} \equiv \sqrt{5}(\overline{T}_{j1} + \overline{T}_{j2} + \overline{T}_{j3})\,,\\
&a_{j1} \equiv (2\overline{T}_{j1} - \overline{T}_{j2} - \overline{T}_{j3})\,,\\
&a_{j2} \equiv \sqrt{3}(\overline{T}_{j2} - \overline{T}_{j3})\,,
\end{aligned} \tag{4}$$

$$\mathfrak{z}_{ij} \equiv \begin{pmatrix} -1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{2}\sqrt{3} & \frac{1}{2}\sqrt{3} \\ 0 & \frac{5}{6} & \frac{5}{6} \end{pmatrix}, \tag{5}$$

where $i,j=1,2,3$, and where $n=0,1,2$. These operators satisfy

$$o(1)[X]^I_{\nu n} = (-1)^\nu [X]^I_{\nu n} . \tag{6}$$

Also we recall that

$$t_\nu(2) = (-1)^\nu t_\nu(3) o(1) , \tag{7}$$

where $t_\nu(i)$ is the two-body t matrix for two-body isospin ν which satisfies

$$_\eta\langle \vec{p}_1'\vec{p}_2'\vec{p}_3' | t_\nu(i) | \vec{p}_1\vec{p}_2\vec{p}_3\rangle_\eta = (2\pi)^3 \delta(\vec{p}_i'(\eta) - \vec{p}_i(\eta))\, _\eta\langle \vec{p}_2'\vec{p}_3' | t_\nu | \vec{p}_j\vec{p}_k\rangle_\eta \tag{8}$$

in the notation of Ref. 2(a). Then, defining the operators

$$[\chi]^I_{\nu n} \equiv (e(1) + (-1)^\nu o(1))[X]^I_{\nu n} \tag{9}$$

and

$$\mathcal{S}t_\nu(i) \equiv (e(1) + (-1)^\nu o(1)) t_\nu(i) , \tag{10}$$

using[6] Eqs. (5.8), (5.11), and (5.13) of Ref. 4 and Eqs. (3)–(7) of the present paper, we obtain

$$[\chi]^I_{\nu n} = \sum_{i=1}^{3} G^{Ii}_{\nu n} \mathcal{S}t_\nu(i) - \pi i \sum_{\nu'} \mathcal{S}t_\nu(3) \mathcal{Y}^I_{\nu\nu'} [\chi]^I_{\nu' n} , \tag{11}$$

where $G^{Ii}_{\nu n}$ and $\mathcal{Y}^I_{\nu\nu'}$ are defined in Table V and Eq. (5.6) of Ref. 2(b). If the matrix elements of Eq. (11) are taken with respect to the partial-wave basis $|e_1e_2e_3JM\mu\lambda\rangle_\alpha$, Eq. (6.7) of Ref. 2(b) is recovered and the connection is completed.

As explained in Ref. 7, Eq. (6.7) of Ref. 2(b) is not in convenient form for numerical calculations. However, upon iteration and a change of variables, the resulting equation, (2.7) of Ref 7, is of a more tractable form. We wish to apply this same procedure to the two independent $I = 1$ amplitudes of Eq. (5.11) of Ref. 4. We rewrite these as

$$\overline{T}_{i1} = \phi_{i1} - \pi i \sum_{j=1}^{2} \sum_{\nu=0}^{2} [K^\nu_{ij}(2) t_\nu(2) + K^\nu_{ij}(3) t_\nu(3)] \overline{T}_{j1} , \tag{12}$$

where

$$K^0(2) = \begin{pmatrix} \frac{1}{3} & 2 \\ 0 & 0 \end{pmatrix}, \quad K^1(2) = \begin{pmatrix} 0 & 0 \\ -\frac{1}{2} & 0 \end{pmatrix}, \quad K^2(2) = \begin{pmatrix} -\frac{1}{3} & 0 \\ \frac{1}{2} & 0 \end{pmatrix}, \quad K^0(3) = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ 0 & 0 \end{pmatrix}, \quad K^1(3) = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & -1 \end{pmatrix}, \quad K^2(3) = \begin{pmatrix} -\frac{1}{3} & -\frac{2}{3} \\ \frac{1}{2} & 1 \end{pmatrix}. \tag{13}$$

Then, in the notation of Refs. 2 and 7, defining the matrix elements

$$\overline{T}^{JM\lambda M_I}_{i1}(e_1', e_2', e_3'; e_1, e_2, e_3; \mu', \mu) \equiv {}_\alpha\langle e_1'e_2'e_3'JM\mu'\lambda | \overline{T}_{i1} | e_1e_2e_3JM\mu\lambda\rangle_\alpha \tag{14}$$

and

$$\phi^{JM\lambda M_I}_{i1}(e_1', e_2', e_3'; e_1, e_2, e_3; \mu', \mu) \equiv {}_\alpha\langle e_1'e_2'e_3'JM\mu'\lambda | \phi_{i1} | e_1e_2e_3JM\mu\lambda\rangle_\alpha , \tag{15}$$

we make the change of variables suggested in Ref. 7,

$$\overline{T}^{JM\lambda M_I}_{i1}(e_1', e_2', e_3'; e_1e_2, e_3; \mu', \mu) = \phi^{JM\lambda M_I}_{i1}(e_1', e_2', e_3'; e_1, e_2, e_3; \mu', \mu) + \sum_{l=0}^{\infty} Y^{\mu'}_l(\xi_1', 0) R^{JM\lambda M_I}_{il}(\mathfrak{M}', e_1'; e_1, e_2, e_3; \mu', \mu) , \tag{16}$$

which reduces the dependence of the amplitudes from the three final-state kinematical variables $\mathfrak{M}', e_1', e_2'$ to the two variables $\mathfrak{M}', e_1'$. With this substitution, Eq. (12) becomes

$$\begin{aligned} R^{JM\lambda M_I}_{il}(\mathfrak{M}', e'_1; e_1, e_2, e_3; \mu', \mu) &= Q^{JM\lambda M_I}_{il}(\mathfrak{M}', e_1'; e_1, e_2, e_3; \mu', \mu) \\ &- \frac{\pi i M_{23}'}{p_1' k_{23}'} \sum_{\nu=0}^{2} \delta_{l+\nu,\,\mathrm{even}} t^\nu_l(M_{23}') \\ &\times \sum_{l'=0}^{\infty} \sum_j \sum_{\mu''} \int_{\epsilon_-}^{\epsilon_+} de_1'' Y^{\mu'}_l(\xi'', 0) [d^J_{\mu'\mu''}(D'') + \lambda d^J_{\mu'\mu''}(D'' + \pi)] Y^{\mu''}_{l'}(z'', 0) [(-1)^{l+l'} K^\nu_{ij}(2) + K^\nu_{ij}(3)] \\ &\times R^{JM\lambda M_I}_{jl'}(\mathfrak{M}', e_1''; e_1, e_2, e_3; \mu'', \mu) , \end{aligned} \tag{17}$$

TABLE I. The number of coupled partial-wave-analyzed amplitudes, $\mathfrak{N}_{J\pi}$, required to describe three-to-three scattering for the various J^π channels of the $I=1$ three-pion system in the minimal K-matrix model. The $\nu=0,1,2$ input consists of $t_0^0, t_2^0, t_1^1, t_0^2$, and the $\nu=0,1$ input consists of t_0^0, t_2^0, t_1^1.

	$\nu=0,1,2$ input		$\nu=0,1$ input	
J^π	$\mathfrak{N}_{J\pi}$ of Ref. 7	$\mathfrak{N}_{J\pi}$ of Ref. 4	$\mathfrak{N}_{J\pi}$ of Ref. 7	$\mathfrak{N}_{J\pi}$ of Ref. 4
0^-	4	5	3	5
1^-	2	3	2	3
1^+	6	8	5	8
2^-	7	10	6	10
2^+	3	5	3	5

TABLE II. The number of coupled partial-wave-analyzed amplitudes, $\mathfrak{N}_{J\pi}$, required to describe three-to-three scattering for the various J^π channels of the $I=2$ three-pion system in the minimal K-matrix model. The $\nu=1,2$ input consists of t_1^1, t_0^2 and the $\nu=1$ input consists of t_1^1.

	$\nu=1,2$ input		$\nu=1$ input	
J^π	$\mathfrak{N}_{J\pi}$ of Ref. 7	$\mathfrak{N}_{J\pi}$ of Ref. 4	$\mathfrak{N}_{J\pi}$ of Ref. 7	$\mathfrak{N}_{J\pi}$ of Ref. 4
0^-	2	4	1	2
1^-	1	2	1	2
1^+	3	6	2	4
2^-	3	6	2	4
2^+	1	2	1	2

in analogy with Eq. (2.7) of Ref. 7 and where Q_{il} is the inhomogeneous term. Equation (2.7) of Ref. 7 and Eq. (17) above may be used to generate sets of coupled integral equations for the partial-wave-analyzed scattering amplitudes for the various J^π channels of the $I=1$ system. The results[8] are given in Table I for the two-body input $t_0^0, t_2^0, t_1^1, t_0^2$ and for t_0^0, t_2^0, t_1^1. These show that the formalism of Refs. 2 and 7 leads to fewer coupled amplitudes per channel than does the formalism of Ref. 4 and Eq. (17). Essentially this is due to the fact that for a given value of i in Eq. (12) the kernel contains all subsystem isospins, while for a given ν in Eq. (11), the kernel contains only one subsystem isospin.

A similar analysis may be carried out for the $I=2$ system of Eq. (5.13) of Ref. 4. When the results are used to generate a set of coupled integral equations for each of the J^π channels, again it is found that the formalism of Refs. 2 and 7 leads to fewer coupled amplitudes per channel than does the formalism of Ref. 4. The numbers of coupled equations per channel obtained for the input t_1^1, t_0^2 and for t_1^1 are given in Table II.

As to other three-pion scattering models, such as the nonminimal K-matrix model of Eqs. (5.18) and (5.19) of Ref. 4, a partial-wave analysis may be performed in an analogous way.[9] Since the subsystem isospin structure for each of the two methods remains the same for these more general equations, the obtaining of fewer coupled amplitudes per J^π, I channel in the permutation-symmetry formalism is expected to continue to hold.

Thus, we conclude that the permutation-group methods and the Cartesian tensor methods for generating three-pion amplitudes are equivalent as expected. Further, while the latter method gives a much better insight into the structure of three-pion-to-three-pion scattering and demonstrates clearly which three-pion isospin decomposed amplitudes may be considered to be independent or dependent [i.e., Eqs. (1), (2)], the former method, with its fewer partial-wave-analyzed amplitudes per channel in the minimal K-matrix model, is better suited for numerical calculations.

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[1]K. L. Kowalski, Phys. Rev. D 7, 2957 (1973).

[2](a) J. A. Lock, Phys. Rev. D 12, 3319 (1975); (b) J. A. Lock, *ibid.* 13, 1009 (1976).

[3]In Ref. 1, the unsymmetrized two-body t matrix should have been employed rather than the symmetrized version. However, a compensating factor of 2 was omitted in going from Eq. (6.3) to Eq. (6.7) in Ref. 2(b), leaving Eq. (6.7) numerically correct as written. Further, the last line in Eq. (6.7) should read

$$[\chi_{JM\lambda IM_I}(e_1'', e_2' + e_3' - e_1'', e_1'; e_1, e_2, e_3; \mu'', \mu)]_{\nu' n}.$$

[4]K. L. Kowalski, Phys. Rev. D 13, 2352 (1976).

[5]In Eq. (5.13a) of Ref. 4 the diagonal element of the kernel should be $2\pi i A(3)$, and in Eq. (5.13b) the diagonal element of the kernel should be $2\pi i A(2)$. Further, the second equation in the Appendix should read $\hat{\delta}_{jk} \equiv 2\delta_{jk} + 1$.

[6]Reference 4 uses fully symmetrized two-body t matrices. However, with Kowalski's normalization of Eq. (4.1), the symmetrized t matrices and the unsymmetrized

t matrices of Ref. 2 are numerically equal.

[7] J. A. Lock, Nucl. Phys. A (to be published).

[8] The amplitude $R_{11}^{JM\lambda M_I}(\mathfrak{M}', e_1'; e_1, e_2, e_3; \mu', \mu)$ is not coupled to the other amplitudes and is given by only its inhomogeneous term.

[9] We must exercise some care in handling the sum over intermediate states; we would use the technique of R. Blankenbecler and R. Sugar, Phys. Rev. 142, 1051 (1966).