Comment on gauge theories without anomalies*

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We obtain an expression for the anomaly of a general representation of SU(N).

Unified gauge theories of the weak, electromagnetic, and strong interactions[1] describe elementary-particle interactions in terms of a gauge field theory based on a simple gauge group. If the unified field theory is to be renormalizable, it must be free of triangle anomalies.[2] This is a serious constraint on the theory when the fermion representation in the gauge theory is complex.[3] It seems possible that nature has indeed chosen such a complex representation, since only the left-handed quark fields participate in conventional weak interactions.

In this note we discuss the anomalies of representations of SU(N). We will obtain some explicit formulas which are useful in constructing representations which are complex but free of anomalies.[3] Such representations are essential ingredients in a class of unified gauge theories of the weak, electromagnetic, and strong interactions.[1]

Take t_a, for $a=1$ to N^2-1, to be the generators of the fundamental N-dimensional representation of SU(N),[4] normalized by the commutation relations

$$[t_a, t_b] = if_{abc}t_c\,. \tag{1}$$

The object of interest in the study of anomalies is the trace of the symmetrized product of three generators:

$$d_{abc} = \mathrm{Tr}(\{t_a, t_b\}t_c)\,. \tag{2}$$

If T_a are the generators of an arbitrary matrix representation R, normalized according to

$$[T_a, T_b] = if_{abc}T_c\,, \tag{3}$$

we define the anomaly of the representation R, $A(R)$, by

$$A(R)d_{abc} = \mathrm{Tr}(\{T_a, T_b\}T_c)\,. \tag{4}$$

We can make it obvious that the trace of the symmetrized product of T's has the form shown in Eq. (4). In general, we can write

$$\mathrm{Tr}(\{T_a, T_b\}T_c) = A_{abc}(R)\,. \tag{5}$$

If R is decomposable into a direct sum of representations R_1 and R_2, $R=R_1\oplus R_2$, the following is obvious:

$$A_{abc}(R_1\oplus R_2) = A_{abc}(R_1) + A_{abc}(R_2)\,. \tag{6}$$

If R is a tensor product of representations, $R=R_1\otimes R_2$, it is clear that

$$A_{abc}(R_1\otimes R_2) = D(R_1)A_{abc}(R_2) + D(R_2)A_{abc}(R_1)\,, \tag{7}$$

where $D(R)$ is the dimension of R. But we can form a general representation by decomposing appropriate tensor products of the fundamental representation [satisfying Eq. (2)]. So it is clear from Eqs. (6) and (7) that $A_{abc}(R) = A(R)d_{abc}$.

Thus Eq. (4) is satisfied, and furthermore $A(R)$ is an integer. The anomaly question reduces to the characterization of the integers $A(R)$ for each irreducible representation R of SU(N).

We label the irreducible representations of SU(N) in a slightly unconventional way by $N-1$ positive integers q_i for $i=1$ to $N-1$. The nonnegative integer q_i-1 is the number of columns with i boxes in the Young tableau associated with the given irreducible representation. For example, the fundamental N-dimensional representation, whose Young tableau is a single box, corresponds to $q_1=2$, $q_i=1$ for $i\neq 1$. In terms of the q_i's, the dimension $D(q)$ is a homogeneous polynomial[4]:

$$D(q) = \prod_{j=1}^{N-1}\left[\frac{1}{j!}\prod_{k=j}^{N-1}\left(\sum_{i=k-j+1}^{k} q_i\right)\right]. \tag{8}$$

Our main result can now be stated simply. For an irreducible representation of SU(N) the ratio of the anomaly to the dimension is the following cubic polynomial in q:

$$\frac{A(q)}{D(q)} = \sum_{i,j,k=1}^{N-1} a_{ijk}q_iq_jq_k\,, \tag{9}$$

where a_{ijk} is completely symmetric in i, j, and k, and for $i\leq j\leq k$

$$a_{ijk} = \frac{2(N-3)!}{(N+2)!}\,i(N-2j)(N-k)\,. \tag{10}$$

One can check Eqs. (9) and (10) by using Eqs.

(6) and (7) and the Clebsch-Gordan series for SU(N) to derive recursion relations for the anomalies.

From the basic formula, Eqs. (9) and (10), we can obtain some useful results for special cases. For SU(3)

$$\frac{A(q)}{D(q)}=\tfrac{1}{60}(q_1-q_2)(q_1+2q_2)(2q_1+q_2)\,. \qquad (11)$$

For SU(4)

$$\frac{A(q)}{D(q)}=\tfrac{1}{60}(q_1-q_3)(q_1+q_3)(q_1+2q_2+q_3)\,. \qquad (12)$$

For SU(5) and higher N, the result does not factor in this way into factors linear in q. For any N, for $q_1=m+1$ and $q_i=1$ for $i\neq 1$,

$$\frac{A}{D}=\frac{m(N+m)(N+2m)}{N(N+1)(N+2)}\,. \qquad (13)$$

This representation is the completely symmetric product of m fundamental N-dimensional representations. Multiplying by the dimension, we find

$$A=\frac{(N+m)!(N+2m)}{(N+2)!(m-1)!}\,. \qquad (14)$$

The completely antisymmetric combination of p fundamentals gives the representation with $q_p=2$ ($p\leq N-1$) and $q_i=1$ for $i\neq p$. For this representation

$$\frac{A}{D}=\frac{p(N-p)(N-2p)}{N(N-1)(N-2)} \qquad (15)$$

and

$$A=\frac{(N-3)!(N-2p)}{(N-p-1)!(p-1)!}\,. \qquad (16)$$

This last class of representations is particularly useful for building unified theories. If the N-dimensional representation of an SU(N) unified gauge group transforms under the color SU(3) subgroup as a triplet plus $N-3$ singlets, then the antisymmetric representations contain only color SU(3) triplets, their complex conjugates and singlets, just right to describe a world of quarks and leptons. With the aid of the anomaly formula Eq. (16), it is trivial to check whether a given unified theory with only antisymmetric fermion representations is anomaly-free. If the fermion fields are all written as left-handed fields, the representations must be such that the sum of all the $A(R)$ is zero. For example, in the SU(5) theory, there are two left-handed $\underline{10}$'s and two $\overline{\underline{5}}$'s and $A(10)=-A(\overline{5})=1$ [$\underline{10}$ is $p=2$ and $\underline{5}$ is $p=4$ in Eq. (16)], so that the theory is anomaly-free.

Our results do not resolve one amusing question about the anomalies of representations of SU(N): Are there irreducible representations which are complex but anomaly-free? There are none for SU(3) or SU(4). Equations (11) and (12) imply that for $N=3$ or 4, the anomaly vanishes if and only if $q_i=q_{N-i}$ for all i, in which case the representation is real. We know of no complex, irreducible, and anomaly-free representations for any N, but have no proof that none exist.

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[1]H. Georgi and S. L. Glashow, Phys. Rev. Lett. 32, 438 (1974).

[2]D. Gross and R. Jackiw, Phys. Rev. D 6, 477 (1972).

[3]H. Georgi and S. L. Glashow, Phys. Rev. D 6, 429 (1972).

[4]For a review of representations of SU(N) see A. Pais, Rev. Mod. Phys. 38, 215 (1966).