More SO(3) monopoles*

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(Received 29 March 1976)

Spontaneously broken gauge theories describing gauge bosons coupled in the manner of the Yang-Mills prescription to a Lorentz scalar ϕ transforming as an arbitrary $(2n+1)$-dimensional irreducible representation of the gauge group SO(3) are considered. It is shown that given the topologically stable, static solution of 't Hooft and Polyakov for the isovector $(n = 1)$ field there exists a recipe for constructing solutions to all higher-dimensional fields ϕ. The case $n = 2$ is worked out in some detail. The same recipe is applicable to any other homotopy class where the isovector problem is solved, and the solutions so generated are seen to be the only possible stable ones. Since the above solutions exist only if the vacuum is U(1) symmetric, arguments supporting that contingency for a general rank-n Lagrangian are given. In two space dimensions, the tower of solutions corresponding to the only stable homotopy class are outlined and the case $n = 2$ is described in detail. In all cases the electric potential that may be added in the manner of Julia and Zee is specified.

I. INTRODUCTION

This paper makes a modest contribution to the field opened up by 't Hooft,[1] Polyakov,[2] and Coleman.[3] 't Hooft and Polyakov independently provided us with a very interesting solution to the classical field equations describing an isotriplet $(\vec{\phi})$ of Lorentz scalars and non-Abelian gauge bosons, where the interactions are invariant under the local gauge group SO(3). The novel feature of this symmetry-breaking solution was that at different points on the sphere S^2_∞ at spatial infinity, the vacuum expectation value of the scalar field $\vec{\phi}$ took on different orientations in internal SO(3) space. This asymptotic scalar field was accompanied by a radial magnetic field (corresponding to the surviving massless gauge boson) with a total flux of $4\pi/e$ (e is the gauge coupling), which prompted 't Hooft to call it a monopole and Polyakov a "hedgehog." It was then Coleman (to the best of my knowledge) who showed that the introduction of certain methods from homotopy theory would shed light not only on the 't Hooft-Polyakov monopole but on the entire family of such solutions with nonconstant asymptotic behavior. One could see from these notions, for example, why the flux of the monopole had to be $4\pi/e$ instead of some arbitrary quantity. One could infer that the decay of the monopole into the vacuum under the influence of small perturbations or fluctuations was forbidden. This stability, but for which the possibility of such solutions manifesting themselves as new physical particles would be greatly diminished, does not follow from the fact that the monopole solution extremizes the energy functional and is rather of topological origin. Furthermore, these methods from homotopy theory, which expose the anatomy of topological stability, tell us if a given Lagrangian admits such solutions and provides an elegant scheme for their classification.

I began this investigation to see if these techniques could be employed to answer the following question: Are there static, topologically stable finite-energy solutions for the Lorentz scalar transforming as any other representation of SO(3) besides the isovector, which corresponds to the 't Hooft-Polyakov monopole? The outcome is presented in the following sequence. A concise survey of the relevant elements of gauge and homotopy theories is presented in Sec. II. The solution for the rank-two isotensor in three space dimensions follows in Sec. III. In Sec. IV it is shown that similar solutions exist for all higher-dimensional representations, and the recipe for constructing these is presented. It is shown that these solutions are the only possible ones. The problem is revisited in two space dimensions (Sec. V), and stable solutions are given for all even-rank tensors. The rank-two tensor is once again discussed in some detail. A brief summary follows in Sec. VI.

II. INTRODUCTION TO HOMOTOPY THEORY

The study of topologically stable solutions such as the 't Hooft-Polyakov monopole involves the elements of homotopy theory as applied to gauge theories. A survey of the necessary ingredients will be presented in this section to render the paper self-contained. It will be tailor-made to suit our needs and will be rather brief; an exhaustive and lucid survey may be found in Ref. 3.

Imagine a rubber string parametrized by a variable x ranging from 0 to 2π and a space or manifold Y which may, for example, be points inside a unit three-sphere or those on the surface of a torus, etc. Now mentally "dip" this string in this manifold and tie its ends. Each point x of the string is in contact with a point on the manifold—this defines a map or function $f(x)$: $S^1 \to y$, where the string with its ends 0 and 2π identified is denoted by its topological equivalent, the circle, S^1. Let us restrict ourselves to maps satisfying $f(0) = f(2\pi) = y_0$, a fixed point in Y. We are still free to deform the loop into other configurations, defining other maps $g(x)$, $h(x)$, etc. Given two maps f and g, the product $f \cdot g$ is defined to be

$$\begin{aligned} f \cdot g(x) &= f(2x), \quad 0 \leq x \leq \pi \\ &= g(2x - 2\pi), \quad \pi \leq x \leq 2\pi . \end{aligned} \tag{2.1}$$

Geometrically $f \cdot g$ corresponds to a configuration in which the portion $0-\pi$ of the string lies along f and the portion $\pi-2\pi$ along g. This is clearly yet another way to immerse the loop S^1 in Y. This suggests that a group structure is possible if we define the inverse map

$$f^{-1}(x) = f(2\pi - x) \text{ (which is } f \text{ run backwards)} \tag{2.2}$$

and the constant or identity map

$$c(x) = y_0 \text{ (which is the loop shrunk to the point } y_0\text{)}. \tag{2.3}$$

This is not quite true since $f^{-1}f(x)$, in which the segment $0-\pi$ goes along f and the segment $\pi-2\pi$ retraces this path, is not the same as $c(x)$, the constant map, in which the loop never leaves y_0. However, it is intuitively clear that $f^{-1}f$ can be deformed to $c(x)$, independent of f and the structure of Y. It therefore seems that a group will obtain if we consider as its elements not distinct maps but equivalence classes of maps, where the elements f_1, f_2 of a class $[f]$ are homotopic, i.e., topologically distortable into each other. The class $[e]$ containing the constant map $c(x) = y_0$ will then be the identity element. The expression

$$f_i \underset{y_0}{\simeq} f_j$$

is used to signify that f_i and f_j are two loops starting and ending at y_0 such that one may be continuously distorted into the other. If one parametrizes the intermediate maps by a variable t, conveniently chosen to range from 0 to 1, the collection defines a function $h(x,t)$ called the connecting homotopy, which satisfies

$$h(x,0) = f_i(x), \tag{2.4a}$$

$$h(x,1) = f_j(x), \tag{2.4b}$$

$$h(0,t) = y_0. \tag{2.4c}$$

If we imagine the loop $f_i(x)$ evolving in time (t) into the loop $f_j(x)$, $h(x,t)$ is simply the history of this evolution.

The set $[e],[f],[g],\ldots$ will form a group provided the notion of multiplication, which was well defined for distinct maps f and g, makes sense for classes $[f]$ and $[g]$. The equation

$$[f] * [g] = [f \cdot g] \tag{2.5}$$

defining class multiplication makes sense if two arbitrary choices of representative elements from $[f]$ and $[g]$ yield the same class, i.e., if

$$f_1 g_1 \underset{y_0}{\simeq} f_2 g_2 .$$

But this is certainly true—$f_1 \cdot g_1$ is homotopic to $f_2 \cdot g_2$ since f_1 can be distorted to f_2 and g_1 to g_2 since they are members of the equivalence classes $[f]$ and $[g]$. The desired homotopy connecting $f_1 g_1$ and $f_2 g_2$ is

$$\begin{aligned} h(x,t) &= h_1(2x, t), \quad 0 \leq x \leq \pi \\ &= h_2(2x - 2\pi, t), \quad \pi \leq x \leq 2\pi \end{aligned} \tag{2.6}$$

where h_1 and h_2 distort f_1 and g_1 into f_2 and g_2, respectively.

The group defined above is called the first homotopy group $\pi_1(y_0, Y)$. A famous example is given by $Y = E^2 - (0,0)$, i.e., the plane minus the origin and y_0 any point. Loops that avoid $(0,0)$ can be shrunk to a point and form $[e]$, the identity element; loops that encircle the origin once clockwise (counterclockwise) form another distinct class $[1]$ ($[-1]$); those that go around $(0,0)$ twice form the classes $[2]$ and $[-2]$, etc.

Instead of mapping the one-dimensional interval $0-2\pi$ (with $0 = 2\pi$), i.e., a circle S^1, if we map a unit square (with all points on the perimeter identified), i.e., a two-sphere S^2, we obtain the second homotopy group π_2. Once again the entire perimeter of the square (identified conventionally with the north pole of the sphere S^2) is required to go to some fixed point y_0 of Y when we consider $\pi_2(y_0, Y)$.

For our discussion of gauge theories let us consider a Lagrangian symmetric under a local gauge group G and involving the gauge fields A_μ^a and a scalar multiplet ϕ:

$$L = -\tfrac{1}{4} F_{\mu\nu}^a F^{\mu\nu,a} + T(D_\mu \phi) - V(\phi), \tag{2.7}$$

where as usual $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\mu A_\mu^a + e f^{abc} A_\mu^b A_\nu^c$ and f^{abc} are the structure constants of the Lie algebra of G. The kinetic energy form T involves the covariant derivative

$$D_\mu \phi = \partial_\mu \phi - e \vec{\mathrm{T}} \cdot \vec{\mathrm{A}}_\mu \phi ,$$

where T^a are the generators that act as the space of ϕ. The potential energy $V(\phi)$ involves only nonderivative terms. The ground state or vacuum of such a theory is decided by the parameters of $V(\phi)$. In the simplest case it is given by $\vec{\mathrm{A}}_\mu = \phi = 0$ (or its gauge equivalent). In the more interesting case of spontaneous breakdown, the energy/volume,

$V(\phi)$, is minimized by a constant but nonzero ϕ. Given a ϕ_0 that minimizes $V(\phi)$, we obtain a degenerate family of vacuums $g\phi_0$ ($g \in G$) by group action. There are two ways of describing this family, each with its own merits. The first is to imagine the surface, called the orbit surface, traced out by ϕ_0 as the group acts on it—each point on this surface, which lies in the space of ϕ, is an allowed vacuum. An equivalent way is to label each vacuum by the element of G that generates it from the reference ϕ_0. However, if there is a subgroup H that leaves ϕ_0 invariant, then $g\phi_0 = gh\phi_0$ ($h \in H$), and all elements $g_i h$ are equivalent to the element g_i. If we now identify all $g_i h$ with g_i, we obtain a new manifold called the coset space G/H, the points of which represent the possible vacuums. Note that in the passage $G \to G/H$, the entire subgroup H is identified with the identity element e since $h_i = eh_i$.

As an example, consider an isovector $\vec{\phi}$ of the group SO(3) and its potential $V(\vec{\phi}) = -\frac{1}{2}\mu^2(\vec{\phi}\cdot\vec{\phi}) + \frac{1}{4}\lambda(\vec{\phi}\cdot\vec{\phi})^2$. For the initial vacuum state $\vec{\phi}_0$ we may choose any vector $\vec{\phi}_0$ of length $(\mu^2/\lambda)^{1/2}$. The invariant subgroup $H = U_{\phi_0}(1)$, the rotations around $\vec{\phi}_0$. The orbit space is clearly a sphere of radius $(\mu^2/\lambda)^{1/2}$ and this is topologically equivalent to $SO(3)/U_{\phi_0}(1)$.

Such detailed parametrization of the possible vacuums is pointless if we are interested in just the ground state, which corresponds to just one point in G/H. However, consider a general finite-energy solution. In any spatial direction it must asymptotically tend to some allowed vacuum configuration or else its $V(\phi)$ would exceed that of the vacuum over an infinite volume. Furthermore, if gauge fields were absent, it would have to tend to the same vacuum in all directions—for although $V(\phi)$ is indifferent to rotations in internal space, the derivative terms $\partial_i\phi$ in $T(D\phi)$ will produce a finite energy/volume. The gauge fields can help us avoid such uninteresting asymptotics, for in their presence $D_\mu\phi = \partial_\mu\phi - e\vec{T}\cdot\vec{A}_\mu\phi$, which can be arranged to vanish rapidly enough.

The existence of such solutions, which exploit the wealth of possible vacuums, seems to indicate an increased variety in the particle spectrum of the theory. However, such variety is illusory if these solutions are not stable under small perturbations and fluctuation from collapsing to the ground state. Even if we pick, as did 't Hooft, a solution that extremizes the energy functional, the question of stability remains open till second variations are calculated and found to be positive or zero. Such computations involve detailed dynamics and we prefer to avoid such an undertaking. How then are we to ascertain the status of asymptotically nonconstant solutions such as the 't Hooft-Polyakov monopole?

We can ascertain this by stretching the topological ideas we discussed earlier. Imagine for a moment that we live in two dimensions where spatial infinity is a circle S^1_∞. A given solution assigns to each point on it a vacuum, and thus defined a map f: $S^1_\infty \to G/H$. The ground state corresponds to the trivial map e in which all points on S^1_∞ (in fact, all points in space) map onto some fixed point in G/H. We are asking if a configuration f can dynamically evolve into e, say under small perturbations. The answer is trivial if we incorporate the fact that all dynamical evolutions are continuous topological deformations, where the entry of abrupt spatial or temporal changes (associated with infinite derivations) is forbidden by the finiteness and constancy of the initial energy. Thus f cannot ever evolve into e if the classes $[f]$ and $[e]$ are distant. (Of course if $[f]=[e]$, f may still be stable, but not for topological reasons.)

Thus, given a Lagrangian $L(A, \phi)$, the quest for stable solutions begins with a study of $\pi_1(G/H)$. If it is nontrivial, i.e., if it contains classes $[f] \neq [e]$, the maps corresponding to the classes $[f]$ are topologically stable. From such a class $[f]$, one picks a map and chooses the accompanying gauge fields to satisfy $D\phi = 0$ on S^2_∞. These asymptotic fields must then be continued to finite distances in a way that will be discussed as we go along.

Our classification of solutions by the behavior of ϕ at S^1_∞ is futile if by a gauge transformation one can go from one class $[f]$ to another, $[g]$, in particular $[g]=[e]$. Fortunately any two gauge-equivalent solutions may be shown to be homotopic to each other.[3]

To play this game in three dimensions one simply replaces S^1_∞ by S^2_∞ and $\pi_1(G/H)$ by $\pi_2(G/H)$ in the above. Before commencing, I would like to discuss a very elegant and powerful theorem relating $\pi_2(G/H)$ to $\pi_1(H)$ and $\pi_1(G)$. The proof will be brief and is provided in the interest of completeness. A complete discussion may be found in Ref. 3.

Consider a finite-energy solution in three dimensions corresponding to which is a configuration of ϕ and A on S^2_∞. Let the sphere be spanned by loops P_1, P_2, etc. that start and end on the north pole [Fig. 1(a)]. Since $D_\mu\phi = \partial_\mu\phi - e\vec{T}\cdot\vec{A}_\mu\phi$ must vanish on this sphere, the field $\phi(p_1)$ at a point p_1 on path P_1 is related to the field $\phi(0) = \phi_0$, the reference vacuum at the north pole, by the solution to this Schrödinger-type equation:

$$\phi(p_1) - \left[T\exp\left(+e\int_0^{p_1}\vec{T}\cdot\vec{A}_i\,dx^i\right)\right]\phi(0) = U(P_1, 0, p_1)\phi(0), \qquad (2.8)$$

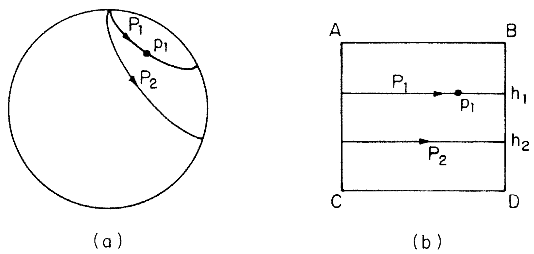

FIG. 1. (a) Paths on S^2_∞, the sphere at infinity. (b) The sphere at infinity represented as a square. The perimeter corresponds to the north pole.

where the integral is along the path P_1 from 0 to p_1 and path ordered by T. Thus to each point p_1 on the sphere is associated an element $U(P_1, 0, p_1)$ of G. Let us now spread out the sphere on a plane placing the loops P_1, P_2, etc. in juxtaposition [Fig. 1(b)]. The four edges of the square correspond to the north pole as they should: AB and CD are zero-length paths that never leave the north pole, while AC and BD are the extremities of paths that start and end on the north pole. The group element associated with the edges AB and CD (the null paths) as well as the edge AC (where the paths begin) is the identity element. On the line BD (where the paths return to the north pole and ϕ is back to ϕ_0) are mapped elements that satisfy $U\phi_0 = \phi_0$, i.e., the elements of the subgroup H that leave ϕ_0 invariant. In the manifold G/H, where all of H shrinks to a point, the square forms a closed surface and corresponds to an element of $\pi_2(G/H)$.

The same square when read vertically instead of horizontally tells another story: As we follow the line BD from B to D, we describe a loop in H starting and ending at the identity. Thus each element of $\pi_2(G/H)$ generates an element of $\pi_1(H)$. This element belongs to kernel (ker) $\pi_1(H) \to \pi_1(G)$, i.e., it is a loop on H that can be deformed to a point in G. The proof is simple [see Fig. 1(b)]. As we move to the left reading along lines parallel to BD, we see the loop in H smoothly evolving to the point loop at the identity by the time we reach AC. Thus the square, which defined an element of $\pi_2(G/H)$, is also the connecting homotopy in G that transforms an element of $\pi_1(H)$ into the identity element of $\pi_1(G)$. Conversely each element of kernel $\pi_1(H) \to \pi_1(G)$ generates an element of $\pi_2(G/H)$. Since this correspondence can be shown to be one to one[3] we have the theorem

$$\pi_2(G/H) = \ker \pi_1(H) \to \pi_1(G). \tag{2.9}$$

We are interested in the following corollary of this theorem: If the vacuum completely breaks the symmetry G, H is trivial, $\pi_1(H)$ and $\pi_2(G/H)$ are trivial, and there are no topologically stable solutions in three dimensions.

III. THE SECOND-RANK ISOTENSOR IN THREE DIMENSIONS

In our search for static topologically stable, finite-energy solutions we adopt the strategy outlined in Sec. II: For the scalar ϕ, we seek a class $[f]$ of maps $S^2_\infty \to G/H$ distinct from the identity $[e]$, pick $f \in [f]$ from this gauge-equivalent set, and determine the gauge fields (on S^2_∞) such that $D\phi = 0$. The behavior of the fields at finite distances will be discussed later. In three dimensions the existence of $[f] \neq [e]$ is decided by $\pi_2(G/H)$. From the theorem $\pi_2(G/H) = \text{kernel}\,\pi_1(H) \to \pi_1(G)$ we see that unless H is nontrivial (so it can contain nontrivial loops) we cannot even get started. Cases where the vacuum completely breaks $G = SO(3)$ do not interest us.

Of the possible $(2n+1)$-dimensional representations the case $n=0$ never interests us: All maps are necessarily trivial, and G/H is a point. The case of the isovector ($n=1$), which we will refer to as $\vec{\phi}$, with the potential

$$V(\vec{\phi}) = -\tfrac{1}{2}\mu^2 \vec{\phi}\cdot\vec{\phi} + \tfrac{1}{4}\lambda(\vec{\phi}\cdot\vec{\phi})^2, \tag{3.1}$$

is familiar. The vacuum is any vector of length $|\phi| = (\mu^2/\lambda)^{1/2}$ and the set of rotations around the vacuum vector, conventionally chosen along the z axis of internal space, form the subgroup $H = U_z(1)$. Instead of asking next if $\ker \pi_1(H) \to \pi_1(G)$ is nontrivial, we directly consider the orbit surface: a sphere $S^2_{G/H}$ of radius $(\mu^2/\lambda)^{1/2}$. Consider the map in which the point $(\theta\phi)$ of S^2_∞ falls on the point $(\theta\phi)$ of $S^2_{G/H}$. In the language of Ref. 3, S^2_∞ covers $S^2_{G/H}$ like skin of the orange covers the orange and in these picturesque terms the stability is obvious: No smooth deformations of the skin can shrink it to a point (the trivial map). This map corresponds to the 't Hooft-Polyakov monopole.

When we turn to higher tensors the situation looks bleak. Consider the second-rank tensor which we will denote by the symbol Φ written as a traceless, symmetrized outer product of two isovectors $\vec{\phi}_1$ and $\vec{\phi}_2$:

$$\Phi^{ij} = \vec{\phi}_1^{\,i}\,\vec{\phi}_2^{\,j} + \vec{\phi}_2^{\,i}\,\vec{\phi}_1^{\,j} - \tfrac{2}{3}\delta^{ij}(\vec{\phi}_1\cdot\vec{\phi}_2). \tag{3.2a}$$

Such a tensor has no axis of invariance in general; if you rotate around $\vec{\Phi}^1$ you affect $\vec{\phi}^2$ and vice versa. Thus the vacuum completely breaks SO(3) unless it miraculously corresponds to the case $\vec{\phi}^1 \parallel \vec{\phi}^2$ [for at least a range of parameters of $V(\Phi)$] in which case $H = U(1)$. But this is precisely what happens. Let us represent the vacuum Φ_{ij} as a 3×3 symmetric traceless matrix and consider it in diagonal form:

$$\Phi_0=\begin{pmatrix}-a & 0 & 0\\ 0 & -b & 0\\ 0 & 0 & (a+b)\end{pmatrix}. \tag{3.2b}$$

In the Lagrangian density [Eq. (2.7)], $T(D_\mu\Phi)=\frac{1}{4}\operatorname{Tr}(D_\mu\Phi D^\mu\Phi)$, $D_\mu\Phi=\partial_\mu\Phi-e[\vec{T}\cdot\vec{A}_\mu,\Phi]$, where $\vec{T}$ are the 3×3 generators. The potential

$$V=-\tfrac{1}{2}\mu^2\operatorname{Tr}\Phi^2-\tfrac{1}{3}\gamma\operatorname{Tr}\Phi^3+\tfrac{1}{4}\lambda(\operatorname{Tr}\Phi^2)^2 \tag{3.3}$$

is the most general admissible one since $(\operatorname{Tr}\Phi^2)^2=2\operatorname{Tr}\Phi^4$ and $\det\Phi$ is expressible in terms of the other terms.[4] The stationary point satisfies

$$\frac{\partial V}{\partial a}=0=[-\mu^2-\gamma b+2\lambda(a^2+ab+b^2)](2a+b) \tag{3.4a}$$

and

$$\frac{\partial V}{\partial b}=0=[-\mu^2-\gamma a+2\lambda(a^2+ab+b^2)](2b+a). \tag{3.4b}$$

Thus either $(2a+b)=0$ or $(2b+a)=0$ or $a=b$. In any case there is a U(1) subgroup. Choosing $a=b=[\gamma\pm(\gamma^2+24\mu^2\lambda)^{1/2}]/12\lambda$

$$\Phi_0=\begin{pmatrix}-a & 0 & 0\\ 0 & -a & 0\\ 0 & 0 & +2a\end{pmatrix}, \tag{3.5}$$

which is invariant under rotations about the z axis. The other two options pick the x and y axes and are completely equivalent.

In the notation of Eq. (3.2a), we can see $\vec{\phi}^1\parallel\vec{\phi}^2$ if $a=b$:

$$\Phi_0^{ij}=\vec{\phi}_0^i\vec{\phi}_0^j-\tfrac{1}{3}\delta_{ij}\vec{\phi}_0\cdot\vec{\phi}_0, \tag{3.6}$$

with $\vec{\phi}_0=\sqrt{3a}(0,0,1)$. The orbit surface generated by Φ_0 is best visualized in terms of group action on the vector $\vec{\phi}_0$ defined in Eq. (3.6). It is again a sphere but with antipodal points identified—since $\vec{\phi}$ and $-\vec{\phi}$ generate the same Φ. We denote it by $\not{S}^2$; the slash reminds us that antipodal points are identified. The map $S^2_\infty\to\not{S}^2$ corresponding to the orange and the skin is clearly stable and in this map Φ at any point on S^2_∞ is obtained from the corresponding isovector $\vec{\phi}$ of 't Hooft and Polyakov via Eq. (3.6). What about the gauge fields? They satisfy $\partial_i\Phi-e[\vec{T}\cdot\vec{A}_i,\Phi]=0$ on S^2_∞. The solution to this Heisenberg-type equation is

$$\Phi_0(p_1,P_1)=\left[T\exp\left(+e\int_0^{p_1}\vec{T}\cdot\vec{A}_i\,dx^i\right)\right]\times\Phi_0\left[T\exp\left(-e\int_0^{p_1}\vec{T}\cdot\vec{A}_i\,dx^i\right)\right]. \tag{3.7}$$

This equation tells us that the tensor at the point p_1 on path P_1 [Fig. 1(a)] is obtained from that at the north pole by the rotation generated by the gauge fields on the path. Since the vector field $\vec{\phi}$ that generates Φ is identical to that of 't Hooft and Polyakov it is clear that the same gauge fields will induce the desired rotation. Thus there is a radial spherically symmetric magnetic field (with a total flux of $4\pi/e$) on S^2_∞. What is the solution at finite distances r? Let us parametrize the field at all distances r as follows (see Appendix):

$$A_i^j=-\epsilon_{ijk}\frac{r^k}{er^2}[1+K(r)],\quad K\to0\text{ as }r\to\infty \tag{3.8}$$

$$\Phi_{ij}=\frac{(3r_ir_j-r^2\delta_{ij})}{r^2}\frac{q(r)}{er},\quad q\to ear\text{ as }r\to\infty. \tag{3.9}$$

One can calculate the energy functional in terms of K and q. Its stationary point is given by the solution to

$$x^2\frac{d^2q}{dx^2}=6K^2q+2qx^2(\bar{\gamma}-6\bar{\lambda})-2\gamma xq^2+12\bar{\lambda}q^3, \tag{3.10}$$

$$x^2\frac{d^2K}{dx^2}=K(K^2-1+9q^2), \tag{3.11}$$

where $\bar{\lambda}=\lambda/e^2$, $\bar{\gamma}=\gamma/e^2a$, and $x=ear$.

Our tensor monopole may be readily converted to a dyon in the manner of Julia and Zee.[5] The philosophy is that if an electric potential A_0^i satisfying $D_0\Phi\equiv0$ is introduced, it is completely decoupled from Φ and enters the Lagrangian via $\frac{1}{2}F^a_{0i}F^a_{0i}=\frac{1}{2}|D_i\vec{A}_0|^2$ and mimics (but for an overall sign) an isotriplet coupled minimally to the gauge fields $\vec{A}_i$. In the present static case we want $D_0\Phi=e[\vec{T}\cdot\vec{A}_0,\Phi]\equiv0$. On S^2_∞, where Φ is obtained from the 't Hooft-Polyakov isovector $\vec{\phi}$ via Eq. (3.6), it is clear that the potential of Julia and Zee which satisfies $\vec{T}\cdot\vec{A}_0\phi=0$ automatically fulfills $[\vec{T}\cdot\vec{A}_0,\Phi]=0$. At finite distances, the asymptotic Φ and $\vec{A}_0$ get modulated by functions only of r that involve no group indices, and the condition $D_0\Phi=0$ continues to hold.

We turn next to the task of constructing topologically stable solutions for higher-rank tensors. It will be convenient for us to employ at times the Cartesian representation for these spherical tensors. We write the rank-n tensor as a traceless symmetrized Kronecker product of n vectors [see Eq. (3.2a) for $n=2$]:

$$\Phi^{ijk\cdots}=\vec{\phi}_1^i\vec{\phi}_2^j\cdots+\text{permutations}-\text{traces}. \tag{3.12}$$

Such an object clearly has $(2n+1)$ degrees of freedom: the product of their lengths plus the $2n$ angles associated with the n vectors. Since it is traceless, it has pure spin n and transforms irreducibly.

IV. HIGHER-RANK ISOTENSORS IN THREE SPACE DIMENSIONS

Let us assume that we have been given a topologically stable map of an isotensor of rank n on S^2_∞ and ask what we can say about it. First we know that there is a nontrivial H. This H has to be U(1) since SO(3) has only one nontrivial subgroup, SO(2) = U(1). Stated differently, the only symmetry a rank-n tensor can have is that of rotations around an axis, and this happens when the vectors $A, B, \ldots, N$ that build it up are aligned along this axis. Thus the most general orbit surface, generated by this underlying vector, is either S^2 (n odd) or $\mathscr{S}^2$ (n even, since even-rank tensors are insensitive to the sign of the underlying vector). There are of course many classes of maps $S^2_\infty \to S^2$ or $\mathscr{S}^2$. The constant map, $S^2_\infty \to$ a point on G/H, is trivial. The identity map (orange and its skin) is the simplest stable map. The rank-n tensor in this case is given by the n-fold traceless Kronecker product of the 't Hooft-Polyakov isovector map. The gauge fields are the same, to ensure $D\phi = 0$, and the flux is $4\pi/e$. The fields at finite distances are found by extremizing the energy functional and change with n. Notice that this is the most general possibility for this class. Next we have maps in which S^2_∞ wraps around S^2 or $\mathscr{S}^2$ m times. The fields for this case have not been explicitly written down by anyone. But we do know the following: (a) Given the isovector distribution, the only possibility for higher-rank fields is the traceless Kronecker product, and (b) the gauge fields will be the same for all n with a flux $4\pi m/e$.

Our result, that the most general topologically stable configurations for the rank-n tensors correspond to those generated from the isovector configuration by formation of traceless n-fold Kronecker products, loses its relevance if these configurations do not have finite energy above the vacuum. [We mean by vacuum a local minimum of $V(\phi)$.] The kinetic energy $\int d^3x\, T(D\phi)$ can be kept finite since we know the gauge fields that make $D\phi = 0$ on S^2_∞. The potential energy $\int d^3x\, V(\phi)$ will diverge unless the U(1)-symmetric tensors we have mapped at infinity correspond to possible vacuums of the theory. We are therefore faced with the question: Are there U(1)-symmetric minima of $V(\phi)$ for all n? We approach the question in two stages: We first ask if $V(\phi)$ admits U(1)-symmetric stationary points in general, and then ask if these points are also minima in ϕ space.

Let us first observe that within the U(1)-symmetric subspace there will be points stationary with respect to variations within the subspace. To see this, let us first write the rank-n tensor as $\phi = (\phi^n, \phi^{n-1}, \ldots, \phi^0, \ldots, \phi^{-n})$ in the usual spherical or "angular momentum" rotation. The component ϕ^0 is invariant under z rotations. In this subspace the potential is

$$V(\phi^0) = -\tfrac{1}{2}\mu^2(\phi^0)^2 - a\gamma(\phi^0)^3 + b\lambda(\phi^0)^4\,, \tag{4.1}$$

where μ^2 is positive (to ensure symmetry breakdown) and $b\lambda$ is positive for the theory to have a stable ground state. Such a potential has three stationary points. The origin is a maximum (due to the negative mass term) and the other two are necessarily local minima. Are these points stationary with respect to variations in other directions as well? The answer is affirmative[3] and the proof[3] is as follows: Consider a change δV in the potential due to a change $\delta\phi$ at a U(1)-symmetric point:

$$\delta V = \frac{\partial V}{\partial \phi^i}\,\delta\phi^i = V_i\,\delta\phi^i\,. \tag{4.2}$$

If $i \neq 0$, $\delta\phi^i$ has isospin along the z axis and V_i must have an opposite amount so that δV the isoscalar has none. But at the U(1)-symmetric point we cannot construct an object with nonzero isospin along z using ϕ^0. Thus $\delta V = 0$ if $V_0 = 0$. Thus the three stationary points in the U(1) subspace are stationary in the entire ϕ space.

We next ask if the two nontrivial points, which were minima with respect to ϕ^0 variations, are minima for arbitrary $\delta\phi$. One can try to show one of two things:

(i) *All* stationary points of $V(\phi)$ are U(1) symmetric. Then the absolute minimum in the entire space has to lie in this subspace—and we know such a minimum must exist in any sensible ϕ^4 theory with negative (mass)^2, since the potential turns negative as we leave $\phi = 0$ and goes to $+\infty$ as $\phi \to +\infty$. In the isotensor case this is what happened. We searched for a general solution to $\delta V = 0$ and found it to be U(1) symmetric. We have not shown this in general—we have shown that U(1)-symmetric stationary points exist but have not excluded others.

(ii) At least one of the minima in the ϕ^0 direction is also a minimum in all directions.

Even if (i) is true, I do not know how I would go about proving it. As for (ii), let us consider the variation in V at a minimum in the ϕ^0 subspace, keeping quadratic terms:

$$\delta V = \tfrac{1}{2}\sum_{m=0}^{n} \frac{\partial^2 V}{\partial\phi^m \partial\phi^{-m}}\,\delta\phi^m\delta\phi^{-m} = \sum_{m=0}^{n} C_m\,\delta\phi^m\delta\phi^{-m}\,, \tag{4.3}$$

where isospin conservation along z has limited the number of nonzero elements of the mass matrix $\partial^2 V/\partial\phi^m\partial\phi^{-m}$. We are given that $C_0 > 0$. We also know that $C_1 = 0$, since it stands for the mass of the Goldstone bosons. (When the generators T_x and T_y, which are linear combinations of $T_\pm$, act on ϕ^0, the Goldstone mode they excite, $T\phi^0$ has spin ± 1 along z.) The other eigenvalues C_i, $i > 1$, have to be proved ≥ 0. I have verified that in the case of the most general $V(\phi)$ for rank-two tensors this is what happens. But I cannot prove it in general and must assume so. In my defense I would like to say that such an assumption will be mandatory later on anyway: When we modulate the asymptotic fields with functions such as $K(r)$, $q(r)$, etc. [Eqs. (3.8) and (3.9)], we only require them (as did 't Hooft for the isovector [1]) to extremize the energy functional. Neither the authors[5] who numerically solved for the functions associated with the 't Hooft-Polyakov monopole, nor those who found the solutions analytically in a certain limit,[6] nor anybody else has performed the extremely difficult task of finding second variations to verify that these functions indeed minimize the energy. It will assume, as is generally done, with an optimism that is enforced, that extrema and minima are synonymous.

V. SOLUTIONS IN TWO SPACE DIMENSIONS

Spatial infinity is now a circle S^1_∞ and we are concerned with the first homotopy group $\pi_1(G/H)$. Unlike in three dimensions, the existence of topologically stable solutions in two dimensions does not require a nontrivial unbroken subgroup H, since $\pi_1(G/H)$ may be nontrivial even if H is. However, we shall consider first the case $H = \mathrm{U}(1)$, since it admits interesting solutions in three dimensions. Subsequently we will only briefly mention the case where the vacuum completely breaks G.

If $H = \mathrm{U}(1)$, the orbit surface is S^2 or $\mathcal{S}^2$ for n odd or even. Clearly $S^2_\infty \to S^2$ is trivial; all loops on a sphere can be shrunk to a point. Thus there are no stable solutions for odd-rank tensors. On the other hand, any wiggly line, from say the north pole to the south on $\mathcal{S}^2$, is a loop [Fig. 2(a)] and a stable one at that. Among these gauge-equivalent loops we choose one that is a great semicircle, which we can arbitrarily choose to lie in the y-z plane [Fig. 2(b)]. The isovector at spatial angle θ on S^1_∞ is given by $\vec{\phi}(\theta) = a(0, \sin\frac{1}{2}\theta, \cos\frac{1}{2}\theta)$ and the rank $n = 2m$ tensor is given by

$$\phi_{ij\cdots n}(\theta) = b[\vec{\phi}_i(\theta)\vec{\phi}_j(\theta)\cdots\vec{\phi}_n(\theta) - \text{traces}], \quad (5.1)$$

where a and b are constants chosen to minimize $V(\phi)$.

What about the gauge fields common to all even

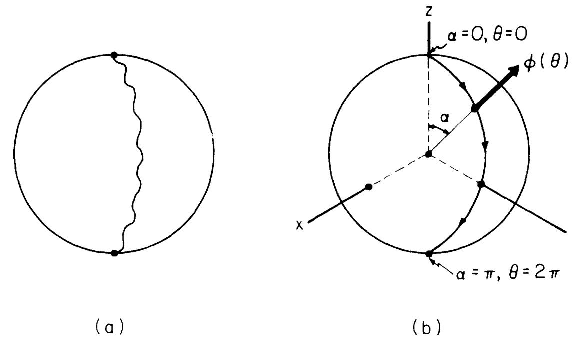

FIG. 2. (a) An arbitrary stable loop on $\mathcal{S}^2_\infty$. (b) Our choice for the stable loop on $\mathcal{S}^2_\infty$.

n? They must fulfill

$$D_\theta\vec{\phi}(\theta) = \left(\frac{1}{r}\frac{\partial}{\partial\theta} - e\vec{T}\cdot\vec{A}_\theta\right)\vec{\phi}(\theta) = 0 \text{ on } S^1_\infty .$$

[In the gauge $\vec{A}_r = 0$ which can always be chosen,[3]

$$D_r\vec{\phi} = \frac{\partial\vec{\phi}}{\partial r} \underset{r\to\infty}{=} 0$$

since

$$\vec{\phi}(r,\infty) \underset{r\to\infty}{\longrightarrow} \phi(\theta).$$

Now $(1/r)(\partial/\partial\theta)\vec{\phi}(\theta) = (1/2r)\vec{e}_\alpha|\vec{\phi}|$, where $\vec{e}_\alpha$ is a unit vector in the direction of increasing α [see Fig. 2(b)]. Thus $A^x_\theta = 1/2re$ will ensure that $D_\theta\vec{\phi} = 0$. The component of $\vec{A}_\theta$ in the y-z plane is arbitrary along $\vec{\phi}$ and is required to vanish in the direction perpendicular to it. We may choose $A^y_\theta = A^z_\theta = 0$. An electric potential $\vec{A}_0$ along $\vec{\phi}$ may be added since it satisfies $D_0\vec{\phi} \equiv 0$.

The continuation of these functions to finite distances is more complicated than in three dimensions and is therefore treated in the Appendix. The case $n = 2$ is once again treated in some detail for illustrative purposes. It is seen that ϕ is described by two functions of r while A^x_θ is described by one. The differential equations obeyed by these functions are derived and shown to admit, as $r\to\infty$, the asymptotic forms prescribed by topological considerations. Thus the theory admits static, finite-energy, topologically stable solutions for $n = 2$. Such a detailed study of higher n is not possible till a specific Lagrangian is assumed.

Since the loop we have considered is the only distinct element of the group $\pi_1(\mathcal{S})$ (up to homotopies), we have considered the most general solution to the case $H = \mathrm{U}(1)$. We consider lastly the case where the minima of $V(\phi)$ completely break G. The manifold G/H is all of SO(3), i.e., a solid sphere of radius π with antipodal points on the surface identified.[7] Once again a line connecting antipodal points is a stable loop. We cannot go any further with our analysis unless a specific Lagrangian is given.

VI. SUMMARY

We set out to find static, topologically stable solutions of finite energy to a Yang-Mills theory describing gauge fields A coupled to a Lorentz scalar ϕ transforming as a $(2n+1)$-dimensional irreducible representation of the gauge group $G=\mathrm{SO}(3)$. Homotopy theory told us that if G is broken down to a subgroup H by the vacuum, stable solutions exist in three (two) space dimensions if $\pi_2(G/H)$ $[\pi_1(G/H)]$ was nontrivial.

In three dimensions the theorem $\pi_2(G/H) = \ker \pi_1(H) \to \pi_1(G)$ told us that a nontrivial H is an essential prerequisite. For the group SO(3), $H=\mathrm{SO}(2)=\mathrm{U}(1)$ was the only possibility. If one represents the rank-n tensor as a traceless, symmetric outer product of n vectors, the U(1)-symmetric possibility corresponds to all these n vectors being parallel, their common direction defining the axis of U(1) rotations. Thus the only possible orbit surface is that generated by the underlying vector—a sphere S^2 for n odd and a sphere $\not{S}^2$ for n even.

Thus, given the 't Hooft-Polyakov isovector field on S^2_∞, the rank-n solutions were given by forming at each point the n-fold traceless outer product. The gauge fields that made the covariant derivative vanish were the same in all cases with a magnetic flux $4\pi/e$. It must be remembered that these tensor distributions were not just *a* possibility, but the *only possibility* for this class. We have of course solved the problem only on S^2_∞ and the behavior at finite r is decided for each n by energy considerations as illustrated in the $n=2$ case described in some detail. The isovector and gauge fields in other homotopy classes, in which S^2_∞ goes around $G/H=S^2$ or $\not{S}^2$ more than once, have not been written down by anyone so far. However, we do know that given this solution, the higher-rank tensors are once again obtained by the same recipe. The gauge fields in each homotopy class would be the same and would carry flux $4\pi m/e$, where m is the number of times S^2_∞ wraps around G/H.

In two dimensions if $H=\mathrm{U}(1)$ we saw that only one stable homotopy class existed and that too only for n even. The case $n=2$ was studied in some detail. It was seen that in two dimensions topological stability was not reflected in the form of a quantized nonzero magnetic flux. We also saw that even if H were trivial there would exist a nontrivial homotopy class.

In all cases discussed above, we presented an electric potential, common to all tensor distributions belonging to a given homotopy class, that could be introduced in the manner of Julia and Zee.

Since topological stability in three dimensions called for $H=\mathrm{U}(1)$ we asked if such U(1)-symmetric vacuums were generally admissible for all n. We found that a necessary (but not sufficient) condition was met. There always existed two nontrivial U(1)-symmetric stationary points. Furthermore, both these were minima with respect to variations within U(1)-symmetric subspace. However, it was not shown that either of these were minima in the other directions as well. Until this fact is demonstrated our analysis remains valid at the one-loophole level. You are invited either to find a homotopy that deforms this loophole to a point or to show that it is a nontrivial element of π_1.

ACKNOWLEDGMENTS

I was introduced to homotopy groups by Madhav Nori of the Institute for Advanced Study. It was then Sidney Coleman who introduced me to the fascinating role they play in gauge theories. I acknowledge with pleasure the debt I owe these gentlemen for their patient help and contagious enthusiasm. Conversations with Howard Georgi and Steve Park at Harvard were very useful. I am grateful to Sander Bais and Joel Primack of U. C. Santa Cruz for correcting a conceptual error in the original version of Sec. V and for motivating the discussion in the Appendix.

APPENDIX

In the main body of this paper, our emphasis was on the specification of the fields at spatial infinity, since the key to topological stability resided there. We will now elaborate on the determination of these fields at finite distances such that the field equations are satisfied.

Now we are not interested in the most general solutions to the Euler-Lagrange equations: We want the simplest solutions with the desired asymptotics dictated by topology. The modesty of our goal permits a simplification of the problem if we exploit once again the theorem first encountered in Sec. IV: If $V(\phi)$ is a function(al) such that $V(\phi) = V(G\phi)$ under a group of transformations G, the stationary points of V within a subspace ϕ^0 such that $\phi^0 = H\phi^0$, where H is a subgroup of G, are stationary points of the full ϕ space. The proof involves a straightforward application of Schur's lemma, as shown in the special case of Sec. IV.

We begin by noting that since the action $S=\int L\,dt$ is invariant under time translations, we may choose H to contain the translations. The only functions invariant under time translations are time-independent ones. Our theorem then tells us that by extremizing the action in the space of static functions, we obtain static solutions to the equations of motion. Furthermore, since the Lagrangian and Hamiltonians differ only by a sign in the absence of time-derivative terms, we may equally well extremize the energy.

We must next enlarge H judiciously, trying to simplify the forms of the functions without depriving them of the flexibility to assume the desired forms at infinity.

For the three-dimensional case we considered, the optimal subgroup is $H=P\exp[\vec{\theta}\cdot(\vec{J}+\vec{I})]$, where P is the parity operation and $R(\theta)=\exp[\vec{\theta}\cdot(\vec{J}+\vec{I})]$ induces equal rotations in space and isospace. The only invariant form for the scalar field is that assumed in the paper, i.e.,

$$\phi_{ijk\cdots}=\frac{(r_i r_j r_k\cdots-\text{traces})}{r^n}f(r)\,, \qquad \text{(A1)}$$

where the intrinsic parity is $(-1)^n$ for rank n. For the gauge field the only possibility is again $A_i^j=\epsilon_{ijk}r^k$. The following argument due to Steve Park should convince those who do not believe this claim. Since internal and external angles are constrained to be equal, treat isospin as spin and expand ϕ in tensor spherical harmonics Y_{JLS}^m. Since ϕ is invariant under $R(\theta)$, $J=m=0$. Since $S=n$, the only possibility is $L=n$, which corresponds precisely to our ansatz. Note also that the orbital parity of $(-1)^L$ is canceled by the intrinsic parity $(-1)^n$. For the same reason, A_i^j is a second-rank tensor and our ansatz is the only one with the desired parity.

In two dimensions, the group H contains P and $\exp[\theta\cdot(J+\frac{1}{2}I_x)]$, where $\mathrm{P}=P\exp(\pi I_z)$, and where P is the operation of reflections on the *spatial* x axis ($\theta\to-\theta$). The scalar fields are assigned even parity. Let us consider in some detail $n=2$. We have in general $\phi^{ij}, A_r^i, A_\theta^j$, $i,j=x,y,z$, which may be functions of r and θ. Let us see what invariance under H implies. If we choose $\theta=2\pi$ in $R(\theta)=\exp[\theta(J+\frac{1}{2}I_x)]$, we find $x\to x$, $y\to -y$, and $z\to -z$ in isospace. This eliminates $\phi^{xy}, \phi^{xz}, A_r^y, A_r^z, A_\theta^y, A_\theta^z$. Invariance under $R(\theta)$ for all θ demands that A_r^x, A_θ^x be functions only of r, while parity forces A_r^x to vanish. So we parametrize $A_\theta^x=(1/2re)[1+L(r)]$. Consider next $\phi^{xx}, \phi^{yz}, \phi^{yy}, \phi^{zz}$. We trade the y and z indices for those of $e_\pm=(y\pm iz)/\sqrt{2}$, which have simple transformations under x rotations. Invariance under $R(\theta)$ requires

$$\begin{aligned}&\phi^{xx}(r,\theta)=m(r)\,,\\ &\phi^{++}(r,\epsilon)=\tfrac{3}{2}e^{i\theta}Q(r)\,,\\ &\phi^{--}(r,\theta)=\tfrac{3}{2}e^{-i\theta}Q^*(r)\,.\end{aligned} \qquad \text{(A2)}$$

The factor $\frac{3}{2}$ is there for convenience. The trace conditions give us $2\phi^{+-}+\phi^{xx}=0$. Under P, $\phi^{++}(r,\theta)\to\phi^{--}(r,-\theta)=\frac{3}{2}Q^*(r)e^{i\theta}$ which equals ϕ^{++} if $Q=Q^*$. Going back to Cartesian components,

$$\phi=\begin{pmatrix} m & 0 & 0\\ 0 & -\frac{1}{2}m+\frac{3}{2}Q\cos\theta & -\frac{3}{2}Q\sin\theta\\ 0 & -\frac{3}{2}Q\sin\theta & -\frac{1}{2}m-\frac{3}{2}Q\cos\theta\end{pmatrix}. \qquad \text{(A3)}$$

Our theorem tells us that if we extremize the energy calculated in terms of m, Q, and L, we obtain solutions to the equations of motion. We find

$$E=2\pi\int r\,dr\left\{\frac{1}{8r^2e}\left(\frac{dL}{dr}\right)^2+\frac{9Q^2L^2}{8r^2}+\frac{1}{8}\left[9\left(\frac{dQ}{dr}\right)^2+3\left(\frac{dm}{dr}\right)^2\right]-\tfrac{1}{4}\mu^2(9Q^2+3m^2)+\tfrac{1}{4}m\gamma(9Q^2-m^2)+\tfrac{1}{16}\lambda(9Q^2+3m^2)^2\right\}. \qquad \text{(A4)}$$

The functions which extremize E obey

$$\frac{d}{dr}\left(\frac{1}{4re^2}\frac{dL}{dr}\right)=\frac{9Q^2L}{4r}\,, \qquad \text{(A5a)}$$

$$\frac{d}{dr}\left(\frac{9r}{4}\frac{dQ}{dr}\right)=\frac{9L^2Q}{4r}-\frac{9\mu^2Qr}{2}+\frac{9mQr\gamma}{2}+\tfrac{1}{8}\lambda(9Q^2+3m^2)18Qr\,, \qquad \text{(A5b)}$$

$$\frac{d}{dr}\left(\frac{3r}{4}\frac{dm}{dr}\right)=-\frac{3\mu^2mr}{2}-\frac{3m^2\gamma r}{4}+\frac{9Q^2\gamma r}{4}+\tfrac{1}{8}\lambda r(9Q^2+3m^2)6m\,. \qquad \text{(A5c)}$$

As $r\to\infty$, these equations admit $L\to 0$, $Q=m=a=$ $[\gamma\pm(\gamma^2+24\lambda\mu^2)^{1/2}]/12\lambda$, i.e., *the fields approach the topologically stable forms prescribed in Sec. V.* Presumably such an analysis can be carried out for higher n, given the Lagrangian.

Let us note incidentally that as $r\to 0$, we find $L\to -1+ar^2$, $Q\to br$, $m\to 2a$. Since m does not depend on θ, it need not vanish at the origin.

*Work supported in part by the National Science Foundation under Grant No. MPS75-20427.

†Work supported by Harvard Society of Fellows.

[1]G. 't Hooft, Nucl. Phys. B79, 276 (1974).

[2]A. M. Polyakov, Zh. Eksp. Teor. Fiz. Pis'ma Red. 20, 430 (1974) [JETP Lett. 20, 194 (1974)].

[3]Sidney Coleman, Lectures at the 1975 International School of Subnuclear Physics, "Ettore Majorana," Harvard University report, 1975 (unpublished). Upon reading a preliminary version of this paper, Professor Coleman drew my attention to the fact that apart from his article on the use of topology in the study of gauge theories (from which I learned the ideas I use here) there existed other expositions: See D. Finkelstein, J. Math. Phys. 7, 1218 (1966); J. Arafune, P. Freund, and C. Goebel, *ibid.* 16, 433 (1975); Yu. S. Tyupkin, V. A. Fateev, and A. S. Shvarts, Zh. Eksp. Teor. Fiz. Pis'ma Red. 21, 91 (1975) [JETP Lett. 21, 42 (1975)]; M. L. Monastrykti and A. M. Perelemov, *ibid.* 21, 94 (1975) [21, 43 (1975)]; A. Patrascioiu, Phys. Rev. D 11, 523 (1975).

[4]H. Georgi and S. L. Glashow, Phys. Rev. D 6, 2977 (1972).

[5]B. Julia and A. Zee, Phys. Rev. D 11, 2227 (1975).

[6]M. K. Prasad and C. S. Sommerfield, Phys. Rev. Lett. 35, 760 (1975).

[7]It is now clear why $H = \mathrm{U}(1)$ was needed for a nontrivial $\pi_2(G/\mathrm{H})$. If H = identity, $G/H = G$ = all of SO(3) which is too large a manifold: All maps of S^2_∞ onto a solid sphere of radius π are collapsible into a point. If H grows to U(1), G/H shrinks to S^2 or $\not{S}^2$, which is just right for S^2_∞. If H grows further [the only choice is $H = \mathrm{SO}(3)$ and this happens for the isoscalar], G/H collapses to a point and all maps are trivial.