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Tests of SU(3) and quark-line rules in $e\bar{e}\to 3P$, PPV reactions and ψ decay*

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It has been shown that reactions $e\bar{e}\to 3P$ and PPV are excellent means for testing SU(3) sum rules as well as the quark-line rule. We can derive various relations involving the cross sections for these reactions if the quark-line rule is valid. The connection with various decay modes of the ψ meson is emphasized. We can also test the absence of the SU(3)-singlet component in the electromagnetic interaction.

I. MOTIVATION

The recent discoveries[1] of ψ and ψ' stimulated renewed interest in the quark-line rule[2] and the related nonet ansatz,[3] since narrow widths of these bosons may be qualitatively understood by means of these rules[4] just as in the case of the ordinary ϕ meson.[5] However, a relatively large experimental decay ratio[6] of

$$\Gamma(\psi\to\phi\pi^+\pi^-)/\Gamma(\psi\to\omega\pi^+\pi^-)=0.20\pm0.10 \qquad (1.1)$$

necessitates either a modification[7] of the rules or a mechanism for creating a large violation of the rules[8-10] for this particular decay mode without disturbing various other successful explanations. The second possibility has been discussed by several authors[8-10] on the basis of various dynamical considerations. In previous papers,[7] I suggested the first alternative of modifying the nonet rule (and hence the quark-line rule) by requiring an additional self-consistency postulate which is relevant essentially only for the decay $\psi\to VPP$. Especially, this modification predicts a unique ratio of

$$\sigma(e\bar{e}\to\phi\pi^+\pi^-)/\sigma(e\bar{e}\to\omega\pi^+\pi^-)=2 \qquad (1.2)$$

and other relations of similar nature (see Sec. III), if we neglect the mass difference between ω and ϕ mesons. Note that the usual quark-line rule demands this same ratio to be zero rather than 2 as in (1.2). Therefore, we may easily distinguish the two alternatives by experimentally measuring the ratio.

Up to now only a few experimental tests of the quark-line rule and the nonet ansatz have been investigated[11,12] for hadronic processes involving ϕ, f, and f' mesons. Also, Cheng[13] noted recently that the quark-line rule enables us to compute the so-called π-N σ term in an agreement with the value obtained by means of dispersion-theoretical methods. However, it may be pointed out that these really test the weaker nonet ansatz rather than the stronger full quark-line rule. One purpose of this note is to show that the reactions

$$e\bar{e}\to VP, PPP, VPP \qquad (1.3)$$

can be used as an excellent means to systematically test the validity of the full quark-line rule as well as the old and new nonet hypotheses. Here, V and P refer to the vector nonet and the pseudoscalar nonet (or octet), respectively. Also, these reactions may be used to test the SU(3) symmetry in a systematical way in the electromagnetic processes. Especially, experimental study of the ratio

$$\sigma(e\bar{e}\to\pi^+\rho^-):\sigma(e\bar{e}\to\pi^0\omega):\sigma(e\bar{e}\to K^0\bar{K}^{0*}) :\sigma(e\bar{e}\to\eta\phi):\sigma(e\bar{e}\to K^+K^{-*}) \qquad (1.4)$$

would lead us to a better understanding[7] of the corresponding decay ratio of

$$\Gamma(\rho^-\to\pi^-\gamma):\Gamma(\omega\to\pi^0\gamma):\Gamma(K^{0*}\to K^0\gamma) :\Gamma(\phi\to\eta\gamma):\Gamma(K^{+*}\to K^+\gamma). \qquad (1.5)$$

The experimental deviation of this decay ratio from the values calculated on the basis of SU(3) and the nonet hypothesis (or the simple quark model) is considerable,[14] although the discrepancy may not be[15,16] as severe as it seems. Also, the U-spin relation

$$\sigma(e\bar{e}\to K^+ + \text{anything})=\sigma(e\bar{e}\to\pi^+ + \text{anything}) \qquad (1.6)$$

does not appear[17] to be well satisfied even for the high-energy kaon. However, this may be due to some kinematical reasons, as we shall discuss in Sec. IV. In view of these facts, systematical experimental checks of SU(3) relations in the electromagnetic process (1.3) would be desirable, since the SU(3) symmetry is known to be pretty well satisfied in other hadronic processes.[11,18]

Another by-product of the present investigation is the fact that many of our relations are also applicable to decay widths of ψ and ψ', if the nonelectromagnetic interaction involved in these decays is an SU(3) singlet. This will be demonstrated shortly below.

For our purpose it is convenient, though not essential, to assume the standard SU(3) quark model in which all SU(3) hadrons are bound states of three quarks q_1, q_2, and q_3. We will not consider the color degree of freedom unless it is otherwise so stated. We shall call these SU(3) hadrons normal hadrons, in contrast to possible charmed hadrons such as ψ and ψ' which are presumed to contain new quarks as their constituents. In this note, we restrict ourselves to discussions of normal hadronic reactions in the lowest order of the electromagnetic coupling constant (i.e., the one-photon-exchange mechanism). We can then replace the initial $e\bar{e}$ system effectively by the hadronic part of the electromagnetic current, which will have the form

$$j_\mu^{\text{em}}(x) = j_\mu^{(3)}(x) + \frac{1}{\sqrt{3}} j_\mu^{(8)}(x) + \theta J_\mu^{(0)}(x) . \tag{1.7}$$

Here, $j_\mu^{(\alpha)}(x)$ $(\alpha = 1, 2, \ldots, 8)$ is the usual octet vector current

$$j_\mu^{(\alpha)}(x) = \tfrac{1}{2} i \bar{q}(x) \gamma_\mu \lambda_\alpha q(x) , \tag{1.8}$$

while $J_\mu^{(0)}(x)$ is an SU(3)-singlet current consisting of only normal quarks q_1, q_2, and q_3, and the multiplicative constant θ has been introduced for convenience. Theoretically, the presence (or absence) as well as the explicit form of the singlet $J_\mu^{(0)}(x)$ depends upon the choice of the specific quark model. For the fractionally charged Gell-Mann–Zweig model the electromagnetic current does not contain the SU(3)-singlet component, so that we can set $\theta = 0$. However, for the integrally charged Han-Nambu quark model, $\theta J_\mu^{(0)}(x)$ is nonzero, representing the color current of the quarks. Experimentally, we see no evidence of any unitary-singlet component in $j_\mu^{\text{em}}(x)$, either in the low-energy phenomena[19] or in the high-energy reactions.[20] Therefore, in practice, we could set $\theta = 0$. However, we keep the singlet term for the following two reasons. First, the special choice $\theta = 0$ will give rise to extra relations which could be experimentally tested.[21] Secondly, if we keep the singlet term, then the resulting SU(3) relations are immediately applicable to the corresponding SU(3) relations among the decay widths of the ψ meson, provided that the effective decay interactions responsible for the normal hadronic decays of the ψ is given by[7]

$$H_\psi(x) = [f j_\mu^{\text{em}}(x) + g \tilde{J}_\mu^{(0)}(x)] \psi_\mu(x) . \tag{1.9}$$

Here $\psi_\mu(x)$ is the field operator of the ψ meson, and the first term proportional to f is due to the virtual electromagnetic interaction, while the second term, which is assumed[7,6] to be an SU(3) singlet, is of nonelectromagnetic origin. Note that the new SU(3)-singlet current $\tilde{J}_\mu^{(0)}(x)$ appearing in (1.9) need not be the same as $J_\mu^{(0)}(x)$ of (1.7). All SU(3) relations for $\sigma(e\bar{e} \to X)$'s will then reproduce the corresponding relations for $\Gamma(\psi \to X)$'s, if we replace all symbols $e\bar{e}$ and σ by ψ and Γ, respectively.

II. $e\bar{e} \to PPP$ AND VP

Here let us consider the reaction

$$e\bar{e} \to P_1 P_2 P_3 , \tag{2.1}$$

where P_1, P_2, and P_3 refer to pseudoscalar octet mesons with four-momenta k_1, k_2, and k_3, respectively. The matrix element for the reaction (2.1) is proportional to

$$M = \langle P_1(k_1) P_2(k_2) P_3(k_3) | j_\mu^{\text{em}}(0) | 0 \rangle , \tag{2.2}$$

where $j_\mu^{\text{em}}(x)$ is given by (1.7). Suppressing all Lorentz indices for simplicity, the SU(3) symmetry together with the charge-conjugation invariance demands that M can be expressed as

$$\begin{aligned} M = {} & \alpha \, \mathrm{Tr}(QP_3P_1P_2 - QP_2P_1P_3) \\ & + \beta \, \mathrm{Tr}(QP_1P_2P_3 - QP_3P_2P_1) \\ & + \gamma \, \mathrm{Tr}(QP_2P_3P_1 - QP_1P_3P_2) \\ & + \tfrac{1}{3} \theta \delta \, \mathrm{Tr}(P_1P_2P_3 - P_3P_2P_1) . \end{aligned} \tag{2.3}$$

Here, Q stands for the charge spurion matrix

$$Q = \lambda_3 + \frac{1}{\sqrt{3}} \lambda_8 \tag{2.4}$$

and P_j $(j = 1, 2, 3)$ refer now to 3×3 matrices representing the jth pseudoscalar octet. In Eq. (2.3) the first three terms result from the octet components in $j_\mu^{\text{em}}(x)$, while the last term proportional to θ is due to the SU(3)-singlet current $\theta J_\mu^{(0)}(x)$ in (1.7). We note that if the interaction Hamiltonian (1.9) is responsible for $\psi \to P_1P_2P_3$ decay, then its matrix element is also written in the form of (2.3) with different numerical coefficients: α', β', γ', and δ'. Therefore, we can discuss both reactions $e\bar{e} \to P_1P_2P_3$ and $\psi \to P_1P_2P_3$ simultaneously, as long as we keep θ nonzero.

The numerical coefficients α, β, γ, and δ are actually functions of three momenta k_1, k_2, and k_3. The Bose statistics for mesons demands

$$\begin{aligned} \alpha(k_1, k_2, k_3) &= -\alpha(k_1, k_3, k_2) , \\ \beta(k_1, k_2, k_3) &= \alpha(k_2, k_3, k_1) , \\ \gamma(k_1, k_2, k_3) &= \alpha(k_3, k_1, k_2) , \end{aligned} \tag{2.5}$$

as well as

$$\begin{aligned} \delta(k_1, k_2, k_3) &= -\delta(k_1, k_3, k_2) \\ &= -\delta(k_2, k_1, k_3) . \end{aligned} \tag{2.6}$$

Note that δ is completely antisymmetric with respect to interchanges of k_1, k_2, and k_3.

From (2.3), it is easy to compute that

$$M(e\bar{e}\to\pi^+\pi^-\pi^0)=(\alpha+\beta+\gamma)+\theta\delta\,, \tag{2.7a}$$

$$M(e\bar{e}\to\pi^+\pi^-\eta)=\sqrt{3}\,(\alpha+\beta-\gamma)\,, \tag{2.7b}$$

$$M(e\bar{e}\to\pi^+K^-K^0)=\sqrt{2}\,(\alpha+\beta-2\gamma)-(1/\sqrt{2}\,)\theta\delta\,, \tag{2.7c}$$

$$M(e\bar{e}\to K^+K^-\pi^0)=(2\alpha+2\beta-\gamma)+\tfrac{1}{2}\theta\delta\,, \tag{2.7d}$$

$$M(e\bar{e}\to K^0\bar{K}^0\pi^0)=(\alpha+\beta+\gamma)-\tfrac{1}{2}\theta\delta\,, \tag{2.7e}$$

$$M(e\bar{e}\to K^+K^-\eta)=\sqrt{3}\gamma+\tfrac{1}{2}\sqrt{3}\,\theta\delta\,, \tag{2.7f}$$

$$M(e\bar{e}\to K^0\bar{K}^0\eta)=-\sqrt{3}\,(\alpha+\beta+\gamma)+\tfrac{1}{2}\sqrt{3}\,\theta\delta \tag{2.7g}$$

for decay matrix elements of $M(e\bar{e}\to P_1P_2P_3)$, if we neglect the common multiplicative constant. We should note that in general we have

$$M(e\bar{e}\to PP'P'')\neq M(e\bar{e}\to P'PP'')\neq M(e\bar{e}\to P''PP')\,. \tag{2.8}$$

For example, we find

$$M(e\bar{e}\to K^-K^0\pi^+)=\sqrt{2}\,(\gamma+\alpha-2\beta)-\frac{1}{\sqrt{2}}\theta\delta\,, \tag{2.8$'$}$$

which differs from $M(e\bar{e}\to\pi^+K^-K^0)$, although the former can be obtained from the latter by the cyclic interchange $\alpha\to\gamma\to\beta\to\alpha$ in view of Eqs. (2.5) and (2.6).

Eliminating four unknown parameters α, β, γ, and $\theta\delta$ from (2.7), we find the following SU(3) relations:

$$\sqrt{3}\,M(e\bar{e}\to\pi^+\pi^-\eta)=\sqrt{2}\,M(e\bar{e}\to\pi^+K^-K^0)+M(e\bar{e}\to\pi^+\pi^-\pi^0)\,, \tag{2.9a}$$

$$\sqrt{3}\,M(e\bar{e}\to\pi^+\pi^-\eta)=2M(e\bar{e}\to K^+K^-\pi^0)-M(e\bar{e}\to\pi^+\pi^-\pi^0)\,, \tag{2.9b}$$

$$\sqrt{3}\,M(e\bar{e}\to K^+K^-\eta)=2M(e\bar{e}\to\pi^+\pi^-\pi^0)-M(e\bar{e}\to K^+K^-\pi^0)\,, \tag{2.9c}$$

$$\sqrt{3}\,M(e\bar{e}\to K^0\bar{K}^0\pi^0)=-M(e\bar{e}\to K^0\bar{K}^0\eta)\,. \tag{2.9d}$$

We may also directly derive these relations from the U-spin consideration. The same formulas hold for $M(\psi\to P_1P_2P_3)$, if the decay interaction Hamiltonian (1.9) is assumed. Neglecting mass differences among SU(3) multiplets for the sake of the simplicity, Eqs. (2.9) give the following SU(3) sum rules among the cross sections:

$$\sigma(e\bar{e}\to K_LK_S\eta)=3\sigma(e\bar{e}\to K_LK_S\pi^0)\,, \tag{2.10a}$$

$$\sigma(e\bar{e}\to\pi^+\pi^-\pi^0)+\sigma(e\bar{e}\to\pi^+\pi^-\eta)=\sigma(e\bar{e}\to K^+K^-\pi^0)+\sigma(e\bar{e}\to K^+K^-\eta)\,, \tag{2.10b}$$

$$\sigma(e\bar{e}\to\pi^+K^-K^0)+2\sigma(e\bar{e}\to K^+K^-\pi^0)=\sigma(e\bar{e}\to\pi^+\pi^-\pi^0)+3\sigma(e\bar{e}\to\pi^+\pi^-\eta)\,. \tag{2.10c}$$

In addition to these identities, Eqs. (2.9) give various triangular inequalities such as

$$[\sigma(e\bar{e}\to\pi^+\pi^-\pi^0)]^{1/2}\le[3\sigma(e\bar{e}\to\pi^+\pi^-\eta)]^{1/2}+2[\sigma(e\bar{e}\to K^+K^-\pi^0)]^{1/2}\,. \tag{2.11}$$

These relations (2.10) and (2.11) are valid for the ψ decay width, if we replace symbols $e\bar{e}$ and σ by ψ and Γ, respectively.

Next let us assume that $\theta=0$, i.e., that the electromagnetic interaction has no SU(3)-singlet component. Then, we find an additional relation,

$$M(e\bar{e}\to K^0\bar{K}^0\pi^0)=M(e\bar{e}\to\pi^+\pi^-\pi^0)\,, \tag{2.12}$$

which leads to

$$\sigma(e\bar{e}\to K_LK_S\pi^0)=\sigma(e\bar{e}\to\pi^+\pi^-\pi^0)\,. \tag{2.13}$$

This relation cannot be obtained by a simple U-spin consideration.[21] Because of the special ansatz $\theta=0$, this identity has no counterpart for the decay of the ψ meson.

Up to this point, all relations derived above are valid for total as well as differential cross sections. However, if we are only interested in the total integrated cross section, then we have to integrate over final-state phase space Ω involving three momenta k_1, k_2, and k_3. Because of the Bose symmetry condition (2.5), this leads to

$$\int d\Omega\,|\alpha|^2=\int d\Omega\,|\beta|^2=\int d\Omega\,|\gamma|^2\,, \tag{2.14a}$$

$$\int d\Omega\,\alpha^*\beta=\int d\Omega\,\beta^*\alpha=\int d\Omega\,\gamma^*\alpha\,. \tag{2.14b}$$

Moreover, if we assume $\theta=0$, then (2.7) together with (2.14) can give one more relation, so that we have the following five relations for the total integrated cross sections $\sigma_T(e\bar{e}\to P_1P_2P_3)$:

$$\sigma_T(e\bar{e}\to K_LK_S\eta)=3\sigma_T(e\bar{e}\to K_LK_S\pi^0)\,, \tag{2.15a}$$

$$3\sigma_T(e\bar{e}\to K^+K^-\eta)=\sigma_T(e\bar{e}\to K^+K^-\pi^0)\,, \tag{2.15b}$$

$$\sigma_T(e\bar{e}\to K_LK_S\pi^0)=\sigma_T(e\bar{e}\to\pi^+\pi^-\pi^0)\,, \tag{2.15c}$$

$$4\sigma_T(e\bar{e}\to K^+K^-\pi^0)=3\sigma_T(e\bar{e}\to\pi^+\pi^-\pi^0)+3\sigma_T(e\bar{e}\to\pi^+\pi^-\eta)\,, \tag{2.15d}$$

$$2\sigma_T(e\bar{e}\to K^+K^-\pi^0)=2\sigma_T(e\bar{e}\to\pi^+\pi^-\pi^0)+\sigma_T(e\bar{e}\to\pi^+K^-K^0)\,, \tag{2.15e}$$

where we have set $\sigma_T=\int d\Omega\sigma$.

We may remark that if the $3P$-γ vertex operator can be expressed by means of a *local* effective interaction

$$H_{\mathrm{eff}}(x)=G_1\epsilon_{\mu\nu\alpha\beta}A_\mu(x)\,\mathrm{Tr}[\partial_\nu P(x)\partial_\alpha P(x)\partial_\beta P(x)Q]+\theta G_2\epsilon_{\mu\nu\alpha\beta}A_\mu(x)\,\mathrm{Tr}[\partial_\nu P(x)\partial_\alpha P(x)\partial_\beta P(x)]\,, \tag{2.16}$$

then we have $\alpha=\beta=\gamma$, so that we can derive more relations. However, since the final-state interaction among mesons is expected to be considerable, the assumption of the local Hamiltonian will be a poor approximation. The explicit form of Eq. (2.16) with $\theta=0$ in terms of π, K, $\bar{K}$, and η mesons can be found elsewhere.[22]

Returning to the original problem, we note the following. As far as the $e\bar{e}\to P_1P_2P_3$ reaction is concerned, the quark-line rule is automatically satisfied as the consequence of the SU(3) symmetry and the charge-conjugation invariance, since the disconnected quark-line diagrams corresponding to such terms as $\mathrm{Tr}(QP_1)\,\mathrm{Tr}(P_2P_3)$ are forbidden by charge conjugation. However, if we assume P_j ($j=1,2,3$) to represent now the pseudoscalar nonets rather than the pure octets, this is no longer the case. Indeed, a disconnected quark-line term

$$(\mathrm{Tr}P_1)\,\mathrm{Tr}(QP_2P_3-QP_3P_2) \tag{2.17}$$

is consistent with both SU(3) and charge conjugation. If the quark-line rule or the nonet ansatz is applicable[23] even for the pseudoscalar nonet, then the presence of terms such as (2.17) is not allowed, so that we can still use (2.3) for this instance. In this way, we find

$$M(e\bar{e}\to K^+K^-\eta')=M(e\bar{e}\to\pi^+\pi^-\eta')\,, \tag{2.18a}$$

$$M(e\bar{e}\to\pi^+\pi^-\eta')=\sqrt{2}\,M(e\bar{e}\to\pi^+\pi^-\eta)\,, \tag{2.18b}$$

$$M(e\bar{e}\to K^0\bar{K}^0\eta')=0\,. \tag{2.18c}$$

In this derivation, we neglected the small η-η' mixing so that the η' meson is a pure SU(3) singlet. Actually, (2.18a) is a simple result of SU(3) symmetry, and only (2.18b) and (2.18c) represent the consequences of the nonet hypothesis. Especially, they imply the validity of

$$\sigma(e\bar{e}\to\pi^+\pi^-\eta')=2\sigma(e\bar{e}\to\pi^+\pi^-\eta)\,, \tag{2.19a}$$

$$\sigma(e\bar{e}\to K_LK_S\eta')=0\,. \tag{2.19b}$$

Since we did not assume $\theta=0$, these relations should also be valid for the corresponding decay rates for the ψ meson, if the nonet hypothesis is good for the pseudoscalar meson.

Next, let us investigate the reaction

$$e\bar{e}\to VP\,. \tag{2.20}$$

The most general SU(3)-invariant matrix element consistent with the charge conjugation is now given by

$$\begin{aligned}M(e\bar{e}\to VP)=\alpha'[&\mathrm{Tr}(QVP)+\mathrm{Tr}(QPV)]\\&+\beta'(\mathrm{Tr}V)\,\mathrm{Tr}(QP)+\gamma'(\mathrm{Tr}P)\,\mathrm{Tr}(QV)\\&+\delta'\,\theta\,\mathrm{Tr}(VP)\,,\end{aligned} \tag{2.21}$$

where α', β', γ', and δ' are new sets of constants. On the basis of (2.21), we can derive many relations, for example

$$\begin{aligned}\sigma(e\bar{e}\to\rho^+\pi^-)&=\sigma(e\bar{e}\to\rho^0\pi^0)\\&=\sigma(e\bar{e}\to\rho^-\pi^+)\,,\end{aligned} \tag{2.22a}$$

$$\sigma(e\bar{e}\to K^{+*}K^-)=\sigma(e\bar{e}\to\rho^+\pi^-)\,. \tag{2.22b}$$

Actually, the validity of (2.22a) depends upon only the weaker SU(2) invariance. If we assume $\theta=0$ in addition, then we find extra relations

$$\sigma(e\bar{e}\to K^{0*}\bar{K}^0)=4\sigma(e\bar{e}\to K^{+*}K^-)\,, \tag{2.23a}$$

$$\sigma(e\bar{e}\to\rho^0\eta)=3\sigma(e\bar{e}\to\rho^0\pi^0)\,. \tag{2.23b}$$

Moreover, when we impose the quark-line rule or the nonet ansatz for both vector and pseudoscalar mesons, then we can ignore all terms in (2.21) other than the first term (i.e., we set $\theta=\beta'=\gamma'=0$). This gives

$$\begin{aligned}\sigma(e\bar{e}\to\omega\pi^0)&=9\sigma(e\bar{e}\to\rho^0\pi^0)\\&=27\sigma(e\bar{e}\to\omega\eta)\,,\end{aligned} \tag{2.24a}$$

$$\sigma(e\bar{e}\to\omega\eta')=2\sigma(e\bar{e}\to\omega\eta)\,, \tag{2.24b}$$

$$\begin{aligned}6\sigma(e\bar{e}\to\phi\eta')&=3\sigma(e\bar{e}\to\phi\eta)\\&=8\sigma(e\bar{e}\to\rho^0\pi^0)\,,\end{aligned} \tag{2.24c}$$

$$\sigma(e\bar{e}\to\phi\pi^0)=0\,, \tag{2.24d}$$

$$\begin{aligned}\sigma(e\bar{e}\to\rho^0\eta')&=2\sigma(e\bar{e}\to\rho^0\eta)\\&=6\sigma(e\bar{e}\to\rho^0\pi^0)\,,\end{aligned} \tag{2.24e}$$

where we assumed ideal ω-ϕ mixing[3] and neglected the small mixing between η and η'. The relations (2.24) are intimately related to the corresponding identities among $\Gamma(V\to P\gamma)$. However, the decay widths for $\Gamma(V\to P\gamma)$ have been discussed by many authors[3,15,16] and we will not go into detail.

Finally, in ending this section, we will briefly comment on the reactions

$$e\bar{e}\to PP\,,\quad e\bar{e}\to VV\,, \tag{2.25a}$$

$$e\bar{e}\to B\bar{B}\,. \tag{2.25b}$$

It is well known[24] that in the exact-SU(3) limit we have

$$M(e\bar{e}\to K^+K^-)=M(e\bar{e}\to\pi^+\pi^-)\,, \tag{2.26a}$$

$$M(e\bar{e}\to K^0\bar{K}^0)=0\,, \tag{2.26b}$$

and the corresponding relations for $K\to K^*$ and $\pi\to\rho$. Experimentally, the relation (2.26a) appears to be well satisfied.[25] A slight improvement of (2.26) is possible, if we assume the vector-dominance model. In that case, we obtain only one relation,

$$M(e\bar{e}\to K^+K^-)-M(e\bar{e}\to K^0\bar{K}^0)=M(e\bar{e}\to\pi^+\pi^-)\,, \tag{2.27}$$

as has been noted by Gourdin.[25] Also, we mention that we can correlate[26] $\sigma(e\bar{e}\to VP)$ and $\sigma(e\bar{e}\to PP)$ with each other if we assume SU(6) invariance. With respect to $e\bar{e}\to B\bar{B}$, we simply mention[7] here that

$$\sigma(e\bar{e}\to p\bar{p})=\sigma(e\bar{e}\to \Sigma^+\bar{\Sigma}^+). \tag{2.28}$$

If we assume $\theta=0$, then we should have[21]

$$4\sigma(e\bar{e}\to \Lambda\bar{\Lambda})=\sigma(e\bar{e}\to n\bar{n}). \tag{2.29}$$

Returning to Eq. (2.27), we can prove that (2.27) can be derived if the combined effect of both electromagnetic and SU(3)-violating interactions has negligible contributions from decouplets and the 27-plet components with a large octet enhancement.

III. $e\bar{e}\to VPP$

Here we shall consider a reaction

$$e\bar{e}\to VP_1P_2, \tag{3.1}$$

where V is a vector nonet, while P_1P_2 refer now to two pseudoscalar octets (*not* nonets) with four-momenta k_1 and k_2, respectively. Hereafter, we assume the ideal ω-ϕ mixing[3] and neglect small η-η' mixing for simplicity.

The most general SU(3)-invariant matrix element consistent with the charge-conjugation invariance is the linear combination of the following nine terms:

$$S_1=\mathrm{Tr}(jVP_1P_2)+\mathrm{Tr}(jVP_2P_1)+\mathrm{Tr}(jP_1P_2V)+\mathrm{Tr}(jP_2P_1V)+\mathrm{Tr}(jP_1VP_2)+\mathrm{Tr}(jP_2VP_1), \tag{3.2a}$$

$$S_2=\mathrm{Tr}(jVP_1P_2)+\mathrm{Tr}(jVP_2P_1)+\mathrm{Tr}(jP_1P_2V)+\mathrm{Tr}(jP_2P_1V)-2\mathrm{Tr}(jP_1VP_2)-2\mathrm{Tr}(jP_2VP_1), \tag{3.2b}$$

$$S_3=2\mathrm{Tr}(jV)\mathrm{Tr}(P_1P_2)-\mathrm{Tr}(jP_1)\mathrm{Tr}(VP_2)-\mathrm{Tr}(jP_2)\mathrm{Tr}(VP_1), \tag{3.2c}$$

$$S_4=\mathrm{Tr}(jV)\mathrm{Tr}(P_1P_2)+\mathrm{Tr}(jP_1)\mathrm{Tr}(VP_2)+\mathrm{Tr}(jP_2)\mathrm{Tr}(VP_1), \tag{3.2d}$$

$$S_5=(\mathrm{Tr}V)[\mathrm{Tr}(jP_1P_2)+\mathrm{Tr}(jP_2P_1)], \tag{3.2e}$$

$$S_6=(\mathrm{Tr}j)[\mathrm{Tr}(VP_1P_2)+\mathrm{Tr}(VP_2P_1)], \tag{3.2f}$$

$$S_7=(\mathrm{Tr}j)(\mathrm{Tr}V)\mathrm{Tr}(P_1P_2), \tag{3.2g}$$

$$A_1=\mathrm{Tr}(jVP_1P_2)+\mathrm{Tr}(jP_2P_1V)-\mathrm{Tr}(jVP_2P_1)-\mathrm{Tr}(jP_1P_2V), \tag{3.2h}$$

$$A_2=\mathrm{Tr}(jP_1)\mathrm{Tr}(VP_2)-\mathrm{Tr}(jP_2)\mathrm{Tr}(VP_1). \tag{3.2i}$$

Here, j is the 3×3 spurion matrix corresponding to the electromagnetic current (1.7) and has a form of

$$j=Q+\theta\delta\lambda_0,\quad \lambda_0=(\tfrac{2}{3})^{1/2}\underline{1}$$
$$Q=\lambda_3+(1/\sqrt{3})\lambda_8 \tag{3.3}$$

where δ is an arbitrary constant, representing the unknown matrix element of the SU(3)-singlet current $J_\mu^{(0)}(x)$. Again, if the electromagnetic current does not contain any SU(3)-singlet term, then we can set $\theta=0$ and $j=Q$. If we wish, we can separate out the contribution from the singlet current, and the resulting expression is given in the Appendix.

In Eqs. (3.2), all S_α $(\alpha=1,2,3,\ldots,7)$ are symmetric with respect to interchange of P_1 and P_2, while A_1 and A_2 are antisymmetric for $P_1\leftrightarrow P_2$. Therefore, Bose symmetry demands that we have to multiply symmetric wave functions of momenta k_1 and k_2 to all S_α $(\alpha=1,\ldots,7)$ and antisymmetric wave functions to A_1 and A_2. Also, we remark that the specific choice for the special combinations given in (3.2) has been motivated by the following consideration[27]: If we consider a permutation group Σ_3 which consists of six permutation operations among three objects, V, P_1, and P_2, then S_1 and S_4 are invariant, i.e., they correspond to singlet representations of Σ_3. However, two pairs, (S_2,A_1) and (S_3,A_2), belong to two-dimensional representations of the group Σ_3. We need not classify S_5, S_6, and S_7 since the assumption $\mathrm{Tr}P_1=\mathrm{Tr}P_2=0$ but $\mathrm{Tr}V\neq 0$ precludes a symmetrical treatment of these terms by means of the group Σ_3. Of course, if we wish, it is possible to do so for these terms by adding terms proportional to $\mathrm{Tr}P_1$ and $\mathrm{Tr}P_2$, as we have done in the previous paper.[7] We remark that S_2, A_1, A_2 are independent of the parameter θ.

As we noted elsewhere,[7,27] we have an SU(3) identity equation

$$S_1=S_4+S_5+S_6-S_7 \tag{3.4}$$

so that actually eight out of nine terms in Eq. (3.2) are linearly independent.

Expressing Eqs. (3.2) in terms of individual particle operators π, K, $\bar{K}$, η, ω, ρ, K^*, $\bar{K}^*$, and ϕ (see Appendix), we can find large numbers of SU(3) relations

$$M(e\bar{e}\to K^{+*}\bar{K}^0\pi^-)=M(e\bar{e}\to \rho^+K^0K^-), \tag{3.5a}$$

$$M(e\bar{e}\to K^{0*}\bar{K}^0\pi^0)=\sqrt{3}M(e\bar{e}\to K^{0*}\bar{K}^0\eta), \tag{3.5b}$$

$$\sqrt{2}M(e\bar{e}\to \phi K^+K^-)=M(e\bar{e}\to \omega\pi^+\pi^-)-M(e\bar{e}\to \rho^0\pi^+\pi^-), \tag{3.5c}$$

$$\sqrt{2}M(e\bar{e}\to \phi K^0\bar{K}^0)=M(e\bar{e}\to \omega K^0\bar{K}^0)-M(e\bar{e}\to \rho^0K^0\bar{K}^0), \tag{3.5d}$$

$$\sqrt{2}M(e\bar{e}\to \phi\pi^+\pi^-)=M(e\bar{e}\to \omega K^+K^-)-M(e\bar{e}\to \rho^0K^+K^-), \tag{3.5e}$$

$$M(e\bar{e}\to K^{0*}\pi^+K^-) = M(e\bar{e}\to \phi K^+K^-) - M(e\bar{e}\to \phi\pi^+\pi^-)$$
$$= \sqrt{2}[M(e\bar{e}\to \rho^0K^+K^-) - M(e\bar{e}\to \rho^0\pi^+\pi^-)]$$
$$= \sqrt{2}[M(e\bar{e}\to \omega\pi^+\pi^-) - M(e\bar{e}\to \omega K^+K^-)], \quad (3.5f)$$

$$M(e\bar{e}\to \rho^+K^-K^0)$$
$$= \sqrt{2}[M(e\bar{e}\to K^{+*}K^-\pi^0) - M(e\bar{e}\to \rho^+\pi^-\pi^0)]$$
$$= (\tfrac{2}{3})^{1/2}[M(e\bar{e}\to \rho^0\pi^0\eta) - M(e\bar{e}\to K^{+*}K^-\eta)], \quad (3.5g)$$

$$M(e\bar{e}\to \rho^+\pi^-\eta)$$
$$= M(e\bar{e}\to \rho^0\pi^0\eta)$$
$$= \sqrt{3}M(e\bar{e}\to \rho^+\pi^-\pi^0) - 2M(e\bar{e}\to K^{+*}K^-\eta)$$
$$= \tfrac{1}{2}\sqrt{3}M(e\bar{e}\to K^{+*}K^-\pi^0) - \tfrac{1}{2}M(e\bar{e}\to K^{+*}K^-\eta), \quad (3.5h)$$

as well as a few others involving more than two neutral final particles:

$$4M(e\bar{e}\to V^0K^0\bar{K}^0)$$
$$= M(e\bar{e}\to V^0\pi^0\pi^0) + 3M(e\bar{e}\to V^0\eta\eta)$$
$$- \sqrt{3}[M(e\bar{e}\to V^0\pi^0\eta) + M(e\bar{e}\to V^0\eta\pi^0)], \quad (3.6)$$

where V^0 stands for any of ρ^0, ω, and ϕ. Finally, we have

$$M(e\bar{e}\to \tilde{\rho}^0\pi^0\pi^0) = -M(e\bar{e}\to \tilde{\rho}^0\eta\eta), \quad (3.7a)$$

$$\sqrt{3}[M(e\bar{e}\to \tilde{\omega}\pi^0\pi^0) - M(e\bar{e}\to \tilde{\omega}\eta\eta)] = 2M(e\bar{e}\to \tilde{\omega}\pi^0\eta)$$
$$= 2M(e\bar{e}\to \tilde{\omega}\eta\pi^0), \quad (3.7b)$$

$$\sqrt{3}[M(e\bar{e}\to \tilde{\phi}\pi^0\pi^0) - M(e\bar{e}\to \tilde{\phi}\eta\eta)] = 2M(e\bar{e}\to \tilde{\phi}\eta\pi^0)$$
$$= 2M(e\bar{e}\to \tilde{\phi}\pi^0\eta), \quad (3.7c)$$

where $\tilde{\rho}^0$, $\tilde{\omega}$, and $\tilde{\phi}$ are defined by

$$\tilde{\rho}^0 = \tfrac{1}{2}(\omega - \rho^0) - (1/\sqrt{2})\phi,$$
$$\tilde{\omega} = \tfrac{1}{2}(\omega - \rho^0) + 1/\sqrt{2}\,\phi, \quad (3.8)$$
$$\tilde{\phi} = (1/\sqrt{2})(\omega + \rho^0).$$

As an independent check of these relations, I have verified also that these equations follow directly from the U-spin consideration. All equations (3.5)–(3.7) are similarly valid for the corresponding decays $\psi \to VP_1P_2$, if we replace $e\bar{e}$ by ψ. From (3.5)–(3.7), we find the following relations among differential cross-sections

$$\sigma(e\bar{e}\to K^{+*}\bar{K}^0\pi^-) = \sigma(e\bar{e}\to \rho^+K^0K^-), \quad (3.9a)$$

$$\sigma(e\bar{e}\to K^{0*}\bar{K}^0\pi^0) = 3\sigma(e\bar{e}\to K^{0*}\bar{K}^0\eta), \quad (3.9b)$$

$$2\sigma(e\bar{e}\to \rho^0\pi^0\eta) + \sigma(e\bar{e}\to K^{+*}K^-\eta)$$
$$= \sigma(e\bar{e}\to K^{+*}K^-\pi^0) + \sigma(e\bar{e}\to \rho^+K^-K^0), \quad (3.9c)$$

$$2\sigma(e\bar{e}\to \rho^+\pi^-\pi^0) + \sigma(e\bar{e}\to \rho^+K^-K^0)$$
$$= \sigma(e\bar{e}\to K^{+*}K^-\pi^0) + 3\sigma(e\bar{e}\to K^{+*}K^-\eta), \quad (3.9d)$$

$$\sigma(e\bar{e}\to \rho^0\pi^+\pi^-) + \sigma(e\bar{e}\to \omega\pi^+\pi^-) + \sigma(e\bar{e}\to \phi\pi^+\pi^-)$$
$$= \sigma(e\bar{e}\to \rho^0K^+K^-) + \sigma(e\bar{e}\to \omega K^+K^-) + \sigma(e\bar{e}\to \phi K^+K^-), \quad (3.9e)$$

as well as many triangular inequalities, which we will not bother to write down.

If we assume $\theta = 0$ in addition, i.e., if the electromagnetic interaction does not contain any SU(3) singlet, then we find many more relations, such as

$$M(e\bar{e}\to \phi K^0\bar{K}^0) = -\sqrt{2}\,M(e\bar{e}\to \omega\pi^+\pi^-), \quad (3.10a)$$

$$M(e\bar{e}\to K^{0*}\bar{K}^0\pi^0) = \sqrt{3}\,M(e\bar{e}\to \rho^0\pi^0\eta), \quad (3.10b)$$

$$-2\sqrt{2}M(e\bar{e}\to \phi\pi^+\pi^-) = M(e\bar{e}\to \rho^0K^0\bar{K}^0)$$
$$+ M(e\bar{e}\to \omega K^0\bar{K}^0), \quad (3.10c)$$

$$2M(e\bar{e}\to \rho^0K^+K^-) = M(e\bar{e}\to \rho^0\pi^+\pi^-)$$
$$+ M(e\bar{e}\to \rho^0K^0\bar{K}^0), \quad (3.10d)$$

$$M(e\bar{e}\to \rho^0\eta\eta) = M(e\bar{e}\to \rho^0K^0\bar{K}^0). \quad (3.10e)$$

From (3.5) and (3.10), we find then additional relations among annihilation cross-sections:

$$\sigma(e\bar{e}\to \phi K_SK_S) = \sigma(e\bar{e}\to \phi K_LK_L)$$
$$= \sigma(e\bar{e}\to \phi K^0\bar{K}^0)$$
$$= 2\sigma(e\bar{e}\to \omega\pi^+\pi^-), \quad (3.11a)$$

$$\sigma(e\bar{e}\to K^{0*}\bar{K}^0\pi^0) = 3\sigma(e\bar{e}\to \rho^0\pi^0\eta), \quad (3.11b)$$

$$4\sigma(e\bar{e}\to \phi\pi^+\pi^-) + 2\sigma(e\bar{e}\to \omega\pi^+\pi^-)$$
$$= \sigma(e\bar{e}\to \rho^0K^0\bar{K}^0) + \sigma(e\bar{e}\to \omega K^0\bar{K}^0), \quad (3.11c)$$

$$\sigma(e\bar{e}\to \rho^0K^0\bar{K}^0) + \sigma(e\bar{e}\to \rho^0\pi^+\pi^-)$$
$$= 2\sigma(e\bar{e}\to \rho^0K^+K^-) + \sigma(e\bar{e}\to K^{0*}\pi^+K^-). \quad (3.11d)$$

Next, let us test the hypothesis of the usual quark-line rule in which we consider only connected quark-line diagrams. Then, we need take into account only three terms S_1, S_2, and A_1 in (3.2). Furthermore, we restrict ourselves to the case $\theta = 0$. Then, from Table I of the Appendix, we obtain the following remarkably simple consequences:

$$\sigma(e\bar{e}\to \rho^0K^0\bar{K}^0) = \sigma(e\bar{e}\to \omega\pi^+\pi^-)$$
$$= \sigma(e\bar{e}\to \omega K^0\bar{K}^0)$$
$$= \sigma(e\bar{e}\to K^{0*}\pi^0\bar{K}^0)$$
$$= \tfrac{1}{2}\sigma(e\bar{e}\to \phi K^0\bar{K}^0)$$
$$= 3\sigma(e\bar{e}\to K^{0*}\eta\bar{K}^0), \quad (3.12a)$$

$$\sigma(e\bar{e}\to \phi\pi^+\pi^-) = \sigma(e\bar{e}\to \phi\pi^0\eta) = 0, \quad (3.12b)$$

$$\sigma(e\bar{e} \to \rho^0 K^+K^-) = \sigma(e\bar{e} \to \omega K^+K^-)\,, \tag{3.12c}$$

$$\sigma(e\bar{e} \to \phi K^+K^-) = \sigma(e\bar{e} \to K^{0*}\pi^+K^-)\,, \tag{3.12d}$$

$$\begin{aligned}\sigma(e\bar{e} \to \rho^0\pi^0\pi^0) &= 9\sigma(e\bar{e} \to \rho^0 K^0\bar{K}^0)\\ &= 9\sigma(e\bar{e} \to \rho^0\eta\eta)\\ &= 27\sigma(e\bar{e} \to \rho^0\eta\pi^0)\\ &= 81\sigma(e\bar{e} \to \omega\eta\eta)\\ &= 3\sigma(e\bar{e} \to \omega\pi^0\eta)\,.\end{aligned} \tag{3.12e}$$

These will be the test of the full quark-line rule in the exact-SU(3) limit. Especially, we note that $\sigma(e\bar{e} \to \phi\pi^+\pi^-) = 0$.

However, the SU(3) quark-line rule may not be unambiguous for the present case because of the SU(3) identity relation (3.4) which states that a sum of all connected quark-line diagrams is equal to a sum of all disconnected quark-line diagrams. Note that such a situation does not arise for $e\bar{e} \to P_1P_2P_3$, as we see from Eq. (2.3). In connection with an explanation of the decay rate ratio (1.1), I suggested[7] that we should modify the nonet rule (and hence the quark-line rule) so that a special combination S_1 should not appear at all to be self-consistent with the spirit of the quark-line rule. Our modified nonet ansatz[7] then implies that we must consider a linear combination of S_2, S_3, S_4, A_1, and A_2. In that case, we find the following different set of predictions with $\theta = 0$:

$$\begin{aligned}\sigma(e\bar{e} \to \rho^0 K^0\bar{K}^0) &= 9\sigma(e\bar{e} \to \omega\pi^+\pi^-)\\ &= 9\sigma(e\bar{e} \to \omega K^0\bar{K}^0)\\ &= \tfrac{9}{2}\sigma(e\bar{e} \to \phi\pi^+\pi^-)\\ &= \tfrac{9}{2}\sigma(e\bar{e} \to \phi K^0\bar{K}^0)\,,\end{aligned} \tag{3.13a}$$

$$\sigma(e\bar{e} \to \phi K^+K^-) = 2\sigma(e\bar{e} \to \omega K^+K^-)\,, \tag{3.13b}$$

$$\begin{aligned}&2\sigma(e\bar{e} \to \rho^0 K^+K^-) + \sigma(e\bar{e} \to K^{0*}K^-\pi^+)\\ &\quad = \sigma(e\bar{e} \to \rho^0 K^0\bar{K}^0) + \sigma(e\bar{e} \to \rho^0\pi^+\pi^-)\,,\end{aligned} \tag{3.13c}$$

$$\begin{aligned}&6\sigma(e\bar{e} \to \omega K^+K^-) + 3\sigma(e\bar{e} \to \omega\pi^+\pi^-)\\ &\quad = \sigma(e\bar{e} \to \rho^0\pi^+\pi^-) + \sigma(e\bar{e} \to K^{0*}K^-\pi^+)\,,\end{aligned} \tag{3.13d}$$

in contrast to (3.12). Especially, we note the validity of (1.2). We may easily distinguish two sets of predictions (3.12) and (3.13) by experimentally measuring these cross sections.

We can find various relations for $\Gamma(\psi \to VPP)$ for the modified nonet ansatz. However, since some of the relations are already discussed elsewhere,[7] we will not go into detail here.

IV. GENERAL CASE

We consider here the general case on the basis of the U-spin invariance, or more simply the Weyl reflection symmetry[28] W_{23}, in which we interchange q_2 and q_3. Noting that under the Weyl reflection W_{23} the electromagnetic current is invariant and pseudoscalar octet mesons transform as

$$\begin{aligned}&\pi^\pm \longleftrightarrow K^\pm\,, \quad K^0 \longleftrightarrow \bar{K}^0\,,\\ &\pi^0 \to \tfrac{1}{2}\pi^0 + \tfrac{1}{2}\sqrt{3}\,\eta\,,\\ &\eta \to \tfrac{1}{2}\sqrt{3}\,\pi^0 - \tfrac{1}{2}\eta\,,\end{aligned} \tag{4.1}$$

we find the following relations:

$$M(e\bar{e} \to \pi^+ X) = M(e\bar{e} \to K^+\tilde{X})\,, \tag{4.2}$$

$$M(e\bar{e} \to \pi^+\pi^- X) = M(e\bar{e} \to K^+K^-\tilde{X})\,, \tag{4.3}$$

$$M(e\bar{e} \to \pi^0 X) = \tfrac{1}{2}M(e\bar{e} \to \pi^0\tilde{X}) + \tfrac{1}{2}\sqrt{3}\,M(e\bar{e} \to \eta\tilde{X})\,, \tag{4.4a}$$

$$M(e\bar{e} \to \eta X) = \tfrac{1}{2}\sqrt{3}\,M(e\bar{e} \to \pi^0\tilde{X}) - \tfrac{1}{2}M(e\bar{e} \to \eta\tilde{X})\,, \tag{4.4b}$$

where $\tilde{X}$ is the Weyl transform of X, i.e.,

$$\tilde{X} = W_{23}X \tag{4.5}$$

for any single-particle or multiparticle state X. Now let M be a set consisting of some particle states (X's) which are invariant as a whole under W_{23}, i.e.,

$$W_{23}M = M\,. \tag{4.6}$$

Then, summing over all possible states X belongs to M, Eqs. (4.2) and (4.3) lead to

$$\sum_{X\in M}\sigma(e\bar{e} \to \pi^+ X) = \sum_{X\in M}\sigma(e\bar{e} \to K^+ X)\,, \tag{4.7}$$

$$\sum_{X\in M}\sigma(e\bar{e} \to \pi^+\pi^- X) = \sum_{X\in M}\sigma(e\bar{e} \to K^+K^- X)\,. \tag{4.8}$$

If we choose the set M to consist of only two states, π^0 and η, then (4.8) immediately reproduces Eq. (2.10b). Also, if M represents all possible states, then (4.7) gives (1.6), i.e.,

$$\sigma(e\bar{e} \to \pi^+ + \text{anything}) = \sigma(e\bar{e} \to K^+ + \text{anything})\,. \tag{4.9}$$

This relation has been experimentally found[17] to be poorly satisfied at $\sqrt{s} = 3.8$ GeV and $\sqrt{s} = 4.8$ GeV, even for high-energy kaons. However, the validity of (4.9) presupposes the validity of Weyl equalities among individual exclusive reactions such as

$$\sigma(e\bar{e} \to \pi^+\pi^+\pi^-\pi^-) = \sigma(e\bar{e} \to K^+K^+K^-K^-)\,. \tag{4.10}$$

In view of a large mass difference between the pion and the kaon, we do not expect that this relation is well satisfied even at $\sqrt{s} = 4.8$ GeV. We note that the relation (3.5a) or (3.9a) is also an immediate consequence of the Weyl symmetry. Perhaps in view of the relatively small mass difference involved in $K^*\bar{K}\pi$ and $\rho K\bar{K}$ final states, experimental verification of (3.9a) is more suitable to test the SU(3) symmetry.

From (4.4a) and (4.4b), we can derive the following triangular inequalities:

$$\tfrac{1}{3}\sigma_1 \le \sigma_2 \le 3\sigma_1 , \tag{4.11}$$

where we have set

$$\sigma_1 = \sum_{X\in M} \sigma(e\bar{e} \to \pi^0 X), \tag{4.12a}$$

$$\sigma_2 = \sum_{X\in M} \sigma(e\bar{e} \to \eta X). \tag{4.12b}$$

If we assume that the electromagnetic current $j_\mu^{\text{em}}(x)$ does not contain the SU(3)-singlet component, i.e., $\theta = 0$, then we can say something more. In that case, $j_\mu^{\text{em}}(x)$ satisfies

$$(1 + W_{12} + W_{13}) j_\mu^{\text{em}}(x) = 0, \tag{4.13}$$

where W_{12} and W_{13} are Weyl reflections interchanging $q_1 \leftrightarrow q_2$ and $q_1 \leftrightarrow q_3$, respectively. This implies that we should have an identity

$$M(e\bar{e} \to X) + M(e\bar{e} \to W_{12}X) + M(e\bar{e} \to W_{13}X) = 0 \tag{4.14}$$

for any state X. As a matter of fact, all special relations obtained in the previous sections for the case $\theta = 0$ can be derived from systematic investigation of (4.14). Especially, replacing X by $K^0 + X$ in (4.14), and noting (4.1), this gives

$$M(e\bar{e} \to K^0X) + M(e\bar{e} \to K^+X') + M(e\bar{e} \to K^-X'') = 0, \tag{4.15}$$

where we have set

$$X' = W_{12}X, \quad X'' = W_{23}W_{13}X. \tag{4.16}$$

Therefore, if M is now a set which is invariant under all Weyl reflections W_{12}, W_{13}, and W_{23}, as well as the charge conjugation C, i.e.,

$$W_{12}M = W_{13}M = W_{23}M = M = CM , \tag{4.17}$$

then (4.15) leads to a triangular relation

$$\sum_{X\in M} \sigma(e\bar{e} \to K^0 X) \le 4 \sum_{X\in M} \sigma(e\bar{e} \to K^+X). \tag{4.18}$$

This is the best bound we can obtain in view of (2.23a). Similarly, we can derive

$$\begin{aligned}\tfrac{1}{2}\sum_{X\in M} \sigma(e\bar{e} \to \phi X) &\le \sum_{X\in M} [\sigma(e\bar{e} \to \omega X) + \sigma(e\bar{e} \to \rho^0 X)] \\ &\le 5 \sum_{X\in M} \sigma(e\bar{e} \to \phi X)\end{aligned} \tag{4.19}$$

under the same conditions.

Up to now, we assumed the validity of the SU(3) symmetry. However, we can derive some useful identities on the basis of the weaker SU(2) groups, i.e., the charge independence alone. Consider the reaction

$$e\bar{e} \to \text{odd numbers of pions}. \tag{4.20}$$

Then, because of the G parity, only the isoscalar component of $j_\mu^{\text{em}}(x)$ can contribute to the reaction. Especially, the average number of π^+ mesons must be equal to that of the π^0's when we take the average over all final states with a given number of pions. In this way, it is easy to find

$$\sigma_T(e\bar{e} \to 2\pi^+2\pi^-\pi^0) = 2\sigma_T(e\bar{e} \to \pi^+\pi^-3\pi^0), \tag{4.21}$$

$$\begin{aligned}2\sigma_T(e\bar{e} \to 3\pi^+3\pi^-\pi^0) &= \sigma_T(e\bar{e} \to 2\pi^+2\pi^-3\pi^0) \\ &\quad + 4\sigma_T(e\bar{e} \to \pi^+\pi^-5\pi^0),\end{aligned} \tag{4.22}$$

$$\begin{aligned}\sigma_T(e\bar{e} \to 4\pi^+4\pi^-\pi^0) &= \sigma_T(e\bar{e} \to 2\pi^+2\pi^-5\pi^0) \\ &\quad + 2\sigma_T(e\bar{e} \to \pi^+\pi^-7\pi^0)\end{aligned} \tag{4.23}$$

for the total integrated reaction cross sections $\sigma_T = \int d\Omega\sigma$ involving five, seven, and nine mesons. Since we need not assume $\theta = 0$ for this derivation, these relations should be valid also for the corresponding decay rates for the ψ meson by replacing symbols $e\bar{e}$ and σ_T by ψ and Γ_T, respectively.

Many other equalities and inequalities for reactions $e\bar{e} \to$ pions on the basis of the SU(2) have been extensively studied by Pais[29] and others. These are also applicable to the corresponding pionic decays of the ψ mesons.

APPENDIX

Here we shall compute matrix elements of the reaction $e\bar{e} \to VP_1P_2$. It is convenient to separate out contributions of the SU(3)-singlet terms from the rest. For this purpose, let us define S_1', S_3', S_4', and S_5', respectively, by setting $\theta = 0$, i.e., $j = Q$ in Eqs. (3.2). Note that S_2, A_1, and A_2 are really independent of θ, so that we may set

$$S_2' \equiv S_2, \quad A_1' \equiv A_1, \quad A_2' \equiv A_2 . \tag{A1}$$

The identity (3.4) is now rewritten as

$$S_1' = S_4' + S_5' . \tag{A2}$$

In order to compensate for the θ-dependent terms, we introduce

$$S_6' = \text{Tr}(VP_1P_2) + \text{Tr}(VP_2P_1) , \tag{A3}$$

$$S_7' = (\text{Tr}V)\text{Tr}(P_1P_2) , \tag{A4}$$

which represent the contributions from the SU(3)-singlet current. The S_1, S_3, S_4, S_5, S_6, and S_7 are given by

TABLE I. Relative numerical values for matrix elements $M(e\bar{e}\to VP_1P_2)$. The expressions for S_1', S_2', S_3', S_4', S_5', S_6', S_7', A_1', and A_2' are defined in the Appendix.

	$e\bar{e}\to VP_1P_2$	S_1'	S_2'	S_3'	S_4'	S_5'	S_6'	S_7'	A_1'	A_2'
1	$\rho^0\pi^0\pi^0$	9	0	0	9	0	0	0	0	0
2	$\rho^0\pi^+\pi^-$	3	4	6	3	0	0	0	0	0
3	$\rho^0K^+K^-$	3	2	6	3	0	3	0	0	0
4	$\rho^0K^0\bar{K}^0$	3	0	6	3	0	-3	0	0	0
5	$\rho^0\eta\eta$	3	0	6	3	0	0	0	0	0
6	$\rho^0\eta\pi^0$	$\sqrt{3}$	0	$-\sqrt{3}$	$\sqrt{3}$	0	$2\sqrt{3}$	0	0	1
7	$\rho^+\pi^0\pi^-$	3	-2	-3	3	0	0	0	-2	$\sqrt{3}$
8	$\rho^+K^0K^-$	0	$\sqrt{2}$	0	0	0	$3\sqrt{2}$	0	$\sqrt{2}$	0
9	$\omega\pi^+\pi^-$	3	0	2	1	2	6	$\sqrt{2}$	0	0
10	ωK^+K^-	3	2	2	1	2	3	$\sqrt{2}$	0	0
11	$\omega K^0\bar{K}^0$	-3	0	2	1	-4	3	$\sqrt{2}$	0	0
12	$\omega\eta\eta$	1	0	0	3	-2	2	$\sqrt{2}$	0	0
13	$\omega\pi^0\eta$	$3\sqrt{3}$	0	$-\sqrt{3}$	$\sqrt{3}$	$2\sqrt{3}$	0	0	0	1
14	$\phi\pi^+\pi^-$	0	0	$-2\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}$	0	1	0	0
15	ϕK^+K^-	0	$-2\sqrt{2}$	$-2\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}$	$3\sqrt{2}$	1	0	0
16	$\phi\eta\eta$	$-4\sqrt{2}$	0	0	$-3\sqrt{2}$	$-\sqrt{2}$	$4\sqrt{2}$	1	0	0
17	$\phi\pi^0\eta$	0	0	$\sqrt{6}$	$-\sqrt{6}$	$\sqrt{6}$	0	0	0	$-\sqrt{2}$
18	$\phi K^0\bar{K}^0$	$-3\sqrt{2}$	0	$-2\sqrt{2}$	$-\sqrt{2}$	$-2\sqrt{2}$	$3\sqrt{2}$	1	0	0
19	$K^{+*}\pi^0K^-$	3	-1	-3	3	0	3	0	-1	$\sqrt{3}$
20	$K^{+*}\eta K^-$	$\sqrt{3}$	$-\sqrt{3}$	$-\sqrt{3}$	$\sqrt{3}$	0	$-\sqrt{3}$	0	$-\sqrt{3}$	1
21	$K^{0*}\pi^0\bar{K}^0$	3	0	-3	3	0	-3	0	0	$\sqrt{3}$
22	$K^{0*}\eta\bar{K}^0$	$\sqrt{3}$	0	$-\sqrt{3}$	$\sqrt{3}$	0	$-\sqrt{3}$	0	0	1
23	$K^{+*}\pi^-\bar{K}^0$	0	$\sqrt{2}$	0	0	0	$3\sqrt{2}$	0	$-\sqrt{2}$	0
24	$K^{0*}\pi^+K^-$	0	$-2\sqrt{2}$	0	0	0	$3\sqrt{2}$	0	0	0

$$S_1=S_1'+\sqrt{6}\;\theta\delta S_6', \tag{A5}$$

$$S_3=S_3'+2(\tfrac{2}{3})^{1/2}\theta\delta S_7', \tag{A6}$$

$$S_4=S_4'+(\tfrac{2}{3})^{1/2}\theta\delta S_7', \tag{A7}$$

$$S_5=S_5'+2(\tfrac{2}{3})^{1/2}\theta\delta S_7', \tag{A8}$$

$$S_6=\sqrt{6}\;\theta\delta S_6', \tag{A9}$$

$$S_7=\sqrt{6}\;\theta\delta S_7'. \tag{A10}$$

Now, we express all S_μ' and A_μ' in terms of π, K, $\bar{K}$, η, ρ, ω, ϕ, K^*, and $\bar{K}^*$. Then we evaluate matrix elements for $e\bar{e}\to VP_1P_2$. The result is shown in Table I, where the numerical entries in each column are suitably normalized. As an additional check, the identity (A2) has been individually verified, as we see from the table. Since SU(2) invariance together with the G conjugation demands

$$M(e\bar{e}\to\omega\pi^0\pi^0)=M(e\bar{e}\to\omega\pi^+\pi^-), \tag{A11}$$

$$M(e\bar{e}\to\phi\pi^0\pi^0)=M(e\bar{e}\to\phi\pi^+\pi^-), \tag{A12}$$

$$M(e\bar{e}\to\rho^+\eta\pi^-)=M(ee\to\rho^0\eta\pi^0), \tag{A13}$$

we did not list these reactions in the table.

Also, we notice that $M(e\bar{e}\to VP'P)$ can be obtained from $M(e\bar{e}\to VPP')$ by changing the signs of the A_1' and A_2' terms because of the Bose symmetry. Besides, the charge conjugation demands

$$M(e\bar{e}\to VP_1P_2)=M(e\bar{e}\to\bar{V}\bar{P}_1\bar{P}_2), \tag{A14}$$

where $\bar{V}$ and $\bar{P}$ represent the antiparticle states of V and P, respectively. From these operations, we can calculate matrix elements of all other reactions not listed in the table.

The relations (3.10) are easily obtained from Table I by omitting contributions from S_6' and S_7', since they represent the effect of the unitary-singlet current $\theta J_\mu^{(0)}(x)$. Similarly, (3.12) follows by considering S_1', S_2', and A_1', while (3.13) results by taking into account S_2', S_3', S_4', A_1', and A_2', which alone are allowed by the new nonet hypotheses. Note that S_3', S_4', and A_2' correspond to disconnected quark-line diagrams, while only S_1', S_2', and A_1' give the connected quark-line terms for the case $\theta=0$. Thus, relations (3.12) really test the full quark-line rule, which is stronger than the usual nonet hypothesis.

Note added in proof. If the nonet ansatz or the quark-line rule is applicable to the pseudoscalar nonet P_9, then we predict the validity of

$$\sigma(e\bar{e} \to \eta' + \text{pions}) = 2\sigma(e\bar{e} \to \eta + \text{pions})$$

for any final state containing η or η' but any given number of pions of a given type, such as $\pi^+\pi^+\pi^-\pi^-$. This generalizes Eqs. (2.19a) and (2.24b). The same relation holds also for the corresponding strong-interaction reaction

$$\sigma(p\bar{p} \to \eta' + \text{pions}) = 2\sigma(p\bar{p} \to \eta + \text{pions}),$$

under the same condition.

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[4]E.g., H. Harari, in *Proceedings of the 1975 International Symposium on Lepton and Photon Interactions at High Energy, Stanford, California,* edited by W. T. Kirk (SLAC, Stanford University, Stanford, 1976), p. 317.

[5]See Ref. 3. Note that the small decay rate for $\phi \to \pi\rho$ may be explained as a unitarity correction of the rule. See J. Pasupathy, Phys. Rev. D 12, 2929 (1975); K. Akama and S. Wada, Phys. Lett. 61B, 279 (1976).

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[9]K. Ninomiya [University of Nagoya Report No. DPNU-1-76, 1976 (unpublished)] assumes that a large violation of the quark-line rule comes from a Pomeron-exchange mechanism between two vector mesons and two pions. This corresponds to a consideration of the $\mathrm{Tr}(Vj)\mathrm{Tr}(P_1P_2)$ term.

[10]W. F. Palmer and S. S. Pinsky [Ohio State University Report No. COO-1545-171, 1975 (unpublished)] attribute the large violation of the quark-line rule to a large deviation from the ideal mixing among members of the scalar nonet. See also M. Chaichan and M. Hayashi, Phys. Lett. 61B, 178 (1976).

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[19]E.g., see S. Okubo, Prog. Theor. Phys. Suppl. 37–38, 114 (1964); M. S. Chanowitz, Phys. Rev. Lett. 35, 977 (1975) and references quoted therein.

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[21]For example, the Coleman-Glashow relation $\mu(\Lambda) = \frac{1}{2}\mu(n)$ for the magnetic moments of the baryons is a special result of $\theta = 0$. If we do not assume it, this relation is no longer valid, and we obtain only a weaker U-spin relation of $\mu(\Lambda) = \frac{1}{3}[\mu(\Sigma^0) + 2\mu(n)]$, with $\mu(\Sigma^0) = \frac{1}{2}[\mu(\Sigma^+) + \mu(\Sigma^-)]$. So far, both relations for $\mu(\Lambda)$ are equally well (or equally badly) satisfied. Therefore, the experimental deviation from $\mu(\Lambda) = \frac{1}{2}\mu(n)$ is likely due to the SU(3) violation rather than the presence of the SU(3) singlet current.

[22]S. Okubo and B. Sakita, Phys. Rev. Lett. 11, 50 (1963).

[23]If the nonet ansatz or the quark-line rule is applicable to the pseudoscalar nonet $(\pi, K, \bar{K}, \eta, \eta')$, then Lipkin (see Ref. 12) derives the relation

$$\sigma(K^-p \to \eta Y) + \sigma(K^-p \to \eta' Y) = \sigma(\pi^-p \to K^0 Y) + \sigma(K^-p \to \pi^0 Y)$$

for the forward meson production cross section, assuming no exotic exchange in the t channel. This relation is badly satisfied experimentally, as Lipkin points out. In contrast, he obtains the relations (with ideal ω-ϕ mixing) $\sigma(K^-p \to \omega Y) = \sigma(K^-p \to \rho^0 Y)$ and $\sigma(K^-p \to \phi Y) = \sigma(\pi^-p \to K^{0*} Y)$ for the forward vector-meson productions, which are in good agreement with experiment.

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[27]Note that the definitions of S_3 and S_4 are different from those given in Ref. 7. Also, since we assumed $\mathrm{Tr}P_1 = \mathrm{Tr}P_2 = 0$, we need not consider all terms proportional to $\mathrm{Tr}P_1$ and $\mathrm{Tr}P_2$ in this note.

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