Group-theoretical analysis of the Melosh transformations

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This paper is devoted to a group-theoretical analysis of the Melosh transformations. The first Melosh transformation is derived and uniquely characterized as being minimal (i.e., little-group rotation free) by making use of the de Sitter transformation properties of the Dirac equation. A class of transformations in the lightlike quantization formalism is derived from its spacelike counterparts by making use of its de Sitter structure, which coincides with another given previously. The second Melosh transfomation is considered in the same spirit and it is seen how it can be included in this scheme. Finally, it is shown how all the considered Melosh transformations can be associated to Wigner rotations.

I. INTRODUCTION

The $SU(6)_W$ group arises in physics in two different ways. First, it appears as an approximate symmetry group of the strong interactions[1] under which hadrons are classified into multiplets and can be naively considered—forgetting about exotic states—as made up of two or three basic entities, the "constituent quarks," which are associated with the basis of its smallest nontrivial representations. Secondly, it appears as a subgroup composed of "good" operators[2] (i.e., those leading to nonvanishing matrix elements between single-particle states in the infinite-momentum frame) of U(12), the group which results when the "current" quark-model algebra is amplified to include scalar, pseudoscalar, and tensor currents. Following Gell-Mann[3] and Melosh[4,5] these two groups are usually called $SU(6)_{W,\text{strong}}$ and $SU(6)_{W,\text{currents}}$. $SU(6)_{W,\text{strong}}$ appears to be useful in classifying hadrons into irreducible representations; $SU(6)_{W,\text{currents}}$ is partly composed of directly measurable quantities (e.g., vector and axial-vector currents).

As is well known, some of the $SU(6)_{W,\text{strong}}$ generators can be identified, through generalized conservation of vector current (CVC), as the space integrals of the time component of the vector currents, but it is not possible to go any further. In the free-quark model (all the considerations in this paper will be made in the free-field case) the expression of the $SU(6)_{W,\text{currents}}$ generators is given by

$$\begin{aligned} F^i &= \int d^4x\,\delta(n_\sigma x^\sigma)\bar{q}(x)(n_\sigma\gamma^\sigma)\frac{\lambda^i}{2}q(x)\,, \quad i=1,\ldots,8\,,\\ F^i_{x,y} &= \int d^4x\,\delta(n_\sigma x^\sigma)\bar{q}(x)(n_\sigma\gamma^\sigma)\beta\frac{\sigma_{x,y}}{2}\frac{\lambda^i}{2}q(x)\,, \quad i=0,1,\ldots,8\,,\\ F^i_z &= \int d^4x\,\delta(n_\sigma x^\sigma)\bar{q}(x)(n_\sigma\gamma^\sigma)\frac{\sigma_z}{2}\frac{\lambda^i}{2}q(x)\,, \quad i=0,1,\ldots,8\,, \end{aligned} \tag{1.1}$$

and the generators of the four-translations are

$$P^\mu = i\int d^4x\,\delta(n_\sigma x^\sigma)\bar{q}(x)(n_\sigma\gamma^\sigma)\frac{\partial}{\partial x_\mu}q(x)\,. \tag{1.2}$$

Let us consider first the case of the spacelike charges. These are conventionally obtained by taking $n_\sigma=(1,\vec{0})$ in the above expressions. Simple inspection of (1.1) shows that only the conserved F^i can be identified with the corresponding set of $SU(6)_{W,\text{strong}}$, the rest of the generators not fulfilling the obviously necessary condition of commutation with the Hamiltonian $H=P^0$, so that in general $F^i_\alpha|0\rangle\neq 0$.

In fact there are general arguments both of phenomenological[6] and fundamental[7] nature which prevent the identification of $SU(6)_{W,\text{currents}}$ with $SU(6)_{W,\text{constituents}}$. However, it is possible to assume, as indicated by Gell-Mann,[3,8] the existence of a unitary transformation relating the two groups, and thus providing a form for the otherwise ill-defined operators of the $SU(6)_{W,\text{strong}}$ approximate symmetry algebra. This program has been pursued by Melosh[4,5,9] (Gomberoff, Horwitz, and Ne'eman[10] have also considered a Melosh-type transformation for the $[U(6)\otimes U(6)]_W$ algebra). The *first* Melosh transformation[4,11] is found in the quark model by assuming the existence of a unitary transformation V relating the strong (W^i_α) and the current (F^i_α) charges in such a way that

$$[W^i_\alpha, H_0] = 0 , \tag{1.3}$$

where

$$W^i_\alpha \equiv V F^i_\alpha V^{-1} \tag{1.4}$$

or, equivalently, by defining a "representation" W ($O_W = V^{-1} O V$) such that $W^i_{\alpha W} = F^i_\alpha$ and

$$[H_W, F^i_\alpha] = 0 . \tag{1.5}$$

(1.5) indicates that V must take away the "transverse" term $\vec{\gamma}_\perp \cdot \vec{\partial}_\perp = \gamma^1\partial^1 + \gamma^2\partial^2$ of H. This problem is similar to the old one of Foldy and Wouthuysen[12] (FW), which suggests the answer

$$V = \exp(iY) , \qquad Y = \tfrac{1}{2} \int d^3x\, q(x)^\dagger \arctan\left(\frac{\vec{\gamma}_\perp \cdot \vec{\partial}_\perp}{m}\right) q(x) , \tag{1.6}$$

leading to

$$H_W = \int d^3x\, q(x)^\dagger (-i\alpha_z \partial_z + \beta\kappa) q(x) \tag{1.7}$$

($\kappa = [m^2 + (\vec{\gamma}_\perp \cdot \vec{\partial}_\perp)^2]^{1/2}$) which satisfies (1.5) and thus to strong charges fulfilling the symmetry condition (1.3). Besides, V preserves the C and P transformation properties of the charges, contains only "good" operators, is an SU(3) singlet, and commutes with J_z.

It is clear, however, that the requirement of the transformed charge being conserved [(1.5)] does not uniquely specify the transformation. Obviously, any unitary transformation commuting with H_W could be added to V and still (1.5) would be preserved: this type of ambiguity has already been recognized by Palmer and Rabl,[13] who have discussed the FW and Melosh transformations (also in absence of interactions) using Fock-space methods in the equal-time formalism. For instance, the Gomberoff *et al.* transformed Hamiltonian[10] is the same as that of FW, although both transformations are different (Sec. II). Thus, other conditions are needed in addition to (1.5) to determine the desired V.

Let us now turn to the lightlike or null-plane current charges[14] ($\hat{F}^i_\alpha$; we shall use a caret for null-plane quantities). As is well known these are specially suitable in dealing with the infinite-momentum limit; they can be formally obtained by boosting the spacelike charges to infinite momentum. Quantizing the theory as is customary[15] on the hyperplane $x^+ = (x+t)/\sqrt{2} = 0$ [light-cone variables are defined in the form $x^\mu \to (x^+ = (x+t)/\sqrt{2},\ x, x^- = (t-z)/\sqrt{2})$] the $\hat{F}^i_\alpha$ are given by (1.1) with $n_\sigma = (1/\sqrt{2})(1,0,0,1)$, and Y now becomes[4]

$$\hat{Y} = \frac{1}{\sqrt{2}} \int d^4x\, \delta(x^+) q_+(x)^\dagger \arctan\left(\frac{\vec{\gamma}_\perp \cdot \vec{\partial}_\perp}{m}\right) q_+(x) , \tag{1.8}$$

where $q_+ \equiv P_+ q = \frac{1}{2}(1+\alpha^3)q$. However, as de Alwis[16] first pointed out, the $\hat{F}^i_\alpha$ already commute with the lightlike Hamiltonian $\hat{H} \equiv \hat{P}^-$, so that the original motivation to introduce V vanishes unless the postulate is made that V should be abstracted first from the finite-momentum theory. By using spin arguments, Melosh[5] has recently proposed a *second* transformation[17] given by

$$\hat{Y}_{II} = \sqrt{2} \int d^4x\, \delta(x^+) q_+(x)^\dagger \arctan \frac{\vec{\gamma}_\perp \cdot \vec{\partial}_\perp}{m + |p^0 + p^3|} q_+(x) . \tag{1.9}$$

Owing to the process followed for its derivation, (1.9) is, unlike (1.8), not invariant under longitudinal boosts. Such invariance is obtained by formally writing Y_{II} in the form[5]

$$\hat{Y}_{II} = \sqrt{2} \int d^4x\, \delta(x^+) q_+(x)^\dagger \times \arctan \frac{\vec{\gamma}_\perp \cdot \vec{\partial}_\perp}{|p^0 + p^3| M/\sqrt{2}P^+ + m} q_+(x) , \tag{1.10}$$

where P is the over-all momentum of the system, $2P^+P^- - P^2 = M^2$; in the rest frame $M/P^+\sqrt{2} = 1$ and (1.10) gives again (1.9).

In terms of lightlike charges it appears that (1.9) [or (1.10)] is better motivated than (1.6). However, it seems interesting to analyze both transformations and the ambiguities mentioned earlier within a general framework to see how the form of the transformation can be constrained; after all, and despite the better motivation of (1.9),[18] any of the Melosh transformations can be considered as just a free-quark-model approximation to the V which is required in the physical world. The aim of this paper is to carry out such an analysis by making use of the de Sitter properties of the Dirac equation. Our procedure is tantamount to assuming that any physically meaningful transformation (acting on the spacelike quantized Dirac field) is associated to an element of this group. This is not surprising since such transformation should preserve some type of spinor scalar product and in fact Bracken and Cohen[19,20] have shown how the usual transformations (e.g., FW, Cini-Touschek,[21] and Chakrabarti[22]) are related to elements of SO(4,1). As a result of this approach we shall find in Sec. II that the first Melosh transformation is *uniquely* characterized as being the minimal transformation, in a sense to be appropriately defined, which leads to (1.5); in the same way, we shall find the relation between the FW and the Gomberoff *et al.* transformation. Because our procedure relies on the spacelike formalism, the derivation of transformations in the lightlike formalism is restricted to those cases for which the boosting process leaves their group structure unaltered. In this way, the application of the constraints imposed by the de Sitter group to the transformation will lead to a general form of $\hat{V}$ which will be seen to coincide with that given by Eichten *et al.*[23] (See also de Alwis[16] and Osborn.[24]) This will be done in Sec. III. It will also be shown there how the transformation (1.10) is related to the de Sitter group, although owing to the presence of the momentum operator P which does not act on the field $q(x)$, (1.10) cannot be directly interpreted as a transformation of the Dirac equation. Finally, we shall briefly examine in Sec. IV how the transformations (1.8), (1.9), and (1.10) can be understood as "Wigner" rotations.

II. THE FIRST MELOSH TRANSFORMATION AS A MINIMAL TRANSFORMATION

In this section we characterize the class of transformations which eliminate the transverse $\vec{\gamma}_\perp \cdot \vec{\partial}_\perp$ component from the free-quark Hamiltonian. This type of problem, as already mentioned, has no unique solution, but the first Melosh transformation will arise as the "minimal" transformation of its class.

Our considerations are best made by writing the Dirac equation in a form which makes it easy to use its SO(4,1) transformation properties; to avoid cumbersome notation, we shall use an unquantized formalism which is sufficient for our purposes here. The connection between the Dirac equation and the de Sitter group has been analyzed many times; more recently it has been considered by De Vos and Hilgevoord,[25] and by DeVries[26] and Bracken and Cohen[19] in connection with the so-called canonical transformations of the Dirac equation. This is done by writing the equation in the form (we summarize here those results which are relevant for our purposes and refer, e.g., to the papers quoted above for details)

$$\Gamma_\alpha p^\alpha \psi \equiv \Gamma(p)\psi = 0 \quad (\alpha = 0, 1, \ldots, 4), \tag{2.1}$$

where the five-vector p^α is defined by $p^\alpha = p^\mu$ (for $\alpha = \mu = 0, 1, 2, 3$), $p^4 = -m$, and

$$\Gamma_\mu = \gamma_5 \gamma_\mu, \quad \Gamma_4 = \gamma_5 \quad (\gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3). \tag{2.2}$$

The Clifford algebra is now defined by

$$\{\Gamma_\alpha, \Gamma_\beta\} = 2g_{\alpha\beta}, \quad g_{\alpha\beta} = \mathrm{diag}(1, -1, -1, -1, -1), \tag{2.3}$$

and the matrices

$$T_{\alpha\beta} = \frac{i}{4}[\Gamma_\alpha, \Gamma_\beta] \tag{2.4}$$

satisfy the typical pseudo-orthogonal commutation rules

$$[T_{\alpha\beta}, T_{\gamma\delta}] = i(g_{\alpha\gamma}T_{\beta\delta} + g_{\beta\delta}T_{\alpha\gamma} - g_{\alpha\delta}T_{\beta\gamma} - g_{\beta\gamma}T_{\alpha\delta}) \tag{2.5}$$

which here correspond to the SO(4,1) algebra.

Obviously, and owing to the fact that $p^4 = -m$ is constant, (2.1) is not invariant under the whole de Sitter group of transformations. However, by applying the transformation

$$Q(S) = \exp(\tfrac{1}{2} i \omega^{\alpha\beta} T_{\alpha\beta}) \tag{2.6}$$

($\omega^{\alpha\beta} = -\omega^{\beta\alpha}$ and real) associated to an element $S_\alpha^\beta \in \mathrm{SO}(4,1)$ ($S_0^0 \geq 1$, $\det S = 1$) to (2.1) we find, from

$$\Gamma(S^{-1}p) = \Gamma(p') = Q(S)\Gamma(p)Q(S)^{-1}, \tag{2.7}$$

that, with $\psi_Q \equiv Q\psi$,

$$\Gamma(p')\psi_Q = 0. \tag{2.8}$$

In this way the different "canonical" forms of the Dirac equation can be classified. Bracken and Cohen[19] speak of "$p^{(\alpha)}$ forms" according to which component (and scalar product) is preserved.

In our case, since we look for a *unitary* transformation acting on the Hamiltonian H, we are interested in a $p^{(0)}$ form. Such a transformation is to be found in the SO(4) subgroup of SO(4,1) and will belong to a *class* of transformations determined by the form of the transformed Hamiltonian. Since to fulfill (1.5) we want it to lack the $\vec{\gamma}_\perp \cdot \vec{p}_\perp$ part, S will bring

$$(p^0, \vec{0}_\perp, p^3, -\kappa) \tag{2.9}$$

to $(p^0, \vec{p}_\perp, p^3, -m)$; this *determines* ($p^\alpha p_\alpha = 0$) the value of $\kappa = (m^2 + \vec{p}_\perp^{\,2})^{1/2}$. Any two transformations performing such an operation will differ by an element belonging to the little group of (2.9). The general little group of a five-vector p_α, $p_\alpha^{\,2} = 0$, is the Euclidean group in three dimensions (this is analogous to the case of a lightlike vector, for which one finds the Euclidean group in two dimensions). Of it (and again in analogy with the physical situation found for the lightlike vector) only the rotation part matters here. This is because since the Euclidean group is noncompact, its unitary representations have to represent the "translations" trivially if they operate—as they do in the present case—in a finite-dimensional vector space. Accordingly, the class of transformations we are interested in is one of the quotient set SO(4)/R, R being the group of rotations which leaves the vector $(\vec{0}, p^3, -\kappa)$ invariant; given a transformation, all others are obtained by adding suitable elements of the little group. The canonical way of doing it is to find first the transformation which is little-group rotation-free; such a transformation is obviously the simplest. We now show that this is precisely the first Melosh transformation.

To characterize the little-group rotation-free transformation one has to proceed in a geometrical way since all directions are equivalent under SO(4). We define such a miminal transformation, in the three-dimensional space in which it effectively operates, as the rotation in the plane determined by $(\vec{0}_\perp, -\kappa)$ and $(\vec{p}_\perp, -m)$ which carries one vector onto the other. Such a transformation can be written locally in the form

$$R^i{}_j = (1-\tfrac{1}{2}\epsilon^2)\delta^i{}_j - (1-\tfrac{1}{4}\epsilon^2)^{1/2}\eta^i{}_{jm}\epsilon^m + \tfrac{1}{2}\epsilon^i\epsilon_j\,, \tag{2.10}$$

where η_{ijk} is the Levi-Cività tensor and $\vec{\epsilon}$ is the vector which determines the rotation, its modulus being related to the rotation angle θ through

$$2\sin\tfrac{1}{2}\theta = \epsilon\,.$$

In our case, $\vec{\epsilon} = (\epsilon/|\vec{p}_\perp|)(p^2, -p^1, 0)$, and a short calculation shows that

$$\theta = 2\arctan\frac{|\vec{p}_\perp|}{(\kappa+m)} \tag{2.11}$$

or, equivalently,

$$\theta = \arctan\frac{|\vec{p}_\perp|}{m}\,. \tag{2.12}$$

It is now simple, from (2.10), to find the infinitesimal generator of the transformation, with the result

$$\begin{pmatrix} 0 & 0 & -p^1/|\vec{p}_\perp| \\ 0 & 0 & -p^2/|\vec{p}_\perp| \\ p^1/|\vec{p}_\perp| & p^2/|\vec{p}_\perp| & 0 \end{pmatrix} \tag{2.13}$$

which in the spinorial representation we are interested in leads to

$$T = -\frac{p^1}{|p_\perp|}T_{14} - \frac{p^2}{|p_\perp|}T_{24} = -\tfrac{1}{2}i\frac{\vec{\gamma}_\perp\cdot\vec{p}_\perp}{|\vec{p}_\perp|}\,. \tag{2.14}$$

Thus we find

$$Q = iY\,,\quad Y = -\tfrac{1}{2}i\frac{\vec{\gamma}_\perp\cdot\vec{p}_\perp}{|p_\perp|}\arctan\frac{|\vec{p}_\perp|}{m}\,, \tag{2.15}$$

which, as anticipated, corresponds to the first Melosh transformation ($QHQ^{-1} = H_W$).

By using similar arguments it is simple to show that the transformation of Gomberoff *et al.*[10] differs from V by the rotation which relates the five-vectors $(p^0, 0, 0, p^3, -\kappa)$ and $(p^0, 0, 0, 0, -\omega)$ (this additional rotation is a "bad" operator, but this is required for their purposes). Because of this, the transformed Hamiltonians of FW and Gomberoff *et al.* coincide in spite of the fact that the transformations are different.

III. GENERAL FORM OF $\hat{V}$

Let us now consider the constraints that the condition of $V = \exp(iY)$ being an element of SO(4,1) imposes on $\hat{V}$. In this section we shall no longer require the condition $[H_W, F] = 0$, so that the following analysis will be generally meaningful only when considering lightlike charges. The general form of $\hat{V}$ in the light-plane quantization will be obtained by infinite boosting of V, the form of which is derived in the spacelike quantization using the formalism of Sec. II where the SO(4,1) structure of the Dirac equation has been exhibited. To do this in a consistent manner, we have to require that the group structure of V be preserved in the boosting process.

Since in the quark model one already has $[\hat{F}^i_\alpha, \hat{H}] = 0$, it is sufficient to require $[\hat{Y}, \hat{H}] = 0$ to obtain $[\hat{W}^i_\alpha, \hat{H}] = 0$. Besides longitudinal boost invariance, we shall require on general grounds the usual conditions for $\hat{V}$:

(a) $\hat{V}$ is SU(3) scalar and unitary;
(b) $\hat{V}$ contains only good operators;
(c) good $\hat{C}$ and R transformations (recall that for lightlike charges a new operation $e^{-i\pi J_2}P$—the "mirror reflection"—replaces the usual definition of parity); and
(d) $[\hat{J}_3, \hat{V}] = 0$.

These constraints determine immediately in our scheme the form of the transformation. It is clear that, from our basic assumption, V will be in general a product of exponentials involving the generators appearing in (2.4). Of them, only four,

$$T_{14} = \tfrac{1}{2}i\gamma_1\,,\quad T_{24} = \tfrac{1}{2}i\gamma_2\,,$$
$$T_{12} = \tfrac{1}{2}i\gamma_1\gamma_2\,,\quad T_{03} = \tfrac{1}{2}i\gamma_0\gamma_3\,, \tag{3.1}$$

are good operators and accordingly will lead to a V containing only good operators. T_{03} will lead to a nonunitary V ($\gamma_0\gamma_3$ generates the boost in the z direction), and condition (c) leaves only T_{14} and T_{24} to enter in the transformation $\hat{V}$. Condition (d) requires T_{14} and T_{24} to appear in the combination

$$\vec{\gamma}_\perp\cdot\vec{p}_\perp \tag{3.2}$$

and the dependence of the angle of rotation on p_1, p_2 to be of the form $\theta(p_1, p_2) = \theta(|\vec{p}_\perp|)$. The general form of the transformation may thus be written as

$$\hat{V} = \exp\left[i\sqrt{2}\int d^4x\,\delta(x^+)q_+(x)^\dagger i\frac{\vec{\gamma}_\perp\cdot\vec{p}_\perp}{|\vec{p}_\perp|}\theta(|\vec{p}_\perp|)q_+(x)\right]\,, \tag{3.3}$$

which is seen to coincide with that obtained by Eichten *et al.*[23] from general arguments (similar forms have also been given by de Alwis[16] and Osborn[24]), although here it has been derived in a very economical way. The V corresponding to $\hat{V}$ will not give rise to conserved W's unless $\theta = \arctan[|\vec{p}_\perp|/(\kappa+m)]$, which corresponds to the first Melosh transformation (Sec. II); any other value for θ requires the use of the lightlike formalism. Although the transformation

$$\exp\left(\theta \frac{\vec{\gamma}_\perp \cdot \vec{p}_\perp}{|\vec{p}_\perp|}\right), \quad \theta = \arctan \frac{|\vec{p}_\perp|}{|p_0+p_3|+m} \tag{3.4}$$

obviously belongs to SO(4,1), the transformation (1.9) cannot be derived in the way explained above since, because of its noninvariance under longitudinal boosts (owing to the presence of $|p^0+p^3|$), it is not possible to "deboost" the transformation to the spacelike formalism where our de Sitter considerations were originally made. However, the boost-invariant form (1.10) can be associated to a rotation of the same generators and angle

$$\theta = \arctan \frac{|\vec{p}_\perp|}{[(p^0+p^3)/(P^0+P^3)]M+m} . \tag{3.5}$$

However, because the appearance of the factor $M/(P^0+P^3)$, (3.5) cannot be interpreted as a transformation of the Dirac equation satisfied by the free-quark field.

IV. MELOSH TRANSFORMATIONS AS WIGNER ROTATIONS

There is a point in our approach which requires further clarification. The rotations of the de Sitter group which correspond to the considered transformations are true O(4) rotations, i.e., involve the fourth spatial axis and accordingly are not rotations of the Lorentz group. How then can they be associated to "Wigner" rotations? The answer is found when the transformations are obtained from the Fock-space expression of the lightlike current charges. This can be written in the general form

$$\hat{F}^i_\alpha = \sum_{r,s} \frac{1}{(2\pi)^3} \int \frac{d^3p}{2\omega} \, (a^{\dagger r}_{\vec{p}} a^s_{\vec{p}} \chi^{\dagger r} S^{-1} \Gamma^i_\alpha S \, \chi^s - b^{\dagger s}_{\vec{p}} b^r_{\vec{p}} \chi^{\dagger r} S^{-1} \Gamma^i_\alpha S \chi^s) , \tag{4.1}$$

where $\Gamma^i_\alpha = (1, \beta\sigma_x/2, \beta\sigma_y/2, \sigma_z/2)\lambda^i/2$, the χ's are four-component Pauli spinors, S is a transformation to be eliminated by means of the Melosh procedure, and the basis used for the Fock-space operators in each case will become apparent in what follows. It is simple to show that S in (4.1) is a rotation which corresponds to a change in the spin basis. For instance, for the first Melosh transformation (1.8) S takes the form (we use the spinorial Dirac representation $D^{1/2,0}+D^{0,1/2}$)

$$S = \frac{\kappa + m + \vec{\gamma}_\perp \cdot \vec{p}_\perp \gamma^3}{[2\kappa(\kappa+m)]^{1/2}} \tag{4.2}$$

and corresponds to the rotation which relates the lightlike basis defined through the Kogut-Soper boost[27]

$$B(p) = \exp\left(-i\frac{\vec{E}_\perp \cdot \vec{p}_\perp}{a}\right) \exp\left(-iK_3 \ln\frac{a}{m}\right) = \frac{aP_+ + mP_- + \gamma^0 \vec{\gamma}_\perp \cdot \vec{p}_\perp P_+}{\sqrt{ma}} \tag{4.3}$$

(where $a = \omega + p^3$, $E = K_1 + J_2$, $E_2 = K_2 - J_1$, $\vec{K}, \vec{J}$ being the usual Lorentz generators) and the basis used in this case in (4.1) defined through the boost

$$B'(p) = \exp\left(-i \ln\frac{a}{\kappa} K_3\right) \exp\left(-i\frac{\vec{K}_\perp \cdot \vec{p}_\perp}{|p_\perp|} \operatorname{arcsinh}\frac{|\vec{p}_\perp|}{m}\right) = \frac{[(a+\kappa)+(a-\kappa)\gamma^0\gamma^3](\kappa+m+\gamma^0\vec{\gamma}_\perp \cdot \vec{p}_\perp)}{2[2\kappa a m(\kappa+m)]^{1/2}} \tag{4.4}$$

because $B^{-1}B' = S$.

In the same way, the rotation S for the second Melosh transformation (1.9) is given by[5]

$$S = \frac{a + m + \vec{\gamma}_\perp \cdot \vec{p}_\perp \gamma^3}{[2a(\omega+m)]^{1/2}} \tag{4.5}$$

and turns out to be $B^{-1}\Lambda$, where Λ is the Hermitian boost now used in (4.1),

$$\Lambda = \frac{\omega + m + \gamma^0 \vec{\gamma} \cdot \vec{p}}{[2m(m+\omega)]^{1/2}} . \tag{4.6}$$

Such a rotation has been know for some time in connection with the definition of the representations of the Poincaré group in E(2) bases.[28]

In a similar way the rotation associated to (1.10) can be formally written as

$$\exp\left(\theta \frac{\vec{\gamma}_\perp \cdot \vec{p}_\perp}{|\vec{p}_\perp|} \gamma^3\right) = \frac{aM + mA + \vec{\gamma}_\perp \cdot \vec{p}_\perp \gamma^3 A}{[2aA(mM + p^0P^0 - p^3P^3)]^{1/2}} , \tag{4.7}$$

where $A \equiv P^0 + P^3$ and θ is given by (3.5). Despite the presence of A, it can be shown again that (4.7) corresponds to a Wigner rotation. This has been done by Bucella, Savoy, and Sorba[29] from the study of the representations of the Poincaré group in the meson case; the rotation determined by (4.7) arises as part of the Clebsh-Gordan coefficient[30] which determines the hadron state from the product of the Poincaré group representations defining the quark and antiquark states,

$$B^{-1}(p) L(p \leftarrow P) B(P) , \tag{4.8}$$

where $L(p \leftarrow P)$, the pure Lorentz transformation connecting the unit vectors P^μ/M and p^μ/m, is given by[31]

$$\frac{(Mm+P\cdot p)+[p^0(\vec{\gamma}\cdot\vec{\mathrm{P}})-P^0(\vec{\gamma}\cdot\vec{\mathrm{p}})]\gamma^0}{[2mM(Mm+P\cdot p)]^{1/2}}. \tag{4.9}$$

Direct calculation of (4.8) in the case $\vec{\mathrm{P}}_\perp=0$ (collinearity) gives (4.7). For $P^\mu=(M,\vec{0})$, (4.8) and (4.7) reduce to $B^{-1}\Lambda$ and (4.5) as expected.

V. CONCLUSIONS

In this paper we have studied the set of transformations which lead the ordinary Dirac Hamiltonian H to the form H_W described in the text. Among the transformations of this class [one of the set SO(4)/R] we have shown how the first Melosh transformation can be characterized by the fact of being "minimal," i.e., little-group rotation-free.

A class of suitable transformations $\hat{V}$ in the light-plane quantization formalism has been obtained through the usual conditions plus imposing a SO(4,1) structure on the spacelike transformation V which, through infinite boosting, gives rise to $\hat{V}$. As a result, the class of transformations obtained is seen to be the same as that obtained by Eichten *et al.* using more general arguments. The second Melosh transformation has to be modified (through the inclusion of the factor $M/P^+\sqrt{2}$) if it is to be included in the scheme; this corresponds to the explicit consideration of the hadron state, made up from the individual quarks in the sense of Bucella *et al.*[29]

Finally, we have shown how all the mentioned Melosh transformations in the light-plane formalism can be associated to Wigner rotations.

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$$L(a \leftarrow b) = \left(\frac{1+\lambda}{2}\right)^{1/2} + \left(\frac{2}{1+\lambda}\right)^{1/2} \tfrac{1}{4}[\gamma\cdot a, \gamma\cdot b],$$

where $a^2=b^2=1$, $(ab)^2=\lambda^2>1$; see, for instance, H. Bacry, *Leçons sur la Théorie des Groupes et les Symêtries de les Particules Elêmentaires* (Gordon and Breach, New York, 1967).