PHYSICAL REVIEW D VOLUME 14, NUMBER 4 15 AUGUST 1976

Inconsistencies in the symmetric tensor field description of spin-2 particles in an external homogeneous magnetic field

P. M. Mathews, M. Seetharaman, and J. Prabhakaran*
Department of Theoretical Physics, University of Madras, Madras-600025, India
(Received 5 January 1976)

We present a systematic investigation of the problem of a spin-2 particle, described by a symmetric tensor wave function, moving in a homogeneous magnetic field (h.m.f.). An interaction involving a multiple of the Federbush term, besides the minimal interaction, is considered. By explicit solution of the wave equation with an external h.m.f., we show that the energy spectrum of the spin-2 particle, like that of the spin-3/2 particle (Rarita-Schwinger theory with minimal coupling) spills over into the complex plane. This happens even for arbitrarily small magnetic fields if the coupling is minimal, while the onset of the trouble is delayed till the field strength rises to $2m^2/3e$ if the Federbush term is included. Our results also throw new light on the problem of the number of constraints, and bring into focus the associated breakdown of Lorentz invariance. We demonstrate that the correct number of constraints exists (irrespective of whether the Federbush term is present or not) if the electromagnetic field is a pure h.m.f.; but the number of constraints is too few for any other electromagnetic field (even one obtainable from a pure h.m.f. by a Lorentz transformation) unless the standard Federbush term is included. The results of our analysis are discussed in relation to a similar analysis for lower spins as well as their implications for higher-spin theories.

I. INTRODUCTION

A number of investigations have been made in recent years into the question of the consistency (or otherwise) of the familiar relativistic wave equations for particles of spin $s>1$ when coupled to external fields. The classic work of Johnson and Sudarshan,[1] which initiated these studies, showed that the local field theory of Rarita and Schwinger[2] for spin-$\frac{3}{2}$ particles cannot be consistently quantized with positive-definite metric when minimal coupling with an external electromagnetic field is introduced. Since then the method of quantization used in their work and alternative methods have been critically examined[3] on the one hand, and on the other studies at the basic c-number level have been made, resulting in the revelation of different types of inconsistencies in particular theories.

Inconsistencies in Lagrangian field theories at the c-number level received wide attention following the work of Velo and Zwanziger,[4] who showed, by an examination of the characteristic surfaces[5] of the Rarita-Schwinger equation with the minimal coupling to the electromagnetic field, that the propagation of the spin-$\frac{3}{2}$ field so described is noncausal. Further work[6] on these lines has brought to light a variety of theories of spin 1, $\frac{3}{2}$, and 2 with specific interactions wherein noncausal propagation occurs, and a couple of (spin-$\frac{3}{2}$) theories which are free of it.[7,8] Another type of trouble is typified by the stationary-state problem of charged spin-1 particles, having anomalous magnetic moment, moving in a constant homogeneous magnetic field (h.m.f.). As was shown by Tsai and collaborators,[9] the energy spectrum of such a particle is not purely real if the magnetic field is very large. The use of a new method introduced by Mathews[10] not only enabled this result to be demonstrated with much greater ease, but also made it possible to make similar calculations for $s>1$. Explicit solutions have in fact been obtained for the "stationary states" or "normal modes" of a spin-$\frac{3}{2}$ particle in an h.m.f.,[11,12] and it has been found in the case of the Rarita-Schwinger formalism with minimal coupling that complex eigenfrequencies appear[13,14] if the magnetic field strength $\mathcal{H}$ exceeds $3m^2/2e$.

Our objective in the present paper is to investigate the consistency problem for spin-2 particles (described by a symmetric second-rank tensor field) coupled to the electromagnetic field. In particular we apply the method of Refs. 10–12 to obtain the exact solutions and energy spectrum for spin-2 particles in an h.m.f. in the case of minimal coupling as well as in the presence of an additional interaction which we refer to as being of the generalized Federbush type (see below). We find that the energy spectrum spills over into the complex plane in all cases. This happens only for high magnetic fields if the nonminimal term is chosen to coincide with that of Federbush,[15] but with minimal coupling the trouble is present for arbitrarily low field strengths.

Apart from these results, our work gives new insight into another kind of pathological behavior, namely the extra degrees of freedom which crop up in the presence of the electromagnetic field. Attention was focused on this phenomenon by the work of Federbush.[15] Starting with a Lagrangian leading to a system of first-order differential equations for a 50-component wave function (made up of a symmetric second-rank tensor $\psi_{\mu\nu}$ and a third-rank tensor $\Gamma_{\mu\nu\lambda}=\Gamma_{\nu\mu\lambda}$), Federbush showed that on introducing minimal electromagnetic coupling the number of independent components rises from 10 (for a free spin-2 particle and its antiparticle) to 12, and then proposed the "Federbush term," whose introduction removes this difficulty. Actually the Federbush equations, when reduced to the second-order form, coincide with the spin-2 equations obtained by Fierz and Pauli much earlier.[16] This was recently demonstrated explicitly by Hagen,[17] who also showed that the very same second-order equations follow also from a *minimally* coupled first-order theory involving a 30-component wave function (a symmetric $\psi_{\mu\nu}$ together with a suitably defined third-rank tensor with 20 components) which had been given earlier by Chang.[18] Nath[19] and Tait[20] have directly generalized the Fierz-Pauli formulation to obtain a one-parameter family of Lagrangians which lead to the correct number of degrees of freedom with minimal coupling. However, this formulation does not succeed in achieving consistency at the second-quantized level, as shown by Nath himself. Unlike the Fierz-Pauli equations, the second-order equations obtained from Wentzel's Lagrangian[21] with minimal coupling yield an excessive number of degrees of freedom, as noted by Velo and Zwanziger[6] while proving the noncausality of propagation in the latter case. The difference between these two sets of equations results from an extra term (referred to as the Federbush term) in the Lagrangian of the second-order formulation. Much of the confusion resulting from the lack of a definite correspondence between minimality and the number of degrees of freedom can be traced to the derivative-ordering ambiguity in the definition of minimality itself when second-order derivatives are present.[22] The explicit solutions given in this paper serve to display clearly another facet of the consistency problem related to the appearance of extra degrees of freedom: It brings into focus the associated breakdown of Lorentz invariance. We show that when the external field is an h.m.f., there are only five independent components (and five first time derivatives) among the $\psi^{\mu\nu}$ for all couplings of the generalized Federbush type, but that for any other electromagnetic field the number of independent components becomes six except when the coupling coincides with that of Federbush; i.e., in all but this particular case, a Lorentz transformation which changes an h.m.f. into a more general (crossed electric and magnetic) field will also increase the number of degrees of freedom.

The plan of this paper is as follows. In the next section we write down the spin-2 Lagrangian including a term (the "generalized Federbush term") which becomes equal to the Federbush term when the constant factor α appearing in it is set equal to unity. We solve the equation in Sec. II B for the case when the external field is an h.m.f. Since the method of solution is basically the same as what we have employed in treating the spin-1 and spin-$\frac{3}{2}$ case in earlier papers,[10-12] we give here only the essential steps. The energy spectrum is to be obtained by the solution of certain quadratic and cubic equations and it is noted in particular that there are only five branches of E^2 (indicating just five degrees of freedom) irrespective of the value of α. The nature of the spectrum in the two most interesting cases $\alpha=0$ (minimal coupling) and $\alpha=1$ (with Federbush term) is discussed in Sec. II C. In Sec. II D we remark briefly on the value of the g factor in the two cases, and then go on in Sec. II E to verification of the number of independent degrees of freedom from the equations of motion direct. We conclude with Sec. III, wherein the main results of the paper are discussed against the background of what is known in the case of lower spins, and their implications analyzed.

II. SPIN-TWO PARTICLE IN AN h.m.f.

A. Equation of motion

The spin-2 particle in an electromagnetic field will be described here by a symmetric tensor field $\psi^{\mu\nu}=\psi^{\nu\mu}$ obeying second-order differential equations following from the Lagrangian density

$$\begin{aligned}L&=(\pi_\lambda\psi_{\mu\nu})^\dagger(\pi^\lambda\psi^{\mu\nu})-2(\pi_\lambda\psi_{\mu\nu})^\dagger(\pi^\mu\psi^{\lambda\nu})\\&\quad+(\pi^\mu\psi_{\mu\nu})^\dagger(\pi^\nu\psi)+(\pi_\mu\psi)^\dagger(\pi_\lambda\psi^{\mu\lambda})\\&\quad-(\pi_\mu\psi)^\dagger(\pi^\mu\psi)-m^2(\psi^\dagger_{\mu\nu}\psi^{\mu\nu}-\psi^\dagger\psi)\\&\quad-2(\tfrac{1}{2}ie\alpha)\psi^\dagger_{\mu\nu}F^{\mu\rho}\psi_\rho{}^\nu,\end{aligned}\tag{1}$$

wherein

$$\psi\equiv\psi^\mu{}_\mu$$

and (2)

$$\pi_\mu=i\partial_\mu+eA_\mu.$$

The term involving the electromagnetic field $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ explicitly is of the Federbush type,[15] but the real parameter α is left arbitrary here. It reduces to the Federbush term when $\alpha=1$, while $\alpha=0$ corresponds to "minimal coupling." The Euler-Lagrange equations obtained from (1) by the standard variational procedure are

$$L^{\mu\nu} \equiv (\pi^2 - m^2)(\psi^{\mu\nu} - g^{\mu\nu}\psi) - \pi_\lambda(\pi^\mu\psi^{\lambda\nu} + \pi^\nu\psi^{\lambda\mu}) + \tfrac{1}{2}(\pi^\mu\pi^\nu + \pi^\nu\pi^\mu)\psi + g^{\mu\nu}\pi_\rho\pi_\lambda\psi^{\rho\lambda} - \tfrac{1}{2}ie\alpha(F^\mu{}_\rho\psi^{\rho\nu} + F^\nu{}_\rho\psi^{\rho\mu}) = 0. \tag{3}$$

In the following we restrict ourselves to the case when $F_{\mu\nu}$ corresponds to a homogeneous and constant magnetic field.

B. Solution of the equation in an h.m.f.

We take the direction of the h.m.f. to be along the z axis so that $F_{12} = -F_{21} = \mathcal{H}$ and all other components of $F_{\mu\nu}$ are zero. Further, we set[23] $\pi_3 = 0$. Then three of the equations of the set (3), namely those corresponding to $(\mu,\nu) = (0,3)$, $(1,3)$, and $(2,3)$, get decoupled from the rest:

$$L^{03} \equiv (\pi_1{}^2 + \pi_2{}^2 + m^2)\psi^{03} + \pi_0(\pi_1\psi^{31} + \pi_2\psi^{32}) = 0, \tag{4a}$$

$$L^{13} \equiv (\pi_0{}^2 - m^2 - \pi_1{}^2 - \pi_2{}^2)\psi^{31} + \pi_0\pi_1\psi^{03} + \pi_1(\pi_1\psi^{31} + \pi_2\psi^{32}) + ie\mathcal{H}(\tfrac{1}{2}\alpha - 1)\psi^{23} = 0, \tag{4b}$$

and

$$L^{23} \equiv (\pi_0{}^2 - m^2 - \pi_1{}^2 - \pi_2{}^2)\psi^{23} + \pi_0\pi_2\psi^{03} + \pi_2(\pi_1\psi^{31} + \pi_2\psi^{32}) - ie\mathcal{H}(\tfrac{1}{2}\alpha - 1)\psi^{31} = 0. \tag{4c}$$

The remaining seven equations of the set (3) will be taken up after solving Eqs. (4).

We seek solutions with the time dependence e^{-iEt} and so we replace $\pi_0 \equiv p_0$ by E. Further, we observe that the operators a, $a^\dagger$ defined by

$$a = (2e\mathcal{H})^{-1/2}\pi_+, \quad a^\dagger = (2e\mathcal{H})^{-1/2}\pi_-, \quad \pi_\pm = \pi_1 \pm i\pi_2 \tag{5}$$

obey an algebra identical to that of the harmonic-oscillator annihilation and creation operators, and we exploit this fact as in Refs. 10–12 to reduce the differential equations (4) to algebraic equations. First we rewrite these equations in terms of a, $a^\dagger$ and the "number operator"

$$N \equiv a^\dagger a \tag{6}$$

as

$$[1 + (2N+1)\xi]\phi_3 + \epsilon(\tfrac{1}{2}\xi)^{1/2}(a\phi_- + a^\dagger\phi_+) = 0, \tag{7a}$$

$$[\epsilon^2 - 1 - (2N+1)\xi]\phi_+ + \epsilon(2\xi)^{1/2}a\phi_3 + \xi a(a\phi_- + a^\dagger\phi_+) + \xi(\tfrac{1}{2}\alpha - 1)\phi_+ = 0, \tag{7b}$$

and

$$[\epsilon^2 - 1 - (2N+1)\xi]\phi_- + \epsilon(2\xi)^{1/2}a^\dagger\phi_3 + \xi a^\dagger(a\phi_- + a^\dagger\phi_+) - \xi(\tfrac{1}{2}\alpha - 1)\phi_- = 0, \tag{7c}$$

wherein we have written

$$\xi = e\mathcal{H}/m^2, \quad \epsilon = E/m, \quad \phi_3 = \psi^{03}, \tag{8}$$

and

$$\phi_\pm = \psi^{13} \pm i\psi^{23}.$$

If we now introduce the number eigenstates $|n\rangle$ defined by

$$N|n\rangle = n|n\rangle, \quad a|n\rangle = \sqrt{n}\,|n-1\rangle, \quad a^\dagger|n\rangle = (n+1)^{1/2}|n+1\rangle, \quad n = 0,1,2,\ldots \tag{9}$$

and take the functions ϕ_3, ϕ_+, and ϕ_- to be expressed in terms of these, then it becomes evident on inspection of Eqs. (7) that they have solutions of the form

$$\phi_3 = c_3|n\rangle, \quad \phi_+ = c_+|n-1\rangle, \quad \phi_- = c_-|n+1\rangle, \quad n = 1,2,\ldots \tag{10}$$

and also the special solutions

$$\phi_3 = c_3|0\rangle, \quad \phi_+ = 0, \quad \phi_- = c_-|1\rangle, \tag{11a}$$

and

$$\phi_3 = \phi_+ = 0, \quad \phi_- = c_-|0\rangle. \tag{11b}$$

Here the c's are undetermined constants. On substituting (10) into Eqs. (7) and using Eqs. (9), we are led to a set of three algebraic equations. These equations have nontrivial solutions only if

$$\begin{vmatrix} 1 + (2n+1)\xi & \tfrac{1}{2}\epsilon\rho_n & \tfrac{1}{2}\epsilon\rho_{n+1} \\ \epsilon\rho_n & \epsilon^2 - 1 - n\xi + \tfrac{1}{2}\alpha\xi & \tfrac{1}{2}\rho_n\rho_{n+1} \\ \epsilon\rho_{n+1} & \tfrac{1}{2}\rho_n\rho_{n+1} & \epsilon^2 - 1 - (n+1)\xi - \tfrac{1}{2}\alpha\xi \end{vmatrix} = 0, \tag{12}$$

where

$$\rho_n = (2n\xi)^{1/2}. \tag{13}$$

On simplification, (12) reduces to a quadratic equation in ϵ^2,

$$\epsilon^4 - \epsilon^2[2(2n+1)\xi + 2 + \xi^2(1+\tfrac{1}{2}\alpha)] + [(2n+1)\xi + 1][(2n+1)\xi + 1 - \tfrac{1}{2}\alpha(1+\tfrac{1}{2}\alpha)\xi^2] = 0, \tag{14}$$

which requires that

$$\epsilon^2 = (2n+1)\xi + 1 + \tfrac{1}{2}(1+\tfrac{1}{2}\alpha)\xi^2 \pm (1+\tfrac{1}{2}\alpha)\xi[(2n+1)\xi + 1 + \tfrac{1}{4}\xi^2]^{1/2}. \tag{15}$$

It is evident from Eq. (15) that ϵ^2 is positive for both the values of α in which we are interested, namely, $\alpha = 0$ and $\alpha = 1$. The eigenvalues ϵ are then real irrespective of the external field strength $\mathcal{H}$.

In the case of the special solutions (11a) and (11b) we have respectively the following solutions for ϵ^2:

$$\epsilon^2 = (1+\xi)[1 + \xi(1+\tfrac{1}{2}\alpha)] \tag{16a}$$

and

$$\epsilon^2 = 1 + \tfrac{1}{2}\alpha\xi. \tag{16b}$$

Here again, ϵ^2 is positive for any ξ if $\alpha = 0$ or 1. Thus no difficulties arise from the three equations so far considered.

Turning now to the remaining seven members of the set (3), namely those corresponding to $(\mu, \nu) = (3,3)$, $(0,0)$, $(0,1)$, $(0,2)$, $(1,2)$, $(1,1)$, and $(2,2)$, we find that they have the following forms:

$$L^{33} \equiv (\pi_0{}^2 - m^2)(\psi^{11} + \psi^{22}) + (\pi_1{}^2 + \pi_2{}^2 + m^2)\psi^{00} - [\pi_1{}^2\psi^{22} + \pi_2{}^2\psi^{11} - (\pi_1\pi_2 + \pi_2\pi_1)\psi^{12}] + 2\pi_0(\pi_1\psi^{01} + \pi_2\psi^{02}) = 0, \tag{17a}$$

$$L^{00} \equiv m^2(\psi^{11} + \psi^{22}) + (m^2 + \pi_1{}^2 + \pi_2{}^2)\psi^{33} + \pi_1{}^2\psi^{22} + \pi_2{}^2\psi^{11} - (\pi_1\pi_2 + \pi_2\pi_1)\psi^{12} = 0, \tag{17b}$$

$$L^{01} \equiv m^2\psi^{01} - \pi_2(\pi_1\psi^{02} - \pi_2\psi^{01}) - \pi_0(\pi_1\psi^{22} - \pi_2\psi^{12}) - \tfrac{1}{2}ie\alpha\mathcal{H}\psi^{02} - \pi_0\pi_1\psi^{33} = 0, \tag{17c}$$

$$L^{02} \equiv m^2\psi^{02} + \pi_1(\pi_1\psi^{02} - \pi_2\psi^{01}) + \pi_0(\pi_1\psi^{12} - \pi_2\psi^{11}) + \tfrac{1}{2}ie\alpha\mathcal{H}\psi^{01} - \pi_0\pi_2\psi^{33} = 0, \tag{17d}$$

$$L^{12} \equiv (\pi_0{}^2 - m^2)\psi^{12} + \pi_0(\pi_1\psi^{02} + \pi_2\psi^{01}) + \tfrac{1}{2}(\pi_1\pi_2 + \pi_2\pi_1)(\psi^{00} - \psi^{33}) + \tfrac{1}{2}ie(\alpha - 1)\mathcal{H}(\psi^{22} - \psi^{11}) = 0, \tag{17e}$$

$$L^{11} \equiv (\pi_0{}^2 - m^2)\psi^{22} + (\pi_2{}^2 + m^2)\psi^{00} + (\pi_0{}^2 - \pi_2{}^2 - m^2)\psi^{33} + 2\pi_0\pi_2\psi^{02} - ie(\alpha - 1)\mathcal{H}\psi^{12} = 0, \tag{17f}$$

$$L^{22} \equiv (\pi_0{}^2 - m^2)\psi^{11} + (\pi_1{}^2 + m^2)\psi^{00} + (\pi_0{}^2 - \pi_1{}^2 - m^2)\psi^{33} + 2\pi_0\pi_1\psi^{01} + ie(\alpha - 1)\mathcal{H}\psi^{12} = 0. \tag{17g}$$

We now introduce the notations

$$\begin{aligned} &\chi_\pm = \psi^{01} \pm i\psi^{02}, \quad \omega_0 = \psi^{00}, \quad \omega_3 = \psi^{33}, \\ &\theta_\pm = \psi^{11} - \psi^{22} \pm 2i\psi^{12}, \quad \theta_3 = \psi^{11} + \psi^{22}. \end{aligned} \tag{18}$$

In terms of these combinations, the equations (17a)–(17g) become

$$\epsilon(2\xi)^{1/2}(a^\dagger\chi_+ + a\chi_-) + [1 + (2a^\dagger a + 1)\xi]\omega_0 + \tfrac{1}{2}\xi(a^\dagger)^2\theta_+ + \tfrac{1}{2}\xi a^2\theta_- + [\epsilon^2 - 1 - \tfrac{1}{2}(2a^\dagger a + 1)\xi]\theta_3 = 0, \tag{19a}$$

$$[1 + (2a^\dagger a + 1)\xi]\omega_3 - \tfrac{1}{2}\xi(a^\dagger)^2\theta_+ - \tfrac{1}{2}\xi a^2\theta_- + [1 + \tfrac{1}{2}(2a^\dagger a + 1)\xi]\theta_3 = 0, \tag{19b}$$

$$[1 + \xi(aa^\dagger - \tfrac{1}{2}\alpha)]\chi_+ - \xi a^2\chi_- - \epsilon(2\xi)^{1/2}a\omega_3 + \tfrac{1}{2}\epsilon(2\xi)^{1/2}(a^\dagger\theta_+ - a\theta_3) = 0, \tag{19c}$$

$$\xi(a^\dagger)^2\chi_+ - [1 + \xi(a^\dagger a + \tfrac{1}{2}\alpha)]\chi_- + \epsilon(2\xi)^{1/2}a^\dagger\omega_3 - \tfrac{1}{2}\epsilon(2\xi)^{1/2}(a\theta_- - a^\dagger\theta_3) = 0, \tag{19d}$$

$$2\epsilon(2\xi)^{1/2}a^\dagger\chi_- + 2\xi(a^\dagger)^2(\omega_0 - \omega_3) + [\epsilon^2 - 1 - (\alpha - 1)\xi]\theta_- = 0, \tag{19e}$$

$$[\epsilon^2 - 1 + (\alpha - 1)\xi]\theta_+ + (2\xi)^{1/2}2\epsilon a\chi_+ + 2\xi a^2(\omega_0 - \omega_3) = 0, \tag{19f}$$

$$\epsilon(2\xi)^{1/2}(a^\dagger\chi_+ + a\chi_-) + [2\epsilon^2 - 2 - (2a^\dagger a + 1)\xi]\omega_3 + [2 + (2a^\dagger a + 1)\xi]\omega_0 - (\epsilon^2 - 1)\theta_3 = 0. \tag{19g}$$

One can see readily from an inspection of this set of equations that it has solutions of the form

$$\chi_+ = b_+|n-1\rangle, \quad \chi_- = b_-|n+1\rangle, \quad \omega_3 = b_3|n\rangle, \quad \omega_0 = b_0|n\rangle, \theta_+ = b_+'|n-2\rangle, \quad \theta_- = b_-'|n+2\rangle, \quad \theta_3 = b_3'|n\rangle, \quad n = 2, 3, \ldots \tag{20}$$

Apart from these solutions for general $n > 1$, the following special cases also exist:

$$\chi_+ = b_+|0\rangle, \quad \chi_- = b_-|2\rangle, \quad \omega_3 = b_3|1\rangle, \quad \omega_0 = b_0|1\rangle, \quad \theta_+ = 0, \quad \theta_- = b_-'|3\rangle, \quad \theta_3 = b_3'|1\rangle, \tag{21a}$$

$$\chi_+ = 0, \quad \chi_- = b_-|1\rangle, \quad \omega_3 = b_3|0\rangle, \quad \omega_0 = b_0|0\rangle, \quad \theta_+ = 0, \quad \theta_- = b_-'|2\rangle, \quad \theta_3 = b_3'|0\rangle, \tag{21b}$$

$$\chi_+ = 0, \quad \chi_- = b_-|0\rangle, \quad \omega_3 = \omega_0 = 0, \quad \theta_+ = 0, \quad \theta_- = b_-'|1\rangle, \quad \theta_3 = 0, \tag{21c}$$

$$\chi_+ = 0, \quad \chi_- = 0, \quad \omega_3 = \omega_0 = 0, \quad \theta_+ = 0, \quad \theta_- = b_-'|0\rangle, \quad \theta_3 = 0. \tag{21d}$$

Substitution of (20) into Eqs. (19) leads to a set of seven linear, homogeneous equations in the b's and the

b''s. For the existence of nontrivial solutions of these equations, it is necessary that

$$\begin{bmatrix} \epsilon^2-1+(\alpha-1)\xi & 0 & 0 & 2\epsilon\rho_{n-1} & 0 & -\rho_n\rho_{n-1} & \rho_n\rho_{n-1} \\ 0 & \epsilon^2-1-(\alpha-1)\xi & 0 & 0 & 2\epsilon\rho_{n+2} & -\rho_{n+1}\rho_{n+2} & \rho_{n+1}\rho_{n+2} \\ 0 & 0 & \epsilon^2-1 & \epsilon\rho_n & \epsilon\rho_{n+1} & 2\epsilon^2-2-(2n+1)\xi & 2+(2n+1)\xi \\ \rho_n\rho_{n-1} & \rho_{n+1}\rho_{n+2} & 4\epsilon^2-4-(4n+2)\xi & 4\epsilon\rho_n & 4\epsilon\rho_{n+1} & 0 & 4+(8n+4)\xi \\ -\rho_n\rho_{n-1} & -\rho_{n+1}\rho_{n+2} & 4+(4n+2)\xi & 0 & 0 & 4+(8n+4)\xi & 0 \\ \epsilon\rho_{n-1} & 0 & -\epsilon\rho_n & 2+2n\xi-\alpha\xi & -\rho_n\rho_{n+1} & -2\epsilon\rho_n & 0 \\ 0 & \epsilon\rho_{n+2} & -\epsilon\rho_{n+1} & -\rho_n\rho_{n+1} & 2+(2n+2)\xi+\alpha\xi & -2\epsilon\rho_{n+1} & 0 \end{bmatrix} = 0. \tag{22}$$

A tedious, but straightforward evaluation of this determinant shows that we get only a cubic equation for ϵ^2 from (22), irrespective of the value of α. However, the form of the cubic is rather involved and so we limit our discussion to the two special cases $\alpha=0$ and $\alpha=1$.

$\alpha=0$. In this case the cubic equation ($y\equiv\epsilon^2$) has the form

$$\begin{aligned}&[-N\xi^2+(3-11\xi^2-6\xi^4)]\,y^3+[3N^2\xi^2+N(-9+31\xi^2+8\xi^4)+(-9+16\xi^2+18\xi^4)]\,y^2\\ &\quad+(N+1)[-3N^2\xi^2+N(9-26\xi^2-2\xi^4)+(9-11\xi^2-16\xi^4)]\,y+(N+1)^3[N\xi^2+(-3+6\xi^2)]=0,\end{aligned} \tag{23}$$

where

$$N\equiv(2n+1)\xi. \tag{24}$$

$\alpha=1$. Equation (22) now reduces to

$$\begin{aligned}&(4-9\xi^2)(1+\xi^2)\,y^3-[N(12-10\xi^2-9\xi^4)+12+19\xi^2-6\xi^4]\,y^2\\ &\quad+[N^2(12-5\xi^2)+6N(4+2\xi^2-\xi^4)+(12+17\xi^2-12\xi^4)]\,y-(N+1)(N+1-\xi^2)(4+4N-3\xi^2)=0.\end{aligned} \tag{25}$$

In either case we see that the number of branches of the energy spectrum, i.e., the functions giving the dependence of ϵ^2 on n, is five in all: three arising from Eqs. (23) or (25) and two from Eq. (14). Since this is the right number of solutions to have for a spin-2 particle, it would seem that, in the case when the external field is purely magnetic, the equations for $\psi^{\mu\nu}$ would lead to the correct number of constraint relations. Before investigating this point further, we make a brief examination of the reality properties of ϵ.

C. Analysis of the energy spectrum

In examining the nature of the spectrum of eigenvalues, we begin with the special solutions (21) of the field equations. The values of ϵ^2 corresponding to these are determined, respectively, by the following equations:

$$\begin{aligned}&[3(4-9\xi^2)(1+\xi^2)+26a\xi-3a^2(1+4\xi-\xi^2)]\epsilon^4+3(4+12\xi-3\xi^2-4a\xi-a^2)(1+3\xi)[(1+3\xi-\xi^2)+a(1+3\xi+3\xi^2)]\\ &\quad+[a^3(3+12\xi+9\xi^2)+a^2(6+40\xi+81\xi^2+27\xi^3)\\ &\quad-a(12+86\xi+189\xi^2+84\xi^3-27\xi^4)-(24+108\xi+72\xi^2-90\xi^3+9\xi^4-81\xi^5)]\epsilon^2=0,\end{aligned} \tag{26a}$$

$$\begin{aligned}&[3(1+\xi^2)(2-3\xi)+a(3+4\xi-3\xi^2)]\epsilon^4-\{3(4+4\xi-\xi^3-3\xi^4)+a[12+25\xi+15\xi^2+a(1+\xi)(3+\xi)]\}\epsilon^2\\ &\quad+(1+\xi)(2+2\xi+\alpha\xi)[3(1+\xi-\xi^2)+a(3+3\xi+\xi^2)]=0.\end{aligned} \tag{26b}$$

$$[1+(\tfrac{1}{2}\alpha-2)\xi]\epsilon^2=(1+\tfrac{1}{2}\alpha\xi)(1+a), \tag{26c}$$

and

$$\epsilon^2=1+a, \tag{26d}$$

wherein we have set $a=(\alpha-1)\xi$.

Considering first the case $\alpha=0$ (minimal coupling) we see immediately that negative values of ϵ^2 (imaginary ϵ) appear for sufficiently large values of the external magnetic field. While ϵ^2 of (26d) remains positive up to $\xi=1$, Eq. (26c)—corresponding to the mode (21c)—yields

$$\epsilon^2=\frac{1-\xi}{1-2\xi},$$

which leads to negative ϵ^2 already when ξ exceeds $\frac{1}{2}$. (Note that ϵ^2 becomes infinite at $\xi=\frac{1}{2}$ in that mode.)

The inclusion of the Federbush term ($\alpha=1$) does not enable us to escape this malady, though its onset is delayed until ξ reaches the value $\frac{2}{3}$, as far as the mode (21c) is concerned, Eq. (26c) giving $\epsilon^2=(1+\frac{1}{2}\xi)/(1-\frac{3}{2}\xi)$ for $\alpha=1$.

We see thus that modes with imaginary ϵ (exponential time dependence) are present for large enough ξ, whether or not the Federbush term is included. It remains to be seen whether there exists a critical value of ξ below which *all* modes have real frequencies (as in the spin-$\frac{3}{2}$ case).[11] We examine this question now.

It is known[24] that all the roots of a general cubic equation

$$ax^3+3bx^2+3cx+d=0 \tag{27}$$

are real if and only if

$$G^2+4H^3\leqslant 0, \tag{28}$$

where G and H are given by

$$G=a^2d-3abc+2b^3, \tag{29a}$$

$$H=ac-b^2. \tag{29b}$$

It is clear that if the inequality in (28) is to hold, H should be negative. Consequently, whenever $H>0$, Eq. (27) has complex roots. We now examine the sign of H in the case of Eqs. (23) and (25) corresponding to $\alpha=0$ and $\alpha=1$, respectively.

$\alpha=0$. In this case, one finds from Eq. (23) that

$$\begin{aligned}H=\tfrac{1}{9}\,\xi^2[&(-108+125\xi^2+150\xi^4-36\xi^6)\\&+N(-108+244\xi^2-64\xi^4+36\xi^6)\\&+N^2\xi^2(-25+38\xi^2-28\xi^4)+12N^3\xi^4].\end{aligned} \tag{30}$$

Since the positive term $12N^3\xi^4$ dominates for sufficiently large N, *whatever* ξ *may be*, it follows that H is positive, and hence complex values of ϵ for states with large n are unavoidable.

$\alpha=1$. In this case, Eq. (25) applies, and we have

$$G^2+4H^3=-\frac{1}{(27)^2}\sum_{i=0}^{6}A_i(N\xi^2)^{6-i}, \tag{31}$$

wherein

$$\begin{aligned}A_0&=4a_1^3-a_4^2,\\A_1&=12a_1^2a_2-2a_4a_5,\\A_2&=12a_1^2a_3+12a_1a_2^2-24a_4a_6-a_5^2,\\A_3&=24a_1a_2a_3+4a_2^3-2a_5a_6-2a_4a_7,\\A_4&=12a_1a_3^2+12a_2^2a_3-2a_5a_6-a_7^2,\\A_5&=12a_2a_3^2-2a_6a_7,\end{aligned} \tag{32}$$

and

$$A_6=4a_3^3-a_7^2.$$

Here the quantities $a_1,\ldots,a_7$ have the following forms:

$$\begin{aligned}a_1&=133+45\xi^2+81\xi^4,\\a_2&=432+160\xi^2+12\xi^4-54\xi^6,\\a_3&=\xi^2(432+940\xi^2+51\xi^4-288\xi^6),\\a_4&=1870+6507\xi^2-1215\xi^4-1458\xi^6,\\a_5&=10368+27024\xi^2-9567\xi^4-11583\xi^6\\&\quad+1458\xi^8,\\a_6&=\xi^2(62208+34464\xi^2-55278\xi^4\\&\quad-24435\xi^6+16281\xi^8),\\a_7&=\xi^4(51840+64352\xi^2-31248\xi^4\\&\quad-27405\xi^6+11961\xi^8).\end{aligned} \tag{33}$$

It can be seen by inspection of these expressions that all A_i ($i=0$ to 6) are positive for $0<\xi^2<\frac{4}{9}$, and this observation has been confirmed by numerical computation. Hence, we conclude from (31) that for this range of ξ^2, $G^2+4H^3<0$, so that the cubic (25) has all roots real for this range of values of ξ^2.

We see thus that unlike in the case of minimal coupling, there exists a region of magnetic field strengths ($\xi<\frac{2}{3}$) for which all the modes of the spin-2 particle have real E, if the Federbush term is included. The situation then is akin to that in the Rarita-Schwinger theory. It will be recalled[11] that in the latter case complex energy modes begin to appear when ξ exceeds $\frac{3}{2}$. In the matter of noncausality of propagation also the two are entirely parallel: As shown by Velo (see in Ref. 6) the propagation of the spin-2 field, with $\alpha=1$, like that of the minimally coupled Rarita-Schwinger field,[4] is noncausal in the presence of arbitrarily small external fields.

D. The g factor

An inspection of the small-ξ limit of the energy levels given by Eqs. (23) for $\alpha=0$ and Eqs. (25) for $\alpha=1$ enables us to deduce the g factors in the two cases. In the ground state, the z component of spin has absolute value 2 units, and therefore half the difference between the energy of this state and the ground-state energy of the spinless state (the latter being given by $\epsilon^2=1+\xi$) is $g(e\mathcal{H}/2m)$ in the limit of small magnetic fields. In the light of this observation one finds readily that

$$g=1 \text{ for } \alpha=0 \quad \text{and} \quad g=\tfrac{1}{2} \text{ for } \alpha=1. \tag{34}$$

The pattern of the higher-energy levels, including their multiplicity, is in agreement with these assignments: The levels are just what one would get if the levels of the spinless case were assumed to be split into quintuplets with spacing corresponding to the above g values.

We remark here that the g value in the presence of the Federbush term ($\alpha=1$) is the "canonical" value ($1/s$). It is intriguing to note that while in the Rarita-Schwinger theory (where, with minimal coupling, g has the canonical value $\frac{2}{3}$) complex energy eigenvalues occur for $g\xi>1$, in the spin-2 theory with $\alpha=1$ such eigenvalues occur not for $g\xi\equiv\frac{1}{2}\xi>1$ but for $\frac{3}{2}\xi>1$.

E. Derivation of the constraints in the case of an h.m.f.

We return now to the question of the number of degrees of freedom. Since the basic equations (3) are second-order equations for the ten independent components of the symmetric tensor $\psi^{\mu\nu}$ it would appear *a priori* that there are 20 initial conditions (the initial values of $\psi^{\mu\nu}$ and their first time derivatives) to be specified. However, for a particle with spin $s=2$, one has only $2(2s+1)=10$ independent degrees of freedom available and therefore $20-10=10$ constraints are needed. These constraints should follow from the basic equations (3). It has been known (see, for instance, Velo and Zwanziger in Ref. 6) for some time that with just minimal coupling ($\alpha=0$) to arbitrary external electromagnetic fields, Eqs. (3) do not yield the requisite number of constraints, so that there are apparently more degrees of freedom than there should be. Velo[6] has shown recently that this situation obtains for all $\alpha\neq1$. In view of this, the results we have obtained for a pure h.m.f. appear rather surprising. They seem to indicate that in this special case, the number of degrees of freedom is just the right number, implying thereby that there is no loss of constraints. An examination of the constraints following from (3) seems therefore to be called for, and we proceed to do this now.

We note, first of all, that of the ten equations (4a)–(4c) and (17a)–(17g), four equations, namely $L^{00}=0$ [Eq. (17b)], $L^{01}=0$ [Eq. (17c)], $L^{02}=0$ [Eq. (17d)], and $L^{03}=0$ [Eq. (4a)] are evidently constraints, since they involve no second-order time derivatives. To derive the remaining constraints, we proceed systematically as follows: We differentiate these four constraints with respect to time, and see if the second-order time-derivative terms occurring in any of the resulting four equations can be eliminated in favor of lower-order time derivatives by virtue of the equations of motion. If this is possible, new constraint equations result. We then differentiate these too, and see if the second-order time derivatives can be eliminated. We continue this process, ensuring each time also that the constraints so derived are linearly independent (this can be done by comparing their first-order time-derivative parts), until it is no longer possible to derive any more constraints.

Now differentiating (17b), (17c), (17d), and (4a) with respect to time, we get respectively

$$(\pi_2^2+m^2)\pi_0\psi^{11}+(\pi_1^2+m^2)\pi_0\psi^{22}+(\pi_1^2+\pi_2^2+m^2)\pi_0\psi^{33}-(\pi_1\pi_2+\pi_2\pi_1)\pi_0\psi^{12}=0, \tag{35a}$$

$$(\pi_2^2+m^2)\pi_0\psi^{01}+\pi_2\pi_0^2\psi^{12}-\pi_2\pi_1\pi_0\psi^{02}-\pi_1\pi_0^2(\psi^{22}+\psi^{33})-\tfrac{1}{2}ie\,\alpha\mathcal{H}\pi_0\psi^{02}=0, \tag{35b}$$

$$(\pi_1^2+m^2)\pi_0\psi^{02}+\pi_1\pi_0^2\psi^{12}-\pi_1\pi_2\pi_0\psi^{01}-\pi_2\pi_0^2(\psi^{11}+\psi^{33})+\tfrac{1}{2}ie\,\alpha\mathcal{H}\pi_0\psi^{01}=0, \tag{35c}$$

and

$$(\pi_1^2+\pi_2^2+m^2)\pi_0\psi^{03}+\pi_1\pi_0^2\psi^{31}+\pi_2\pi_0^2\psi^{23}=0. \tag{35d}$$

It is evident that (35a), as it stands, is a constraint. The terms involving π_0^2 in (35b) may be eliminated by means of the equations of motion (17e) and (17f), and when this is done we get the constraint

$$m^2(\pi_1\psi^{22}-\pi_2\psi^{12}-\pi_0\psi^{01})+\tfrac{1}{2}ie(\alpha-4)\mathcal{H}\pi_0\psi^{02}+[\tfrac{1}{2}ie\mathcal{H}\pi_2+\pi_2^2\pi_1-\pi_1(\pi_2^2+m^2)](\psi^{00}-\psi^{33})$$
$$+ie(\alpha-1)\mathcal{H}[\tfrac{1}{2}\pi_2(\psi^{11}-\psi^{22})-\pi_1\psi^{12}]=0. \tag{36}$$

The terms in $\pi_0{}^2$ may be eliminated also from (35c) by using Eqs. (17g) and (17e). We then get

$$m^2\pi_0\psi^{02}+\tfrac{1}{2}ie\mathcal{H}(\alpha-4)\pi_0\psi^{01}+m^2\pi_1\psi^{12}-ie\mathcal{H}(1-\alpha)\pi_2\psi^{12}-(\pi_1{}^2\pi_2-\pi_2{}^2\pi_1-m^2\pi_2-\tfrac{1}{2}ie\mathcal{H}\pi_1)(\psi^{00}-\psi^{33})$$
$$-[\tfrac{1}{2}ie\mathcal{H}(1-\alpha)\pi_1+m^2\pi_2]\psi^{11}+\tfrac{1}{2}ie\mathcal{H}(1-\alpha)\pi_1\psi^{22}=0. \quad (37)$$

Finally, we find that Eqs. (4b) and (4c) may be used to eliminate the $\pi_0{}^2$ terms from (35d); this gives

$$m^2\pi_0\psi^{03}+m^2(\pi_1\psi^{31}+\pi_2\psi^{32})-ie\mathcal{H}(1+\tfrac{1}{2}\alpha)(\pi_1\psi^{23}-\pi_2\psi^{13})=0. \quad (38)$$

We have thus generated four more constraints: (35a), (36), (37), and (38). The next step is to differentiate these, to check whether any further constraints result therefrom. On differentiating (35a), we get

$$\pi_0{}^2[\pi_1{}^2\psi^{22}+\pi_2{}^2\psi^{11}+(\pi_1{}^2+\pi_2{}^2)\psi^{33}]-(\pi_1\pi_2+\pi_2\pi_1)\pi_0{}^2\psi^{12}+m^2\pi_0{}^2(\psi^{11}+\psi^{22}+\psi^{33})=0.$$

In attempting to eliminate $\pi_0{}^2$ terms from this equation, we observe first that on adding the three equations (17a), (17f), and (17g), one gets $\pi_0{}^2(\psi^{11}+\psi^{22}+\psi^{33})$ in terms of lower-order time derivatives. Then, operating on the constraints (17c) and (17d) respectively by $\pi_0\pi_1$ and $\pi_0\pi_2$ and adding, we obtain

$$\pi_0{}^2\{(\pi_1\pi_2+\pi_2\pi_1)\psi^{12}-[\pi_1{}^2\psi^{22}+\pi_2{}^2\psi^{11}+(\pi_1{}^2+\pi_2{}^2)\psi^{33}]\}$$

in terms of lower-order time derivatives. The net effect of these operations is to form a combination such as $\pi_0{}^2L^{00}+2\pi_0\pi_iL^{0i}+\pi_i\pi_jL^{ij}+\frac{1}{2}m^2(L^{00}-L^{ii})$ whose vanishing gives the explicit equation

$$ie(1-\alpha)\mathcal{H}\pi_0(\pi_2\psi^{01}-\pi_1\psi^{02})-ie(1-\alpha)\mathcal{H}[(\pi_1{}^2-\pi_2{}^2)\psi^{12}-\pi_1\pi_2\psi^{11}+\pi_2\pi_1\psi^{22}]+\tfrac{3}{2}m^4\psi+\tfrac{3}{2}e^2\mathcal{H}^2(\psi^{00}-\psi^{33})=0. \quad (39)$$

This is the ninth constraint.

The structure of (39) shows at once that if we choose $\alpha=1$, then this constraint does not contain any time-derivative terms at all. This means that the tenth (and the final) constraint is simply the first time derivative of (39) when $\alpha=1$.

If we take $\alpha\neq1$, differentiation of (39) with respect to time gives rise to the terms $\pi_0{}^2\psi^{01}$ and $\pi_0{}^2\psi^{02}$ which require elimination. This can in fact be done. Differentiating the constraints (36) and (37) we get

$$\pi_0{}^2[m^2\psi^{01}-\tfrac{1}{2}ie(\alpha-4)\mathcal{H}\psi^{02}]=m^2\pi_0(\pi_1\psi^{22}-\pi_2\psi^{12})+\pi_0(\tfrac{1}{2}ie\mathcal{H}\pi_2+\pi_2{}^2\pi_1-\pi_1\pi_2{}^2-m^2\pi_1)(\psi^{00}-\psi^{33})$$
$$+\tfrac{1}{2}ie\mathcal{H}(1-\alpha)\pi_0\pi_2(\psi^{11}-\psi^{22})-ie\mathcal{H}(1-\alpha)\pi_0\pi_1\psi^{12} \quad (40)$$

and

$$\pi_0{}^2[m^2\psi^{02}+\tfrac{1}{2}ie(\alpha-4)\mathcal{H}\psi^{01}]=-m^2\pi_0\pi_1\psi^{12}+ie\mathcal{H}(1-\alpha)\pi_0\pi_2\psi^{12}+\pi_0(\pi_1{}^2\pi_2-\pi_2\pi_1{}^2-m^2\pi_2-\tfrac{1}{2}ie\mathcal{H}\pi_1)(\psi^{00}-\psi^{33})$$
$$+[\tfrac{1}{2}ie(1-\alpha)\mathcal{H}\pi_1+m^2\pi_2]\pi_0\psi^{11}-\tfrac{1}{2}ie(1-\alpha)\mathcal{H}\pi_0\pi_1\psi^{22}, \quad (41)$$

and these may be written as

$$\begin{pmatrix} m^2 & -\frac{1}{2}ie\mathcal{H}(\alpha-4) \\ \frac{1}{2}ie\mathcal{H}(\alpha-4) & m^2 \end{pmatrix}\begin{pmatrix}\pi_0{}^2\psi^{01} \\ \pi_0{}^2\psi^{02}\end{pmatrix}=\begin{pmatrix}\text{right-hand side of (40)} \\ \text{right-hand side of (41)}\end{pmatrix}. \quad (42)$$

Multiplying by the inverse of the 2×2 matrix which occurs on the left, one obtains $\pi_0{}^2\psi^{01}$ and $\pi_0{}^2\psi^{02}$ in terms of the lower-order time derivatives. On substituting these into the equation obtained by time differentiation from (39), we obtain the tenth constraint.[25]

Thus we find that it is possible to derive ten constraints in the case of a pure magnetic field, *whatever be the value of* α. The computation of the number of degrees of freedom is now simple. Our initial system of equations allows 20 free initial data (corresponding to 10 components of $\psi^{\mu\nu}$ and their first time derivatives.) They must be restricted by the nine constraints (4a), (17b)–(17d), (35a), (36)–(39), and the tenth constraint (whose explicit form we do not give here) obtained by time differentiation of (39) for $\alpha=1$, or by substituting for $\pi_0{}^2\psi^{01}$ and $\pi_0{}^2\psi^{02}$ from (42) in the equation obtained by time differentiation of (39) for any $\alpha\neq1$. Therefore, we have only $20-10=10$ free initial data, as required for a spin-2 particle. This completes our demonstration that the number of degrees of freedom in the presence of an interaction with an h.m.f. is just what one expects of a spin-2 particle, irrespective of the presence or absence of a generalized Federbush-type nonminimal term in the Lagrangian.

III. DISCUSSION

It is satisfying to note that both the approaches, namely, the explicit solution of the field equation on the one hand, and enumeration of the constraint relations on the other, lead to identical conclusions regarding the number of degrees of freedom of the symmetric tensor field in an interaction with an h.m.f. However, there is one perplexing question which remains: Suppose one imagines a Lorentz transformation to be performed in some direction not parallel to the magnetic field. What was originally a magnetic field will appear in the new frame as a combination of electric and magnetic fields at right angles to each other. The Lorentz transformation should not affect the number of spin degrees of freedom of the spin-2 field. (If this were not so, one could argue that the two inertial frames would become distinguishable by the different values of the number of degrees of freedom; this would be against the principle of relativity.) However, if one goes back to Eq. (3) and examines the number of constraints in the new frame (where an electric field is also present) one finds that for $\alpha=0$ (in fact, for any $\alpha\neq 1$) the number of constraints is only 8 (see Appendix). This loss of constraints implies that the number of degrees of freedom is greater than is required by the spin value. It does not seem possible to reconcile this situation with the result of Lorentz-invariance arguments which lead one to expect that there are no excess degrees of freedom. What one has here is an explicit manifestation of violation of Lorentz invariance in the spin-2 theory with minimal coupling.

It is interesting to reflect on the growing variety and complexity of the problems encountered in relativistic theories of elementary particles as the spin value increases. For spin $\frac{1}{2}$, the Dirac theory is satisfactory in all respects. As the spin increases to 1, one already runs into trouble: With minimal coupling to a Coulomb field a complete set of solutions seems not to exist,[27] while if a nonminimal interaction via an anomalous magnetic moment is assumed, one is faced with the appearance of imaginary energy eigenvalues in the presence of large external magnetic fields. Nevertheless, propagation of the spin-1 field remains causal[28] (Velo and Zwanziger, Ref. 6). In the spin-$\frac{3}{2}$ case, the Rarita-Schwinger theory not only has the drawback of appearance of complex energy eigenvalues in large magnetic fields, but also exhibits breakdown of causality in propagation even in arbitrarily small electromagnetic fields; these happen already with minimal coupling. Other known formulations of spin $\frac{3}{2}$ (Ref. 13) which are free of these difficulties are marred by an indefinite sign for the total charge. Finally, as the spin value goes up to 2, serious problems about the number of degrees of freedom arise on introducing minimal electromagnetic coupling—a type of difficulty which was not present for any other lower spin. Though this particular difficulty is circumvented by the addition of a suitable nonminimal term, the other difficulties (occurrence of complex energy modes, noncausality of propagation) still persist. The spin-2 theory is also inconsistent at the second-quantized level as has been shown by Nath.[19]

Finally, a comment on the minimality of coupling to the electromagnetic field may be in order. As is well known, different formulations which are completely equivalent in the absence of interactions can lead to different consequences when electromagnetic interactions are introduced "minimally." For instance, the Shay-Good equations[29] for spin 1 yield a g factor $\frac{1}{2}$ unlike the Proca formulation which gives $g=1$. When other properties (causality, nature of the energy spectrum, etc.) of the equations are analyzed, one finds that minimality of electromagnetic coupling in the usual sense is no guarantee of good behavior; on the other hand, some of the inconsistencies such as noncausal propagation and pure imaginary energy values disappear when the coupling is so arranged that the g factor is unity. It seems then that the optimal type of coupling is that which leads to $g=1$ in the spin-1 case. For general spin s, there is a long-standing conjecture[30] that the g factor should be $(1/s)$, which may now be considered as a "principle of optimality" in coupling to the electromagnetic field. The case of spin 2 lends support to such a principle, insofar as the absence of the anomaly in regard to the number of degrees of freedom and the existence of a region of magnetic field values for which energy eigenvalues are all real are ensured for just such a coupling as would give $g=1/s=\frac{1}{2}$. It is this coupling (namely that including the Federbush term) which is optimal, despite its being nonminimal.[31] Unfortunately even with this optimal coupling one is a long way from ridding oneself of pathologies such as noncausality of propagation (as indeed is the case already with spin $\frac{3}{2}$). One may in fact legitimately wonder whether there exist any formulations at all of higher-spin fields which are consistent in all respects when external interactions are introduced. Finding a complete answer to this question remains a fascinating and challenging problem.

ACKNOWLEDGMENT

One of us (J.P.) acknowledges the award of a Junior Research Fellowship by the Council of Scientific and Industrial Research, India.

APPENDIX

We verify here that in a frame in which both the electric and magnetic fields are present the number of constraints is only 8, for any $\alpha\neq 1$.

Before proceeding to do this, it might be useful to display how constraints are derived in a covariant way for a general $F_{\mu\nu}$ which is constant. We observe, first of all, that setting either μ or ν equal to zero in (3) leads to four constraints. The next set of four constraints follows from contracting $L^{\mu\nu}$ with π_μ. We thus have the eight constraints

$$L^{\mu 0}=0\,, \tag{A1}$$

$$\begin{aligned} C^\nu &= -m^{-2}\pi_\mu L^{\mu\nu} \\ &= \pi_\mu \psi^{\mu\nu} - \pi^\nu \psi \\ &\quad - ie\, m^{-2}[F_{\mu\lambda}\pi^\lambda \psi^{\mu\nu} + (F^\nu{}_\mu \pi_\lambda + \pi_\lambda F^\nu{}_\mu)\psi^{\mu\lambda} \\ &\qquad + (\tfrac{1}{2}\pi_\mu F^{\mu\nu} + F^{\mu\nu}\pi_\mu)\psi] \\ &= 0\,. \end{aligned} \tag{A2}$$

To derive the ninth constraint, we contract $L^{\mu\nu}$ with $\pi_\nu\pi_\mu$ and compare it with the trace of $L^{\mu\nu}$. Thus we form the combination

$$\begin{aligned} \pi_\nu\pi_\mu L^{\mu\nu} + \tfrac{1}{2}m^2 L^\mu{}_\mu &= ie(1-\alpha)\pi_\nu\pi_\mu(F^{\mu\lambda}\psi_\lambda{}^\nu) + \tfrac{3}{2}m^4\psi \\ &\quad + \tfrac{3}{2}e^2(F_{\mu\nu}F^\nu{}_\rho \psi^{\mu\rho} - \tfrac{1}{2}F_{\nu\mu}F^{\mu\nu}\psi) \\ &= 0\,. \end{aligned} \tag{A3}$$

It is immediately evident from (A3) that when the electric field is nonvanishing, there exists a second-order time-derivative term $ie(1-\alpha)\pi_0{}^2 F_{0i}\psi^{0i}$. We thus find that while the choice $\alpha=1$ reduces (A3) and its first time derivative to the ninth and the tenth constraints, respectively, constraints will not follow from (A3) for any other α, unless $\pi_0{}^2\psi^{0i}$ $(i=1,2,3)$ can be eliminated in terms of lower derivatives. Now if we are considering a reference frame obtained by a boost from another in which only a pure magnetic field exists, the electric and magnetic fields appear at right angles to each other in the new frame. With the magnetic field along the z direction, the electric field may be arranged to be along the y direction, so that the only nonvanishing component of F_{0i} is $F_{02}=\mathcal{E}$. In such a case, the derivation of the ninth constraint necessitates the elimination of $\pi_0{}^2\psi^{02}$ in favor of lower-order time derivatives. To see if this can be done as before by differentiating the other constraints, we note first that with the electric field also present and $\alpha\neq 1$, the constraints corresponding to (36) and (37) have the forms

$$\pi_0[m^2\psi^{01} - \tfrac{1}{2}ie\mathcal{H}(\alpha-4)\psi^{02} - ie\mathcal{E}\,\pi_1\psi^{02} + \tfrac{1}{2}ie\mathcal{E}(\alpha+2)\psi^{12}] + \cdots = 0 \tag{A4}$$

and

$$\pi_0[m^2\psi^{02} + \tfrac{1}{2}ie\mathcal{H}(\alpha-4)\psi^{01} - \tfrac{3}{2}ie\,\mathcal{E}(\psi^{11}+\psi^{33}) + \tfrac{1}{2}ie(\alpha-1)\mathcal{E}(\psi^{00}+\psi^{22})] + \cdots = 0\,, \tag{A5}$$

where the dots stand for terms containing no time derivatives. If we apply π_0 to these equations and solve for $\pi_0{}^2\psi^{02}$, the resulting expressions will evidently contain $\pi_0{}^2\psi^{12}$, $\pi_0{}^2(\psi^{11}+\psi^{33})$, and $\pi_0{}^2(\psi^{22}+\psi^{00})$ and these in turn have to be reexpressed in terms of lower derivatives. The equations of motion do enable us to eliminate $\pi_0{}^2\psi^{12}$, $\pi_0{}^2(\psi^{11}+\psi^{33})$, and $\pi_0{}^2\psi^{22}$, but the term $\pi_0{}^2\psi^{00}$ gives serious trouble. In fact, it is impossible to eliminate it since ψ^{00} does not have any equation of motion at all. However, it may be noted from (A5) that the coefficient $\frac{1}{2}ie(\alpha-1)\mathcal{E}$ will accompany $\pi_0{}^2\psi^{00}$, so that either the choice $\alpha=1$ or the vanishing of the electric field will eliminate this troublesome term altogether. For any $\alpha\neq 1$, the ninth and the tenth constraints would therefore appear to be lost and we would then be left with more than the requisite number of degrees of freedom for the particle.

*Present Address: Physics Department, Loyola College, Madras-600034, India.

[1]K. Johnson and E. C. G. Sudarshan, Ann. Phys. (N. Y.) 13, 126 (1961).

[2]W. Rarita and J. Schwinger, Phys. Rev. 60, 61 (1941).

[3]S. N. Gupta and W. W. Repko, Phys. Rev. 177, 1921 (1969); J. D. Kimel and L. M. Nath, Phys. Rev. D 6, 2132 (1972); G. B. Mainland and E. C. G. Sudarshan, *ibid.* 8, 1088 (1973); 10, 3343 (1974).

[4]G. Velo and D. Zwanziger, Phys. Rev. 186, 1337 (1969).

[5]R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Wiley-Interscience, New York, 1962), Vol. 2.

[6]G. Velo and D. Zwanziger, Phys. Rev. 188, 2218 (1969); B. Schroer, R. Seiler, and A. Swieca, Phys. Rev. D 2, 2927 (1970); P. Minkowski and R. Seiler, *ibid.* 4, 359 (1971); J. Bellisard and R. Seiler, Lett. Nuovo Cimento 5, 221 (1972); G. Velo, Nucl. Phys. B43, 389 (1972); A. Shamaly and A. Z. Capri, Ann. Phys. (N. Y.) 74, 503 (1972); J. D. Jenkins, J. Phys. A 5, 1461 (1972); 6, 1211 (1973); 7, 1129 (1974); L. P. S. Singh, Phys. Rev. D 7, 1256 (1973); W. Tait, Lett. Nuovo Cimento 7, 368 (1973); A. S. Wightman, *Troubles in the External Field Problem for Invariant Wave Equations* (Gordon and Breach, New York, 1971).

[7]C. Fisk and W. Tait, J. Phys. A 6, 383 (1973).

[8]J. Prabhakaran, M. Seetharaman, and P. M. Mathews, J. Phys. A 8, 560 (1975); Phys. Rev. D 12, 3191 (1975).

[9]W. Y. Tsai and A. Yildiz, Phys. Rev. D 4, 3643 (1971); T. Goldman and W. Y. Tsai, *ibid.* 4, 3648 (1971); W. Y. Tsai, *ibid.* 4, 3652 (1971).

[10]P. M. Mathews, Phys. Rev. D 9, 365 (1974).

[11]M. Seetharaman, J. Prabhakaran, and P. M. Mathews, Phys. Rev. D 12, 458 (1975).

[12]M. Seetharaman, J. Prabhakaran, and P. M. Mathews,

Phys. Rev. D 12, 3195 (1975).

[13]However, the authors have shown recently (Refs. 8 and 12) that there exist some formulations of spin-$\frac{3}{2}$ (Bhabha and Gupta, Ref. 14, and Fisk and Tait, Ref. 7) theories which are free of the difficulties associated with noncausal propagation or the occurrence of complex eigenvalues in an h.m.f. But the introduction of an indefinite metric even for free-field quantization becomes inevitable for these theories.

[14]H. J. Bhabha, Philos. Mag. 43, 33 (1952); K. K. Gupta, Proc. R. Soc. London A122, 118 (1954).

[15]P. Federbush, Nuovo Cimento 19, 572 (1961).

[16]M. Fierz and W. Pauli, Helv. Phys. Acta. 12, 297 (1939); Proc. R. Soc. London A173, 211 (1939).

[17]C. R. Hagen, Phys. Rev. D 6, 984 (1972).

[18]S. J. Chang, Phys. Rev. 148, 1259 (1966). However, even this minimal 30-component theory— not to mention the 50-component theory— is subject to unpleasant problems when a *gravitational* interaction is present, despite its good behavior in electromagnetic fields with just minimal coupling. [See C. Aragone and S. Deser, Nuovo Cimento 3A, 709 (1971); C. Aragone, A. Restuccia, and R. Siri, ICTP (Trieste) report, 1973 (unpublished).]

[19]L. M. Nath, Nucl. Phys. 68, 660 (1965).

[20]W. Tait, Phys. Rev. D 5, 3272 (1972).

[21]C. Wentzel, *Quantum Field Theory* (Wiley-Interscience, New York, 1949), p. 205.

[22]S. Rai Choudhury, Lett. Nuovo Cimento 5, 698 (1972).

[23]The results for $\pi_3 \neq 0$ can be obtained by performing a boost along the z direction, which does not affect $\mathcal{K}$.

[24]W. S. Burnside and A. W. Panton, *The Theory of Equations* (Dover, New York, 1970), Vol. I.

[25]It is curious to note that for a particular value of the magnetic field such that $\frac{1}{4}\xi^2(\alpha-4)^2=1$, the 2×2 matrix occurring on the left-hand side of (42) becomes singular, so that one of the constraints appears to be lost. Simultaneously with this the energy eigenvalue belonging to the special solution (21c), namely, the ϵ^2 given by (26c), tends to infinity. This situation reminds one of the spin-$\frac{3}{2}$ case, where it was shown (Ref. 11) that when $\eta \equiv 2e\mathcal{K}/3m^2$ becomes equal to unity, an extra constraint is generated, and simultaneously one of the values of ϵ becomes infinity. However, there is a striking difference between the two cases: In the spin-$\frac{3}{2}$ case, there is one *more* constraint appearing when $\eta=1$, while in the spin-2 case, there is a *loss* of one of the constraints when $\frac{1}{4}\xi^2(\alpha-4)^2=1$.

[26]Johnson and Sudarshan (Ref. 1) had inferred indirectly, from the form of the anticommutators of the field components, that the Rarita-Schwinger theory with minimal coupling violates Lorentz invariance. However, Mainland and Sudarshan (Ref. 3) have recently reconsidered the question and have given a finding that there is no breakdown of Lorentz invariance if the transformation of the external field also is properly taken into account.

[27]H. C. Corben and J. Schwinger, Phys. Rev. 58, 953 (1940).

[28]This statement is for the Proca form of spin-1 equations. In certain other formulations such as the Shay-Good and Kyriakopoulos-Takahashi-Palmer theories, a specific nonminimal coupling is needed to make the g factor equal to unity, and the inclusion of such a term is necessary both to prevent the appearance of complex energy modes and to ensure causality. See, for example, J. Prabhakaran and M. Seetharaman, Lett. Nuovo Cimento 7, 395 (1973); A. Shamaly and A. Z. Capri, Ann. Phys. (N. Y.) 74, 503 (1972); T. Goldman and W. Y. Tsai, Phys. Rev. D 4, 3648 (1971).

[29]D. Shay and R. H. Good, Jr., Phys. Rev. 179, 1410 (1969).

[30]F. J. Belinfante, Phys. Rev. 92, 994 (1953); 92, 997 (1953). See also L. P. S. Singh and C. R. Hagen, Phys. Rev. D 9, 898 (1974); 9, 910 (1974).

[31]Hagen (Ref. 17) has observed that this theory is equivalent to a first-order formulation with minimal coupling, involving a 30-component wave function.