Rapidity amplitudes and their Fourier transforms

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Use of the rapidity y as a parameter in high-energy multiparticle scattering appears to be natural. In this paper we point out that since y is a longitudinal boost angle, it has a natural canonical conjugate variable, namely the boost operator K_3 in the longitudinal direction, and that the eigenvalue k_3 of K_3, unlike y, is an additive quantum number. Hence it might be useful to expand the scattering amplitude in either y or k_3. A field theory in y or k_3 space is developed in which simplicities in y can be translated into coordinate space, through the relationship between k_3 and z. The invariant inclusive cross section, which is known to be a simple function of y, is related to the field number operator. This gives us a way of distinguishing in principle between different coordinate-space properties of particle production.

The invariant inclusive cross sections for high-energy multiparticle production processes are known to be simple functions of the rapidity variable y of the observed particle.[1] The regularity of the cross sections, sometimes called the "central plateau" because they appear to flatten about $y_{\text{c.m.}}=0$, suggests that there should be a regularity in some physical variable canonically conjugate to y.

The quantity which has the "central plateau" flatness or possibly Gaussian shape in y is $\sigma_{\text{inel}}dN/dy$, where σ_{inel} is the inelastic cross section for the inclusive process under study, and dN/dy is the number of particles of the type being measured at y in an interval dy. For example, in the reaction

$$pp\to\pi^+X$$

the quantity plotted as a function of y is $\sigma_{\text{inel}(pp\to\pi^+)}dN_{\pi^+}/dy$.

The rapidity y has several equivalent definitions,

$$y=\sinh^{-1}(p_{\parallel}/m_{\perp})=\tfrac{1}{2}\ln\frac{E+p_{\parallel}}{E-p_{\parallel}}=\ln\frac{E+p_{\parallel}}{m_{\perp}}. \tag{1}$$

If we recall the definition of the boost angle β,

$$\beta=\sinh^{-1}(p/m), \tag{2}$$

we see that the first of these definitions displays the longitudinal boost angle nature of y,[2] except that m is replaced by an effective mass $m_{\perp}$,

$$m_{\perp}=(m^2+p_{\perp}^2)^{1/2}. \tag{3}$$

In general the Poincaré generators of boosts, $K_i=M_{i0}$, are canonically conjugate to the boost angles β_j,

$$[K_i,\beta_j]=i\delta_{ij}. \tag{4}$$

We wish to point out here that a result similar to Eq. (4) holds also for the operators K_3 (we take the 3-direction to be the beam direction; thus $p_3=p_{\parallel}$) and Y, where

$$Y=\ln\frac{H+P_3}{M_{\perp}} \tag{5}$$

and H and $\vec{P}$ are the Hamiltonian and momentum operators. That is,

$$[K_3,Y]=\left[K_3,\ln\frac{H+P_3}{M_{\perp}}\right]=i. \tag{6}$$

Thus K_3 and Y are conjugate operators.[3] Furthermore, although the range of $(H+P_3)/M_{\perp}$ is $(0,\infty)$, the range of $Y=\ln[(H+P_3)/M_{\perp}]$ is $(-\infty,\infty)$, as is the range of K_3. Hence Y and K_3 are self-adjoint conjugate operators in the same sense as P and Q are self-adjoint conjugate operators in quantum mechanics, and the eigenvalues y and k_3 of Y and K_3 are conjugate Fourier transform variables. Even if the range of y is kinematically cut off to $(-Y/2, Y/2)$, we can still do the Fourier transform in a box of length Y to find the k_3 dependence. We emphasize that since K_3 is a generator of the Lorentz group, its eigenvalue k_3 is an additive quantum number, unlike y, which is not.

The asymptotic scattering states can be labeled by complete sets of commuting variables. Two possible sets of labels corresponding to use of Y and K_3 are

$$|m, s, s_3, \vec{p}_\perp, y\rangle$$

and (7)

$$|m, s, s_3, \vec{p}_\perp, k_3\rangle.$$

We note that because

$$[H, Y]=0, \quad (8)$$

a particle can be in both an eigenstate of H and an eigenstate of Y. On the other hand,

$$[H, K_3]=-iP_3, \quad (9)$$

so that a particle cannot simultaneously have a definite value of E and of k_3. However, we point out that the uncertainty relation $\Delta E \Delta k_3 \geq \frac{1}{2}\hbar|\langle P_3\rangle|$ is a weak uncertainty relation, while that for y and k_3 is strong, $\Delta y \Delta k_3 \geq \frac{1}{2}\hbar$.

Both Y and K_3 are constants of the motion, in spite of the fact that K_3 does not commute with H. This is because K_3 contains an explicit t-dependent term, $-P_3 t$, so that the total time derivative of K_3 is zero,

$$\frac{dK_3}{dt}=i[H, K_3]+\frac{\partial K_3}{\partial t}=0. \quad (10)$$

We note that K_3 acts as a dilation operator on the operator $(H+P_3)/M_\perp$ which is the argument of Y,

$$\left[K_3, \frac{H+P_3}{M_\perp}\right]=i\frac{H+P_3}{M_\perp}. \quad (11)$$

This leads us to the relationship

$$\left[K_3, \left(\frac{H+P_3}{M_\perp}\right)^{k_3}\right]=ik_3\left(\frac{H+P_3}{M_\perp}\right)^{k_3}, \quad (12)$$

which implies that the eigenstates of K_3 are associated with the monomials in $(E+p_3)/m_\perp$ and appear multiplied by $[(E+p_3)/m_\perp]^{-ik_3}$ in the expansion

$$|mss_3\vec{p}_\perp y\rangle \sim \int dk_3 \left(\frac{E+p_3}{m_\perp}\right)^{-ik_3} |mss_3\vec{p}_\perp k_3\rangle. \quad (13)$$

As was noted in Ref. 3, K_3 has the interpretation of being like a coordinate, namely

$$K_3 \sim Ez, \quad (14)$$

where E is the energy and z is the third component of the center of energy. During an interaction the total K_3 remains constant, but the K_3 of, for example, the positive pions in the inclusive reaction

$$pp \to \pi^+ X$$

is not separately a constant of the motion. Thus, the scattering matrix must contain a term which generates some k_3 value for the π^+ during the interaction. For the asymptotically separated particles, H is again a free Hamiltonian and $K_3(\pi^+)$ is a constant of the free motion. Thus the asymptotic value of $k_3(\pi^+)$ is equal to the value immediately after the interaction ceases. Because of the coordinate nature of K_3, the spread of values of k_3 reflects the degree of localization of the observed particles when they are produced.

There are several possible models of, for example, pion production at very high energies. If the interaction and production are assumed to take place in a highly Lorentz-contracted disk, then the energy density is localized and the distribution of k_3 for all observed π^+'s should be very sharply peaked. In a Landau-type model, the disk is expanding as the pions form, which would lead to a distribution, probably Gaussian, in the k_3 "positions" of the newly formed pions. In a multi-fireball model, there would be several sharply peaked values of k_3 in the distribution. The constancy of the quantity Ez for all observed pions would also imply that higher-energy pions are produced closer to the center of the collision, in a cascade or bremsstrahlung fashion.

The distribution which is measured (the square of the scattering amplitude) is a function of y which can be Fourier transformed into a function of the conjugate variable k_3. Thus rapidity distributions give information on the localization of the particle production. For example, when the k_3 distribution is a δ function or Gaussian, the rapidity distribution is flat or Gaussian, as seems to be the case experimentally. Note that $\delta(k_3)$ corresponds to the Feynman x distribution.

We turn now to a Fock representation of dN/dy. First, letting the transverse variables $\vec{p}_\perp=0$, but still labeling m as $m_\perp$ to ensure that $E^2=p_3{}^2+m_\perp{}^2$, we can define creation and annihilation operators $a^\dagger(y)$ and $a(y)$ by

$$a(y)=\sqrt{E}\, a(p). \quad (15)$$

Here $a(p)$ is the usual momentum-space (boson) operator satisfying

$$[a(p), a^\dagger(p')]=\delta(p-p'), \quad (16)$$

and p is for now just p_3. Then since

$$\frac{dy}{dp}=\frac{1}{E}, \quad (17)$$

we find that

$$[a(y), a^\dagger(y')]=\delta(y-y'). \quad (18)$$

Thus $a(y)$ is canonically defined.

The expectation value $\langle\tilde{n}|a^\dagger(y)a(y)|\tilde{n}\rangle$ counts the number of π^+'s in the unnormalized n-particle state $|\tilde{n}\rangle$ in an interval dy about y. The (inclusive) expectation value $\langle a^\dagger(y)a(y)\rangle$ includes an average over all possible n-particle final states,

$$\langle a^\dagger(y)a(y)\rangle = \sum_n \langle \bar{n}|a^\dagger(y)a(y)|\bar{n}\rangle, \tag{19}$$

and we may write

$$\frac{dN}{dy} = \langle a^\dagger(y)a(y)\rangle. \tag{20}$$

There are other equivalent ways to write the Fock space representation of dN/dy, but we feel that Eq. (20) is closest to the experimental quantity.[4]

The approximately flat or Gaussian distributions of $\langle a^\dagger(y)a(y)\rangle$ suggest simple shapes also for $\langle a^\dagger a\rangle$ as a function of the eigenvalue k_3 of K_3. In k_3 space $a(k_3)$ is defined by the Fourier transform of $a(y)$,

$$a(k_3) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dy\, e^{-ik_3 y} a(y), \tag{21}$$

so that

$$[a(k_3), a^\dagger(k_3')] = \delta(k_3 - k_3'). \tag{22}$$

That is, $a(k_3)$ is also canonically defined.

Any field ϕ can now be expanded in terms of $a(y)$ or $a(k_3)$ in the conventional way. Writing the boson field $\phi(z,t)$ in coordinate space as an expansion, first in terms of p_3 and E, then in terms of y, then finally in terms of k_3, we find

$$\begin{aligned}\phi(z,t) &= \int_{-\infty}^{\infty}\frac{dp_3}{\sqrt{4\pi E}} e^{i(p_3 z - Et)} a(p_3) \\ &\quad + \text{Hermitian conjugate} \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{dp_3}{\sqrt{2}\,E} e^{i(p_3 z - Et)} a(y) + \text{H.c.} \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk_3 a(k_3)\int_{-\infty}^{\infty}\frac{dp_3}{\sqrt{4\pi}E} e^{i(p_3 z - Et)} e^{ik_3 y} + \text{H.c.}\end{aligned} \tag{23}$$

The term $e^{ik_3 y}$ involves the exponential of a logarithm, so that $\phi(z,t)$ becomes

$$\phi(z,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk_3 a(k_3)\int_{-\infty}^{\infty}\frac{dp_3}{\sqrt{4\pi}E} e^{i(p_3 z - Et)}\left(\frac{E+p_3}{m_\perp}\right)^{ik_3} + \text{H.c.} \tag{24}$$

The integral over p_3 is a function of z, t, and k_3 readily evaluated,[5]

$$\int_{-\infty}^{\infty}\frac{dp_3}{E} e^{i(p_3 z - Et)}\left(\frac{E+p_3}{m_\perp}\right)^{ik_3} = -i\pi e^{-\pi k_3/2}\left(\frac{t+z}{t-z}\right)^{ik_3/2} H_{ik_3}^{(2)}(m_\perp(t^2-z^2)^{1/2}), \tag{25}$$

where $H_{ik_3}^{(2)}$ is a Hankel function of the second kind. This function is a shape function or wave function in coordinate space, determined by the simple relationship between p_3 and y and the simple eigenfunctions of K_3.[6] The appearance of the light-cone variables $t \pm z$ in this function is suggestive of a connection with the light-cone formalism.

We can now let $\vec{p}_\perp$ take on any value and incorporate all three space coordinates into the preceding integrals. This involves defining creation operators $a^\dagger(\vec{p}_\perp, y)$ and $a^\dagger(\vec{p}_\perp, k_3)$, both of which have the canonical commutation relations. The result is the field ϕ to be used as ϕ_{out} or ϕ_{free} in the scattering amplitude (see Ref. 4),

$$\phi(\vec{x},t) = \frac{1}{(2\pi)^{3/2}}\int d^2p_\perp \int_{-\infty}^{\infty} dk_3 a(\vec{p}_\perp, k_3) e^{i\vec{p}_\perp\cdot\vec{x}_\perp} \times\left(-i\frac{\sqrt{\pi}}{2}\right)e^{-\pi k_3/2}\left(\frac{t+z}{t-z}\right)^{ik_3/2} \times H_{ik_3}^{(2)}(m_\perp(t^2-z^2)^{1/2}). \tag{26}$$

Thus the ordinary p_3 shape function $e^{ip_3 z}$ and the p_3 dependence of $a(\vec{p}_\perp, p_3)$ have been replaced by a k_3 shape function $H_{ik_3}^{(2)}$ and k_3 dependence of $a(\vec{p}_\perp, k_3)$.

The Lorentz transformation properties of the shape function of Eq. (25) are particularly simple under a boost in the z direction with boost angle β. The argument of the Hankel function contains $t^2 - z^2$, which is invariant. The phase depending on $t \pm z$ becomes

$$\frac{t+z}{t-z} \to \frac{t\cosh\beta - z\sinh\beta - t\sinh\beta + z\cosh\beta}{t\cosh\beta - z\sinh\beta + t\sinh\beta - z\cosh\beta} = \left(\frac{t+z}{t-z}\right)e^{-\beta}. \tag{27}$$

The only effect of a boost in the z direction is therefore to multiply the shape function by a phase factor $e^{-ik_3\beta/2}$.

Any function of y can be expanded in a Fourier transform in terms of the eigenstates of K_3. We let $F(y)$ be such a function. Then

$$F(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk_3 e^{ik_3 y}\tilde{F}(k_3). \tag{28}$$

We change variables to $\nu = ik_3$, so that

$$F(y) = \frac{-i}{\sqrt{2\pi}}\int_{-i\infty}^{i\infty} d\nu\, e^{\nu y}\tilde{F}(\nu). \tag{29}$$

The substitution of

$$e^y = \frac{E+p_3}{m_\perp} \tag{30}$$

into Eq. (29) gives us the expansion

$$F(y) = -\frac{i}{\sqrt{2\pi}}\int_{-i\infty}^{i\infty} d\nu\left(\frac{E+p_3}{m_\perp}\right)^\nu \tilde{F}(\nu). \tag{31}$$

Now we consider a scattering amplitude for an n-particle final state. We write the amplitude F as a function of p_3 and $\vec{p}_\perp$ for each of the particles,

$$F(p_{1_3}, \ldots, p_{n_3}, \vec{p}_{1_\perp}, \ldots, \vec{p}_{n_\perp}). \tag{32}$$

We change variables from p_3 to y and make the transformation given by Eq. (31) on just one variable y_1,

$$F(y_1, \ldots, y_n, \vec{p}_{1_\perp}, \ldots, \vec{p}_{n_\perp}) = \frac{-i}{\sqrt{2\pi}} \int_{-i\infty}^{i\infty} d\nu_1 \left(\frac{E_1 + p_{1_3}}{m_{1_\perp}}\right)^{\nu_1} \tilde{F}(\nu_1, y_2, \ldots, y_n, \vec{p}_{1_\perp}, \ldots, \vec{p}_{n_\perp}). \tag{33}$$

For high energies E_1 the integrand of Eq. (33) has the form

$$\left(\frac{E_1}{m_{1_\perp}}\right)^{\nu_1} \tilde{F}(\nu_1, y_2, \ldots, y_n, \vec{p}_{1_\perp}, \ldots, \vec{p}_{n_\perp}), \tag{34}$$

as in a Regge representation.

The integral of Eq. (33) may contain singularities, though we do not know their nature. As an example, if the leading singularities in the left half plane were poles, and if y_1 were large and positive, then we could slide the contour to the left to pick up a leading contribution $2\pi i s^{\nu_{\text{pole}}} \text{Res}\tilde{F}(\nu_{\text{pole}})$. It is reasonable, in fact, to expect Regge-type behavior of scattering amplitudes in some kinematic regions (large rapidity).[7]

We have some clues as to the form of the scattering amplitude of Eq. (33). Under a Lorentz transformation along the 3-axis of boost angle β, we have

$$F(y) \to F(y + \beta), \tag{35}$$

or for many variables

$$F(y_1, \ldots, y_n, \vec{p}_{1_\perp}, \ldots, \vec{p}_{n_\perp}) \to F(y_1 + \beta, \ldots, y_n + \beta, \vec{p}_{1_\perp}, \ldots, \vec{p}_{n_\perp}). \tag{36}$$

Thus a Lorentz-invariant function such as the scattering amplitude must be a function of the differences of y's,

$$F = F(y_i - y_j, \vec{p}_{i_\perp}). \tag{37}$$

Furthermore, if the final state consists of identical particles, F must be a symmetric function of the y's and must therefore depend only on the magnitudes of the differences of y's.

$$F = F(|y_i - y_j|, \vec{p}_{i_\perp}). \tag{38}$$

These statements are all consistent with what is known about scattering amplitudes as quadratic functions of four-momenta, which can be verified using the identity

$$(p_0, \vec{p}_\perp, p_3) = (m_\perp \cosh y, \vec{p}_\perp, m_\perp \sinh y). \tag{39}$$

Such symmetry arguments give us constraints on the form of the transformed scattering amplitude $\tilde{F}$ of Eq. (33). For example, if the final state contains three pions we want a symmetric function such as

$$F(y_1, y_2, y_3) = f(|y_1 - y_2|) f(|y_2 - y_3|) f(|y_3 - y_1|), \tag{40}$$

where we drop the variables $\vec{p}_\perp$. Each function $f(\Delta y)$ is given by a Fourier transform of the type

$$f(\Delta y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk_3 e^{ik_3 \Delta y} \tilde{f}(k_3), \tag{41}$$

with

$$\tilde{f}(-k_3) = \tilde{f}(k_3). \tag{42}$$

Then if we expand $F(y_1, y_2, y_3)$ as

$$F(y_1, y_2, y_3) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk_{1_3} e^{ik_{1_3} y_1} \tilde{F}(k_{1_3}, y_2, y_3) \tag{43}$$

we find that

$$\tilde{F}(k_{1_3}, y_2, y_3) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy_1 e^{-ik_{1_3} y_1} f(|y_1 - y_2|) \times f(|y_2 - y_3|) f(|y_3 - y_1|). \tag{44}$$

This expression can be expanded using Fourier transforms of $f(|y_1 - y_2|)$ and $f(|y_3 - y_1|)$ with Fourier conjugate variables k and k', respectively, then integrated over y_1 and k' to give

$$\tilde{F}(k_{1_3}, y_2, y_3) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, \tilde{f}(k) f(|y_2 - y_3|) \tilde{f}(k - k_{1_3}) \times e^{ik(y_3 - y_2) - ik_{1_3} y_3}. \tag{45}$$

In Eq. (45) we have the functional form of the Fourier transform of the scattering amplitude on the boost variable of one of the final particles.

We have shown that there is a well-defined field theory in terms of y or k_3. This means that dN/dy in the invariant inclusive cross section can be properly written as $\langle a^\dagger(y) a(y) \rangle$ and Fourier transformed to dN/dk_3. The shape of dN/dy gives the shape of dN/dk_3, which in turn gives information about the localization properties of the particle production mechanism. The incoming asymptotic states will be eigenstates of $\vec{P}$ and H. During the collision, some property such as energy density is localized, and the outgoing asymptotic states may be expanded in eigenstates of K_3 in a way similar to the angular momentum formalism. Scattering amplitudes may be analyzed as functions of y and k_3 for final particles.

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[1]Different models predict different shapes for this function. The multiperipheral and hydrodynamic models predict, respectively, flat and Gaussian invariant inclusive cross sections in y. For the multiperipheral prediction see C. De Tar, Phys. Rev. D 3, 128 (1971). The hydrodynamic prediction along with fits to the data are in P. Carruthers, Cornell University Report No. CLNS-219, 1973 (unpublished). The present data span too small a region of rapidity space to judge confidently which model gives the better fits, particularly considering experimental difficulties in obtaining the data.

[2]A "fake" rapidity η is often used because it is easier to measure than y. The definitions of η are

$$\eta = \ln\cot(\theta/2)$$

$$= \tfrac{1}{2}\ln\frac{p+p_\parallel}{p-p_\parallel}$$

$$= \ln\frac{p+p_\parallel}{p_\perp},$$

where θ is the laboratory scattering angle of the observed particle. While η approximates y in many cases, it can be seen from the definitions of y and η that only y can be considered a boost angle.

[3]It is interesting to note that in classical physics $K_3 = Ez$, and the canonical transformation [see H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, Massachusetts, 1950), Chap. 8] generated by $F_2 = m_\perp z \sinh P$ takes conjugate variables z, P_3 into conjugate variables Q, P, with $Q = zE = K_3$ and $P = \sinh^{-1}(P_3/m_\perp) = Y$. In this sense K_3 is a coordinatelike operator and Y is a momentumlike operator.

[4]Our unnormalized state $|\tilde{n}\rangle$ is related to the normalized free state $|n\rangle$ via the S matrix,

$$|\tilde{n}\rangle = |n\rangle\,\langle n|S|pp\rangle\,\frac{1}{(\sigma_{\text{inel}})^{1/2}}$$

$$= |n\rangle\,\langle n_{\text{out}}|pp_{\text{in}}\rangle\,\frac{1}{(\sigma_{\text{inel}})^{1/2}},$$

where $\langle n|n\rangle = 1$. It can be shown that

$$\langle n|a^\dagger a|n\rangle = \langle n_{\text{out}}|a^\dagger_{\text{out}}a_{\text{out}}|n_{\text{out}}\rangle,$$

so that our expression (19) is

$$\frac{dN}{dy} = \sum_n \langle\tilde{n}|a^\dagger(y)a(y)|\tilde{n}\rangle$$

$$= \sum_{n,n'} \langle pp_{\text{in}}|n_{\text{out}}\rangle\langle n_{\text{out}}|a^\dagger(y)_{\text{out}}a(y)_{\text{out}}|n'_{\text{out}}\rangle\langle n'_{\text{out}}|pp_{\text{in}}\rangle$$

$$= \langle pp_{\text{in}}|a^\dagger(y)_{\text{out}}a(y)_{\text{out}}|pp_{\text{in}}\rangle\,.$$

The preceding equation may be a more familiar expression for dN/dy to some readers.

[5]*Tables of Integral Transforms* (Bateman Manuscript Project), edited by A. Erdélyi (McGraw-Hill, New York, 1954), Vol. 1; see page 313, number (17); G. N. Watson, *Theory of Bessel Functions* (Cambridge Univ. Press, New York, 1966), Sec. 6.21, number (7).

[6]The phases of this function have been determined for different regions of z and t:

$$\int_{-\infty}^{\infty}\frac{dp_3}{E}\left(\frac{E+p_3}{m_\perp}\right)^{ik_3} e^{i(p_3z-Et)}$$

$$= -i\pi e^{-(\pi/2)k_3}\left(\frac{t+z}{t-z}\right)^{ik_3/2} H^{(2)}_{ik_3}(m_\perp(t^2-z^2)^{1/2}),\quad t^2>z^2,\ t>0$$

$$= 2e^{-(3\pi/2)k_3}\left|\frac{t+z}{t-z}\right|^{\frac{ik_3}{2}} K_{ik_3}(m_\perp(z^2-t^2)^{1/2}),\quad t^2<z^2,\ z>0$$

$$= 2e^{-(\pi/2)k_3}\left|\frac{t+z}{t-z}\right|^{\frac{ik_3}{2}} K_{ik_3}(m_\perp(z^2-t^2)^{1/2}),\quad t^2<z^2,\ z<0$$

$$= -i\pi e^{-(\pi/2)k_3}\left(\frac{t+z}{t-z}\right)^{\frac{ik_3}{2}} H^{(2)}_{ik_3}(e^{-i\pi}m_\perp(t^2-z^2)^{1/2}),\quad t^2>z^2,\ t<0.$$

[7]Professor Paul Fishbane has pointed out to us the similarity of this formalism to the "longitudinal impact parameter" representation used in S.-J. Chang and P. M. Fishbane, Phys. Rev. D 2, 1084 (1970), Sec. V, and P. M. Fishbane and J. D. Sullivan, *ibid.* 6, 3568 (1972), Sec. V. In those papers on deep-inelastic electroproduction and inelastic e^+e^- annihilation, respectively, it was discovered that the structure function W_2 has a particularly striking form in a "longitudinal impact parameter" space corresponding to our k_3 space. Fishbane and Sullivan were able to prove theorems based on the nature of singularities in k_3 space. This reinforces our opinion that the k_3 space representations of scattering functions have real physical meaning.