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Spectral-function sum rules in Lagrangian field theory: An application of the renormalization group*

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Recent methods in applying the renormalization group are used to study the existence of spectral-function sum rules. A spurion expansion closely related to Wilson's original analysis is developed in the context of renormalized perturbation theory. Criteria for the validity of various types of sum rules are developed in terms of the anomalous dimensions of symmetry-breaking operators. Limitations on the order to which the spurion expansion can be carried because of infrared singularities are discussed.

I. INTRODUCTION

The spectral-function sum rules were first proposed by Weinberg.[1] These sum rules equate certain moments of the spectral weight functions of two-point functions for vector and axial-vector currents. The original derivations rested on various *ad hoc* assumptions about the high-momentum behavior of products of currents and the nature of the Schwinger terms of the current commutators.

These Weinberg sum rules have been extended and applied to various aspects of chiral-symmetry breaking.[2-10] In view of the continuing interest in them it may be worthwhile to investigate in a general way the conditions for the sum rules to be valid in renormalizable field theories. This can be done by a combination of various techniques and ideas that have been developed in recent years.

The problem of the Weinberg sum rules is equivalent to the problem of the small-distance or large-momentum behavior of propagators for vector and axial-vector currents. Suppose that our theory has a symmetry limit in which the currents belong to a single irreducible representation of the symmetry group and are conserved. In this limit all the propagators are equal.

Actually, however, the symmetry is broken and not all the propagators are equal. If the symmetry breaking is "soft," the leading small-distance or large-momentum asymptotic behavior of the propagators may be identical or related. Therefore, we can find certain linear combinations of propagators whose asymptotic behavior will be less singular than that of the individual terms. Using the spectral representations, the vanishing of the coefficients of the leading asymptotic terms for these linear combinations can be expressed as the vanishing of certain moments of the equivalent linear combinations of spectral weight functions. These are precisely the Weinberg sum rules. From this point of view they are a type of superconvergence relation.

This approach was originally given a precise form by Wilson, who applied operator-product expansions to study the small-x^2 behavior of the current propagators.[11] By expanding perturbatively in the symmetry-breaking operators, he developed a spurion analysis of the symmetry-breaking terms and used it to discuss the validity of the Weinberg sum rules in terms of the group representation content and of the anomalous dimensions of the symmetry-breaking terms.

Wilson's discussion is not in the context of conventional renormalizable Lagrangian field theory. The appropriate machinery to treat the problem in Lagrangian theory is the renormalization-group or Callan-Symanzik equations.[12,13] With them we can study the behavior of the propagators at large Euclidean momentum. For combinations which decrease fast enough in this region, there will be superconvergence relations which correspond to the spectral-function sum rules.

The form of the renormalization-group equations will depend on the additive counterterms needed to define finite renormalized propagators for the currents. The number of counterterms required depends on the symmetry-breaking terms present. The rules for determining the necessary renormalization subtractions have been systematically developed.[14]

We can get useful results only in the case of so-called soft symmetry breaking, where the symmetry-breaking terms in the Lagrangian have canonical dimensions $\delta \leq 3$. In practice, this means symmetry breaking by mass terms and by scalar fields with nonvanishing vacuum expectation values. It turns out, not surprisingly, that in these cases the propagators approach their symmetric values in the asymptotic spacelike limit. But deriving the sum rules requires knowledge also of nonleading terms which depend on the symmetry-breaking effects. The tools to study this are given by Weinberg's approach to the renormalization group for mass terms[15,16] and by the method of Lee and Weisberger for scalar fields.[17]

When these ingredients are combined in proper proportions and blended, there results a form for the asymptotic behavior of the propagators which is precisely what we require to derive the superconvergence relations. In fact, we obtain effectively Wilson's spurion expansion in a Lagrangian framework and without explicit use of the operator-product expansion.

The detailed applications for specific symmetry groups and specific types of symmetry breaking closely parallel those developed already from the original analysis. Therefore, this paper is limited to the derivation of the general theoretical criteria for the validity of the sum rules without specific applications. The next section deals with the so-called first spectral-function sum rules. In Sec. III, the analysis is extended to the second spectral-function sum rules. Following this, Sec. IV deals briefly with third spectral-function sum rules. Finally, there are some concluding remarks and discussion.

II. FIRST SPECTRAL-FUNCTION SUM RULES

We suppose that we are given a renormalizable Lagrangian field theory. When certain parameters in the theory are set to zero, the Lagrangian is invariant under a group of symmetry transformations and so is the vacuum state. There is a set of conserved Noether currents associated with the generators of the symmetry group. In the usual models the currents are bilinear in the fundamental fields.

For each current we define a covariant propagator or two-point function, including contact terms if necessary, so that the propagator is conserved in the symmetry limit. $J^a_\mu(x)$ denotes a current component with space-time index μ and internal quantum numbers denoted by a. The corresponding momentum-space propagator is

$$\Pi^a_{\mu\nu}(p) = \Pi^a_1(p^2)g_{\mu\nu} + \Pi^a_2(p^2)p_\mu p_\nu . \tag{1}$$

In the symmetry limit, $\Pi^a_1(p^2) = -p^2\Pi^a_2(p^2)$.

The renormalized propagators are defined by introducing counterterms according to the Bogoliubov-Parasiuk-Hepp procedure. The currents have dimension $\delta = 3$ and index of divergence $d = \delta - 4 = -1$. The superficial degree of divergence of the propagator is $D = 2$. Therefore, Π_2, which has a coefficient quadratic in the external momenta, is rendered finite by a single counterterm (in addition to those required to renormalize the Green's functions of the fields themselves). The same subtraction renders Π_1 finite in the symmetry limit.

Specifically, in the symmetry limit we take all renormalized masses to be zero and each scalar field to have a vanishing vacuum expectation value. The theory is defined in terms of some dimensionless coupling constants. For notational simplicity we will assume that there is only one and call it g. The unrenormalized Green's functions are made finite by some symmetry-preserving cutoff procedure characterized by a squared cutoff momentum, Λ^2, and the theory is renormalized at some Euclidean point $p^2 = -\mu^2$.

Using the subscript u to denote unrenormalized quantities, the finite invariant functions are given by

$$\Pi^a_2(p^2; g, \mu^2) = \Pi^a_2(p^2; g_u, \Lambda^2)_u - \Pi^a_2(-\mu^2; g_u, \Lambda^2)_u . \tag{2}$$

The limit $\Lambda \to \infty$ after subtraction is implied.

To obtain a renormalization-group equation for this amplitude, we note that,[18] obviously,

$$\mu^2 \frac{\partial}{\partial \mu^2}\Pi^a_2(p^2; g_u, \Lambda^2)_u \bigg|_{g_u, \Lambda} = 0 ,$$

and we use the chain rule to rewrite the differential operator acting on $\Pi^a_1(p^2; g, \mu^2)$ to obtain

$$\left[\mu^2 \frac{\partial}{\partial \mu^2} + \beta(g)\frac{\partial}{\partial g}\right]\Pi^a_2(p^2; g, \mu^2) = C(g) , \tag{3}$$

where

$$C(g) = \mu^2 \frac{\partial}{\partial p^2}\Pi^a_2(p^2; g_u, \Lambda^2)_u \bigg|_{p^2=-\mu^2} .$$

$C(g)$ is cutoff-independent by power counting, is μ^2-independent by dimensional analysis, and is independent of the quantum numbers a by symmetry. Because the current is conserved and unrenormalized there is no anomalous dimension for the current, $\gamma_J = 0$.

Now suppose the symmetry is broken softly by a term in the Lagrangian of canonical dimension $\delta \le 3$, or index of divergence $d = \delta - 4 < -1$. First, the renormalizations of the partially conserved currents are finite, which means that $\gamma_J = 0$ still. Second, the superficial degree of divergence of any Feynman graph with n insertions of the symmetry-breaking term is $D = 2 + nd$. If $n \ne 0$, $D \le 1$, which means that graphs with symmetry-breaking insertions give cutoff-independent contributions to Π_2. Therefore, the single symmetric subtraction term still renormalizes all the Π^a_2 functions even in the presence of soft symmetry breaking.[19]

The possible kinds of soft symmetry breaking effects are fermion mass terms, meson mass terms, and scalar fields with nonvanishing vacuum expectation value. When the effect of such parameters is included in the renormalization-group analysis,[15-17] the differential equation (3) becomes

$$\left[\mu^2\frac{\partial}{\partial\mu^2}+\beta(g)\frac{\partial}{\partial g}-\tfrac{1}{2}\sum_f\gamma_f(g)m_f\frac{\partial}{\partial m_f}-\tfrac{1}{2}\sum_b\gamma_b(g)M_b\frac{\partial}{\partial M_b}-\tfrac{1}{2}\sum_i\gamma_i(g)v_i\frac{\partial}{\partial v_i}\right]\Pi_2^a(p^2;g,m,M,v,\mu^2)=C(g)\,, \quad (4a)$$

where

m_f stands for fermion masses,

M_b stands for boson masses,

$v_i=\langle 0|\phi_i(0)|0\rangle$ (ϕ is a scalar field),

$\gamma_f(g)$ is the anomalous dimension of a fermion mass insertion in the symmetric theory,

$\gamma_b(g)$ is the anomalous dimension of a boson mass insertion in the symmetric theory,

and

$\gamma_i(g)$ is the anomalous dimension of the scalar field ϕ_i in the symmetric theory.

All the γ's are defined in the symmetric theory and, therefore, are functions only of the dimensionless coupling constant g.

It is convenient to introduce dimensionless variables

$$t=\ln(|p^2|/\mu^2) \quad\text{and}\quad \{x_i\}=\{m_f/\mu\,,M_b/\mu\,,v_i/\mu\}\,.$$

Since Π_2 has zero physical dimension, we get

$$\left\{\frac{\partial}{\partial t}-\beta(g)\frac{\partial}{\partial g}+\tfrac{1}{2}\sum_i[1+\gamma_i(g)]x_i\frac{\partial}{\partial x_i}\right\}\Pi_2^a(t,g,\{x_i\})=-C(g)\,. \quad (4b)$$

The x_i's now appear as additional coupling constants. The solution can be written in a standard form by introducing effective coupling constants $\bar g(t,g)$ and $\bar x_i(t,g,x_i)$, which satisfy the following differential equations and initial conditions:

$$\frac{\partial\bar g}{\partial t}=\beta(\bar g)\,,\quad \bar g(0,g)=g\,,$$
$$\frac{\partial\bar x_i}{\partial t}=-\tfrac{1}{2}[1+\gamma_i(\bar g)]\bar x_i\,,\quad \bar x_i(0,g,x_i)=x_i\,. \quad (5)$$

The solution of Eq. (4b) can be written as

$$\Pi_2^a(t,g,\{x_i\})=\Pi_2^a(0,\bar g(t,g),\{\bar x_i(t,g,x_i)\})-\int_0^t C(\bar g(t',g))dt'\,. \quad (6)$$

$\bar g(t,g)$ is the running coupling constant of the symmetric theory. If $\bar g$ is known and the function $\gamma_i(g)$ is determined, the effective symmetry-breaking constants are easily found from (5):

$$\bar x_i(t,g,x_i)=x_i\exp\left\{-\tfrac{1}{2}\left[t+\int_0^t\gamma_i(\bar g(t',g))dt'\right]\right\}. \quad (7)$$

If $\lim_{t\to\infty}\gamma_i(\bar g(t,g))>-1$, then

$$\lim_{t\to\infty}\bar x_i=0\,.$$

In this case

$$\lim_{t\to\infty}\Pi_2^a(t,g,\{x_i\})=\Pi_2^a(0,\bar g(t,g),\{0\})-\int_0^t C(\bar g(t';g))dt'\,, \quad (8)$$

which is the symmetric Green's function $\Pi_2^a(t,g,\{0\})$.

For the mass terms, the condition $\lim_{t\to\infty}\gamma_i(\bar g)>-1$ is true to any finite order in perturbation theory by Weinberg's power-counting theorem.[20] In an asymptotically free theory $\lim_{t\to\infty}\gamma_i=0$.

For a manifestly covariant theory with positive metric, $\gamma_\phi\geq 0$ follows from the Källén-Lehmann representation for the two-point function. In an asymptotically free gauge theory, $\lim\gamma_\phi=0$.

We assume that, at least, the weaker inequality $\lim\gamma_i>-1$ is satisfied. If not, all attempts to analyze high-momentum behavior by Callan-Symanzik equations and renormalization-group methods are in difficulty.

If we can expand Π_2 in a power series in the $\bar x_i$, the zeroth-order term is the symmetric propagator. The symmetry-breaking terms of order $(\bar x)^n$ behave as

$$(\mu^2/p^2)^{-(n/2)\tau},\quad \tau\equiv t+\int_0^t\gamma(\bar g(t'))dt'\,.$$

The difference between any pair of invariant functions vanishes faster than $|p^2|^{-1/2}$. Hence, the difference satisfies an unsubtracted dispersion relation, and the relevant spectral weight functions must satisfy a convergence condition

$$\int\frac{dp^2}{p^2}[\rho_2^a(p^2)-\rho_2^b(p^2)]=\text{const} \quad (9)$$

(a 0th spectral-function sum rule).

Infrared singularities limit the number of terms which can be expanded as simple powers about the

$x_i = 0$ limit. The basic analysis was given by Weinberg.[15] Boson masses enter only as mass squared. Though the $M = 0$ limit is finite, the first derivative with respect to M^2 at $M^2 = 0$ is infrared-divergent. Therefore, the remainder behaves as $x^2 \ln x^2$.

For fermion lines, we must differentiate at least three times before infrared singularities enter. Similar restrictions hold for scalar tadpole insertions, since they generate fermion mass terms via Yukawa couplings and boson mass terms from ϕ^4 interactions. These limitations on the number of simple power-series terms do not apply if we want to expand the Green's functions about some symmetric point where $x_i \neq 0$, for example, if we want to study mass-splitting effects by expanding about a symmetric mass.

Though fermion masses and scalar tadpoles enter linearly into the theory, only even total powers occur in the expansion of $\Pi_{\mu\nu}$ about $x_i = 0$. This follows straightforwardly from the fact that the trace of an odd number of γ_μ matrices vanishes. (We are restricting ourselves to theories with currents bilinear in spin-$\frac{1}{2}$ and spin-0 fields and with no dimensional coupling constants in the symmetric limit. Strongly interacting vector mesons are assumed to be singlets with respect to the quantum numbers carried by the currents.)

Returning to Eq. (6), we can write

$$\Pi_2^a(t, g, \{x_i\}) = \Pi_2^{\text{sym}}(t, g) + O(\bar{x}^2).$$

Let γ^∞ denote $\lim_{t\to\infty} \gamma(\bar{g}(t, g))$. Then,

$$\bar{x}^2 \sim (|p^2|)^{-1-\gamma^\infty}.$$

If $\gamma^\infty > 0$, the difference between any pair of invariant Π_2 functions vanishes faster than $(1/p^2)$ and we have the superconvergence relation

$$\int dp^2 [\rho_2^a(p^2) - \rho_2^b(p^2)] = 0, \tag{10}$$

which is the first Weinberg spectral sum rule.

For scalar particles in a theory with positive metric, $\gamma_\phi \geq 0$ follows from the Källén-Lehmann representation of the two-point function. There is no similar bound for anomalous dimensions of mass terms.

In asymptotically free theories, the anomalous dimensions approach zero as $1/\ln|p^2|$. The sign of the coefficient of the logarithmic term determines the validity of (10).[21] A superconvergence relation which depends on the sign of logarithmic terms presumably will converge too slowly to be of practical use in low-mass saturation approximations.

If there are no boson mass terms to be considered, we can expand the invariant function:

$$\Pi_2^a(t, g, \{x_i\}) = \Pi_2(t, g, 0) + \tfrac{1}{2} \sum_{jk} \bar{x}_j(t)\bar{x}_k(t) \frac{\partial^2}{\partial x_j \partial x_k} \Pi_2^a(0, \bar{g}(t), \{x_i\}) \Big|_{\{x_i\}=0} + O(\bar{x}^4 \ln \bar{x}^2). \tag{11}$$

If we take a linear combination, $\sum_a C^a \Pi_2^a$, such that

$$\sum_a C^a = 0$$

and (12)

$$\sum_a C^a \sum_{jk} \bar{x}_j(t)\bar{x}_k(t) \frac{\partial^2}{\partial x_j \partial x_k} \Pi_2^a \Big|_{\{x_i\}=0} = 0,$$

this combination vanishes asymptotically as $\exp[-2t(1+\gamma^\infty)]$. Therefore, if $\gamma^\infty > -\frac{1}{2}$,

$$\int dp^2 \sum_a C^a \rho_2^a(p^2) = 0. \tag{13}$$

This bound on γ^∞ is obeyed in asymptotically free theories. If, in fact, $\gamma^\infty > 0$, the next moment also superconverges:

$$\int dp^2 p^2 \sum_a C^a \rho_2^a(p^2) = 0. \tag{14}$$

The linear combinations obeying (12) can be divided into two general classes. The first class obeys (12) independent of the value of the $\bar{x}_i$; that is,

$$\sum_a C^a \frac{\partial^2}{\partial x_j \partial x_k} \Pi_2^a \Big|_{x_i=0} = 0 \quad \text{for all } j,k.$$

Such relations depend only on the group representation content of the currents and the x_i. A typical example is the SU(3) × SU(3) chiral algebra of currents belonging to (1, 8) + (8, 1) with symmetry breaking due to quark mass terms belonging to $(3, \bar{3}) + (\bar{3}, 3)$. The combination of current propagators belonging to (1, 27) + (27, 1) does not couple to the second-order quark mass spurions.

The second class of linear combinations obeying (12) depends explicitly on the values of the $\bar{x}_j(t)$, at least on their relative size or ratios. A typical example is the original Weinberg sum rule for vector and axial-vector isospin currents, which is valid in a model with quark mass splitting with $m_8 = \sqrt{2} m_0$.

In the intermediate renormalization procedure used here the ratio of $\bar{x}_j(t)/\bar{x}_k(t)$ is the same as that of the bare masses m_j^0/m_k^0. The multiplicative renormalization factors are defined for mass vertex insertions in the completely symmetric theory and are the same for all quark masses. So are the respective anomalous dimensions.

III. SECOND SPECTRAL-FUNCTION SUM RULES

Superconvergence relations for Π_1 are called second spectral-function sum rules. The procedure in establishing them is analogous to that for Π_2. First we establish the counterterms needed to define a finite renormalized invariant function when the symmetry is broken.

In the symmetric case, Π_1 is quadratically divergent but Ward identities guarantee that the single subtraction which renders Π_2 finite also renormalizes Π_1. Graphs with n_i symmetry-breaking insertions of type i have superficial degree of divergence $D=2+\sum_i n_i d_i$.

For soft symmetry breaking, $d_i \le -1$ and $D\le 1$. A finite invariant function is defined by

$$\Pi_1^a(p^2,g,\{x_1\})=\Pi_1^a(p^2;g_u,\{m,M,v\}_u,\Lambda^2)_u - \Delta\Pi_1^a(-\mu^2;g_u,\{m,M,v\}_u,\Lambda^2)_\mu + q^2\Pi_2^a(-\mu^2;g_u,\Lambda^2)_\mu\,, \tag{15}$$

where

$$\Delta\Pi_1^a(-\mu^2;\cdots)_u=\Pi_1^a(-\mu^2;g_u,\{m,M,v\}_u,\Lambda^2)_u - \Pi_1^a(-\mu^2;g_u,\{0\},\Lambda^2)_u\,.$$

The renormalization-group equation satisfied by Π_1 is

$$\left(\mu^2\frac{\partial}{\partial\mu^2}+\beta\frac{\partial}{\partial g}-\tfrac{1}{2}\sum_f\gamma_f m_f\frac{\partial}{\partial m_f}-\tfrac{1}{2}\sum_b\gamma_b M_b\frac{\partial}{\partial M_b}-\tfrac{1}{2}\sum_i\gamma_i v_i\frac{\partial}{\partial v_i}\right)\Pi_1^a=-q^2C(g)+\mu^2D^a(g,\{x_i\})\,, \tag{16}$$

where

$$D^a(g,\{x_i\})=\frac{\partial}{\partial p^2}\Delta\Pi_1^a(p^2;\cdots)_u\Big|_{p^2=-\mu^2}.$$

Defining $\Pi_1^a=q^2f^a$ and changing variables as before, one obtains

$$\left[\frac{\partial}{\partial t}-\beta\frac{\partial}{\partial g}+\tfrac{1}{2}\sum(1+y_i)x_i\frac{\partial}{\partial x_i}\right]f^a(t,g,\{x_i\})=C(g)-e^{-t}D^a(g,\{x_i\}), \tag{17}$$

with the solution

$$f^a(t,g,\{x_i\})=f^a(t,\bar g(t,g),\{\bar x_i(t,g,x_i)\})+\int_0^t C(\bar g(t'))dt'-e^{-t}\int_0^t e^{t'}D^a(\bar g(t'),\{\bar x_i(t')\})\,dt'. \tag{18}$$

With the usual assumption, $\gamma^\infty>-1$, it is easy to see that the leading asymptotic term is the symmetric limit. However, simple differences $\Pi_1^a-\Pi_2^b$ do not vanish asymptotically in general. Such differences will obey once-subtracted dispersion relations regardless of the value of γ^∞ unless second-order symmetry-breaking effects cancel. In the latter case, the following discussion applies.

If the propagator can be expanded to second order in powers of the symmetry breaking, then it is sufficient to include in the symmetry-breaking subtraction term just the second-order symmetry-breaking effects. That is,

$$\Delta\Pi_1^a(-\mu^2;\,g,\{m_i\},\Lambda^2)_u=\tfrac{1}{2}\sum_{jk}m_jm_k\frac{\partial^2}{\partial m_j\partial m_k}\Pi_1^a(\cdots)_u\Big|_{m_i=0} \tag{19}$$

and

$$D^a(g,\{x_i\})=\sum_{jk}x_jx_k(D_{jk}^a(g)). \tag{20}$$

Then

$$f^a(t,g,\{x_i\})=f^a(t,g,0)+\tfrac{1}{2}\sum_{jk}\bar x_j(t)\bar x_k(t)\frac{\partial^2}{\partial x_j\partial x_k}f(0,\bar g(t,g),\{x_i\})\Big|_{x_i=0} + e^{-t}\int_0^t e^t\sum_{jk}\bar x_j(t')\bar x_k(t')D_{jk}^a(\bar g(t'))dt'+O(\bar x(t)^4\ln\bar x)\,. \tag{21}$$

If we take linear combinations of invariant functions as before such that both the symmetric and the quadratic symmetry-breaking terms cancel, this combination behaves asymptotically as

$$|p^2|[\bar{x}(t)]^4 \sim |p^2|^{-(1+2\gamma^\infty)}.$$

Therefore, if $\gamma^\infty > -\frac{1}{2}$, such a combination obeys an unsubtracted dispersion relation, but we require $\gamma^\infty > 0$ for the second spectral sum rule to hold,

$$\int dp^2 \sum_a C^a \rho_1^a(p) = 0. \tag{22}$$

This condition is obeyed when the only symmetry breaking is due to scalar fields with nonvanishing vacuum expectation values in a positive-metric theory. In asymptotically free gauge theories the second Weinberg sum rule depends on the lowest-order contributions to the anomalous dimensions of the fermion mass terms.

IV. THIRD SPECTRAL-FUNCTION SUM RULES

Instead of the tensor decomposition of (1), we can write

$$\Pi^a_{\mu\nu} = \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\Pi_1^a(p^2) + p_\mu p_\nu \Pi_0^a(p^2). \tag{23}$$

The absorptive parts of Π_0 and Π_1 come from states of total angular momentum 0 and 1, respectively. Obviously

$$\Pi_0 = \Pi_2 + \Pi_1/p^2. \tag{24}$$

Spectral sum rules for the spin-0 weight function are known as third spectral-function sum rules. They can be derived by straightforward application of the preceding technique. Since Π_0 is a linear combination of Π_1 and Π_2, it is not surprising that these sum rules turn out to be just linear combinations of those previously derived in Secs. II and III.

When $\gamma^\infty > 0$, there is an interesting result for the linear combinations which cancel second-order symmetry-breaking effects. The superconvergence relations for the first moment of ρ_2 [Eq. (13)] and the zeroth moment of ρ_1 [Eq. (22)] can be added to give

$$\int dp^2 \sum_a C^a \rho_0^a(p^2) = 0. \tag{25}$$

Since $p^\mu p^\nu \Pi_{\mu\nu} = (p^2)^2 \Pi_0(p^2)$ is essentially the propagator for the divergence of the current, we can obtain sum rules for the two-point functions of $\partial^\mu J_\mu$ in the same way. This analysis also determines the number of subtractions in the spectral representation for such a propagator. Two are required, but one is determined since the propagator vanishes at $p^2 = 0$. The second depends on our arbitrary renormalization condition for Π_1. If $\Pi^a(p^2)$ denotes the propagator for $\partial^\mu J^a_\mu$,

$$\Pi^a(p^2) = p^2\left[r^a + \frac{p^2}{\pi}\int_0^\infty \frac{\rho_0^a(\kappa^2)}{\kappa^2 - p^2} d\kappa^2\right]. \tag{26}$$

V. DISCUSSION

As mentioned at the beginning, the results of this paper largely reproduce Wilson's spurion analysis in the context of renormalized Lagrangian field theory. However, this method does clarify the role of infrared singularities in limiting the order to which the spurion expansion can be carried.

In contrast with the case of deep-inelastic lepton scattering, the study of the asymptotic behavior of two-point functions in field theory does not require use of Wilson's operator expansion. This is obvious if one remembers the date of the original paper in this field.[22] However, the technical tools to incorporate symmetry breaking including mass effects into the analysis have been developed only recently.

These methods allow us to define and implement precisely the notion of asymptotic symmetry which has been used heuristically and intuitively to justify spectral-function sum rules in the past.

The property of conserved and partially conserved currents which is crucial to this analysis is the absence of anomalous dimensions for such operators. This allows us to relate the number of subtractions in the spectral representation to the number of counterterms required in perturbation theory (with the assumption $\gamma^\infty > -1$). To extend this analysis to propagators for arbitrary composite operators we would have to introduce extra assumptions about their asymptotic dimensions, which could at best be justified from order-by-order power counting in perturbation theory using Weinberg's theorem.[20]

Saturation of such sum rules, which converge only logarithmically, by known low-mass states cannot be expected to be a good approximation. Such sum rules may occur especially in asymptotically free gauge theories and depend for their validity on the lowest-order nontrivial contributions of perturbation theory to the anomalous dimensions of the symmetry-breaking operators. Though not of direct practical use in low-mass saturation approximations, the existence of such marginal superconvergent relations may be important in the theoretical applications of the sum rules. Consider, for example, the pion electromagnetic mass difference calculations, where the Weinberg sum rules are required to get a finite expression.[23] This is to be a model-dependent question, and further investigations of this type of behavior and possible applications are being carried out.

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[19]I am assuming that there remain enough good quantum numbers so that no off-diagonal propagators arise when the full symmetry is broken. The generalization to include this possibility is straightforward. Off-diagonal Π_2's are finite (no subtractions) and off-diagonal Π_1's require a single subtraction.

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[21]In the lowest order of perturbation theory

$$\gamma = bg^2,\ \beta = -ag^3,\ a > 0$$

which results in

$$\bar{x}^2 \underset{t\to\infty}{\sim} e^{-t} t^{-b/2a} .$$

The sign of b is model-dependent.

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