Large-photon-number limit and the essential singularity in finite quantum electrodynamics

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It is shown that the essential singularity in finite quantum electrodynamics can be located by considering only those diagrams with a large number of photons exchanged in the single-fermion loop, without photons emitted and absorbed on a fermion line.

I. INTRODUCTION

The short-distance behavior of quantum electrodynamics (QED)[1] led to the program of Johnson, Baker, and Willey.[2,3] In this program QED is finite if the coefficient of the logarithmically divergent part of the single-fermion-loop contribution to the vacuum polarization vanishes, i.e.,

$$F_1(y)|_{y=\alpha_0}=0\,, \tag{1}$$

where α_0 is the unrenormalized coupling constant.

It was pointed out by Adler[4] that the above eigenvalue equation may hold for the renormalized coupling constant α, and that the zero of F_1 is of infinite order,

$$\left.\frac{d^kF_1}{dy^k}\right|_{y=\alpha}=0\quad (k\geqslant 0)\,; \tag{2}$$

therefore F_1 has an essential singularity at α.[5]

The most straightforward way to find out whether F_1 has an essential singularity at all, and if it has whether it is at α or at $\alpha_0>\alpha$,[6] is to calculate F_1. However, F_1 is an infinite power series in the coupling constant y (Fig. 1), and one is faced with the task of studying a series, the jth term of which is yet unknown.

F_1 is known up to sixth order,[7]

$$F_1(y)=\frac{2}{3}+\frac{y}{2\pi}-\frac{1}{4}\left(\frac{y}{2\pi}\right)^2+\cdots\,; \tag{3}$$

the simplicity of Eq. (3) has not led to much insight regarding higher-order terms. Other attempts at calculating F_1 have been by summing a large class of diagrams[8] and by using a conformally invariant formalism of the five-dimensional hypersphere.[9]

Denoting by q^2 the photon momentum, and by m the fermion mass, the sum of single-fermion-loop contributions to the vacuum polarization is given at high q^2 by[2]

$$\pi_{1,c}(q^2,m,y)=G_1(y)+F_1(y)\ln(-q^2/m^2)\,. \tag{4}$$

From the scaling property of $\pi_{1,c}$ we see that it is enough to discuss F_1 in massless QED ($m=0$). From Eq. (2) it was shown[4] that the $2n$-photon amplitude vanishes in massless QED, and that T_1, which is the single-fermion-loop contribution to the above amplitude, vanished separately, i.e.,

$$T_{1,\mu_1,\ldots,\mu_{2n}}(q_1,\ldots,q_{2n};m,y)|_{m=0;y=\alpha}=0\,, \tag{5}$$

where q_i are the (virtual or real) photon momenta. From Eq. (5), by contracting $n-1$ photon pairs in all possible ways, Eq. (2) is obtained.[4]

It was shown[10] that by contracting $n-1$ photon pairs in less than all possible ways, Eq. (2) simplifies to

$$\left.\frac{d^kH_1}{dy^k}\right|_{y=\alpha}=0\quad (k\geqslant 0)\,, \tag{6}$$

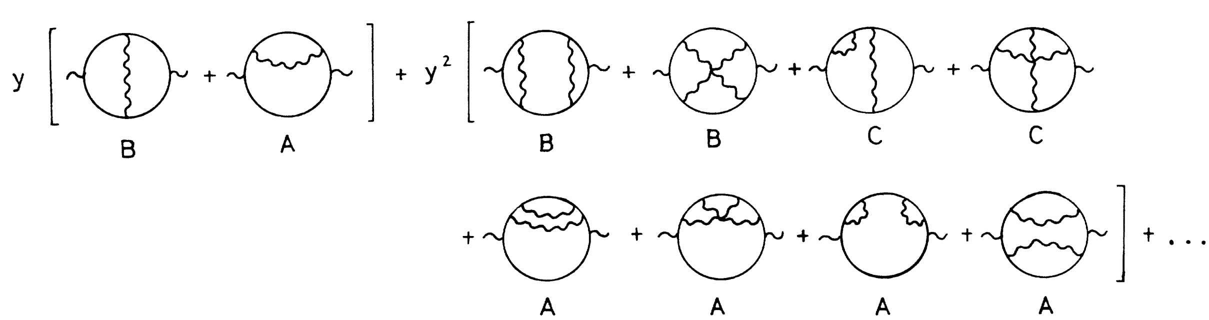

FIG. 1. Topologically distinct diagrams in F_1; the y (coupling constant) dependence is shown, and each diagram is labeled by its type. The zeroth-order diagram was omitted (it does not contribute to ydF_1/dy).

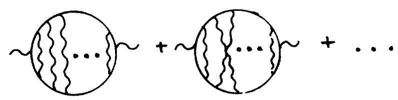

FIG. 2. The jth term in S_1 for large j. It includes all diagrams with photons exchanged between the sides of the loop, without any curved photons.

where H_1 includes all the diagrams with at least one photon exchanged in the loop. It was also pointed out[10] that a more meaningful simplification would not follow without additional input.

In the present paper we do not simplify the eigenvalue equation, but a series is found such that its large-order term is much simpler than the corresponding terms in the eigenvalue functions discussed so far.

The fact that α (see Ref. 6) is an essential singularity of a function S_1 means that it can be located from the $j\to\infty$ behavior of the jth term in the perturbation expansion for the series. Indeed, if the series is convergent around $y=0$ (see Ref. 11) then if we write

$$S_1 = \sum_{j=0} a_j y^j \tag{7}$$

its radius of convergence is given by

$$\alpha = \lim_{j\to\infty} |a_j|^{-1/j}\,; \tag{8}$$

a more useful expression may be

$$\alpha = \lim_{j\to\infty} \left| \frac{a_j}{a_{j+1}} \right| \,. \tag{9}$$

if the limit exists.

The jth term is a sum over permutations of j internal photon lines in the fermion loop. For the series F_1 or even for the simpler series H_1 there are many topologically distinct diagrams in the jth order, and the location of the essential singularity seems extremely difficult. The main result of the present paper is that there exists a series with a much simpler jth term for large j and with an essential singularity at α. The jth term of this series includes, for large j, only those $j!$ diagrams with j photons exchanged between the lower and the upper parts of the loop (Fig. 2); no diagrams with photons emitted and absorbed on the same fermion line of the loop are included. This multiperipheral-like term brings some hope for locating the essential singularity of F_1, if it exists.

In Sec. II Adler's results[4] are briefly summarized, and some terms used in the following sections are defined. In Sec. III the results of Ref. 10 are briefly summarized. In Sec. IV we consider a series obtained from contractions of T_1 orthogonal to those that were used to form H_1. In Sec. V a linear combination of series is formed and its jth term for large j is found. Finally in Sec. VI conclusions are presented.

II. A SERIES FORMED FROM ALL CONTRACTIONS OF T_1

Let us briefly summarize Adler's proof of Eq. (2).[4] Contract $n-1$ external photon pairs in T_1 (the single-fermion-loop contribution to the $2n$-photon amplitude) in all possible ways, thus obtaining $\pi_{1,2n}(q^2;\, m, y)$ which has the same Lorentz structure as $\pi_{1,c}(q^2; m, y)$ (the single-fermion-loop contribution to the photon proper self-energy). By counting all the diagrams in each order (by "order" we always mean the number of internal photons) in $\pi_{1,2n}$ and in $\pi_{1,c}$ Adler obtained[12]

$$\frac{2^{n-1}}{(2n-1)!}\, \frac{(j+n-1)!}{j!}\, \pi_{1,c}^{j+n-1}(q^2; m, m') = \pi_{1,2n}^{j}(q^2; m, m')\,, \tag{10}$$

where $\pi_{1,c}^{j}$ and $\pi_{1,2n}^{j}$ are defined through

$$\pi_{1,c}(q^2; m, y) - \pi_{1,c}(q^2; m', y) = \sum_{j=0}^{\infty} y^j \pi_{1,c}^{j}(q^2; m, m')\,, \tag{11}$$

$$\pi_{1,2n}(q^2; m, y) - \pi_{1,2n}(q^2; m', y) = \sum_{j=0}^{\infty} y^j \pi_{1,2n}^{j}(q^2; m, m')\,. \tag{12}$$

The subtraction is needed to make the logarithmic divergence finite, and is also present in the contraction of T_1. Multiplying Eq. (10) by y^j, summing from $j=0$ to $j=\infty$, and then letting $m, m'\to 0$ with m/m' fixed, Eq. (2) (with $k=n-1$) follows from the vanishing of T_1 at zero fermion mass [Eq. (5)]. The topologically distinct diagrams in F_1 are shown in Fig. 1.

Let us define the following:

Straight photons: photons exchanged between the two fermion lines in the loop.

Curved photons: photons that are emitted and absorbed on the same fermion line in the loop.

Type A diagrams (Fig. 1): diagrams with self-energy insertions only.

Type B diagrams (Fig. 1): diagrams with straight photons only.

Type C diagrams (Fig. 1): diagrams with at least one curved photon and one straight photon.

Good contractions of T_1: contractions of $2(n-1)$ photons such that the zeroth order of T_1 is transformed into diagrams of types B and C.

Bad contractions of T_1: contractions of $2(n-1)$ photons such that the zeroth order of T_1 is transformed into diagrams of type A.

The diagrams used to calculate F_1 are of types

A, B, and C and are obtained from all possible contractions (good and bad) of T_1.

III. A SERIES FORMED FROM THE GOOD CONTRACTIONS OF T_1

In this section we summarize, as an introduction for the next section, the results of Ref. 10. It was shown that by contracting $2(n-1)$ photons in T_1 in all possible good contractions one obtains an object denoted by $P_{1,2n}$ which is defined through its series expansion

$$P_{1,2n}(q^2;m,y)-P_{1,2n}(q^2;m',y) = \sum_{j=0}^{\infty} y^j P^j_{1,2n}(q^2;m,m') . \quad (13)$$

$P^j_{1,2n}$ is a sum over diagrams with $j+n-1$ internal photons ($n-1$ from each contraction, and j internal in T_1). From the definition of a good contraction it is clear the $P^j_{1,2n}$ contains only diagrams of types B and C. It was futhermore shown that all diagrams of types B and C are included in $P^j_{1,2n}$. Denoting by $\pi^{j;\,i}_{1,c}$ the contribution to $\pi^j_{1,c}$ [see Eq. (11)] from all the diagrams with i straight photons we define

$$P^j_{1,c}(q^2;m,m') = \sum_{i=1}^{j} \pi^{j;\,i}_{1,c}(q^2;m,m') . \quad (14)$$

Therefore, $P^j_{1,c}$ includes all diagrams of types B and C while all diagrams of type A are included in $\pi^{j;\,0}_{1,c}$ and

$$\pi^j_{1,c} = \sum_{i=0}^{j} \pi^{j;\,i}_{1,c} . \quad (15)$$

After some combinatorial manipulations [similar but more complicated to those leading to Eq. (10)] it was found that

$$g_n \frac{(j+n-1)!}{(j+n-1)j!} h_{j+n-1} P^{j+n-1}_{1,c}(q^2;m,m') = P^j_{1,2n}(q^2;m,m') . \quad (16)$$

For the exact form[13] of g_n and h_{j+n-1} which are not needed here see Ref. 10. Multiplying Eq. (16) by y^j and summing from $j=0$ to $j=\infty$ Eq. (6) ($k=n-2$) is obtained after letting $m,m'\to 0$ with fixed m/m'. H_1 is the coefficient of the logarithmically divergent part of $P_{1,c}$ where

$$P_{1,c}(q^2;m,y)-P_{1,c}(q^2;m',y) = \sum_{j=1}^{\infty} y^{j-1} h_j P^j_{1,c}(q^2;m,m') . \quad (17)$$

IV. A SERIES FORMED FROM THE BAD CONTRACTIONS OF T_1

. Let $\overline{P}_{1,2n}$ denote the object formed from all possible bad contractions of T_1. It is defined through

$$\overline{P}_{1,2n}(q^2;m,y)-\overline{P}_{1,2n}(q^2;m',y) = \sum_{j=0}^{\infty} y^j \overline{P}^j_{1,2n}(q^2;m,m') , \quad (18)$$

where $\overline{P}^j_{1,2n}$ is a sum over diagrams with $j+n-1$ internal photons; from the definition of a bad contraction only diagrams of types A and C are included in $\overline{P}^j_{1,2n}$ (there should be at least one curved photon).

It is straightforward to prove that all diagrams of types A and C are included in $\overline{P}^j_{1,2n}$. To this end define three numerical coefficients:

$b_{n,j,i}$: the number of diagrams with i straight photons in $\overline{P}^j_{1,2n}$ (examples: $b_{2,1,1}=6$, $b_{2,2,1}=78$).

$c_{n,m}$: the number of diagrams in $\overline{P}^0_{1,2n}$ such that m curved photons are on one side of the fermion loop, and let us call this side (arbitrarily) "up" (examples: $c_{2,0}=2$, $c_{3,2}=9$).

$d_{n,m,j,k,i}$: the number of diagrams in $\overline{P}^j_{1,2n}$ as contributed from a specific $\overline{P}^0_{1,2n}$ with m curved photons up, such that among the j photon lines there are i straight photons and k curved photons up (examples: $d_{2,1,2,1,1}=30$, $d_{3,2,1,0,1}=5$).

These coefficients are related through

$$b_{n,j,i} = \sum_{m=0}^{n-1} \sum_{k=0}^{j-1} c_{n,m} d_{n,m,j,k,i} . \quad (19)$$

One can exactly repeat the proof presented in lemma 2 of Ref. 10 (just set $s=0$) giving

$$c_{n,m} = \frac{n}{2^{n-1}} \frac{(2m)!(2n-2-2m)!}{m!(n-1-m)!} , \quad (20)$$

$$d_{n,m,j,k,i} = \frac{(2m+1)_{2k}(2n-2m-1)_{2j-2i-2k}(2m+2k+1)_i(2n+2j-2m-2k-2i-1)_i}{2^{j-1}k!(j-i-k)!i!} , \quad (21)$$

with Pochhammer's symbol defined as

$$(z)_0=1, \quad (z)_l = z(z+1)\cdots(z+l-1) . \quad (22)$$

Thus both $c_{n,m}\neq 0$ and $d_{n,m,j,k,i}\neq 0$ for all possible values of the indices, which completes the proof.

In analogy with Eq. (14) we define

$$\overline{P}^j_{1,c}(q^2;m,m') = \sum_{i=0}^{j-1} \pi^{j;\,i}_{1,c}(q^2;m,m') ; \quad (23)$$

$\overline{P}^j_{1,c}$ includes all diagrams of types A and C. We now find the number of diagrams in $\bar{P}^j_{1,2n}$ and in $\bar{P}^j_{1,c}$ which will enable us to arrive at an equation similar to Eqs. (10) and (16).

$\bar{P}^j_{1,2n}$: There are[14]

$$n\sum_{l=0}^{n-1}(2l-1)!!(2n-3-2l)!! \tag{24}$$

bad contractions of T_1. Now since there are

$$\frac{(2j+2n-1)!}{2^j j!(2n-1)!} \tag{25}$$

diagrams of order j in T_1 there are

$$\frac{(2j+2n-1)!}{2^j j!(2n-1)!}n\sum_{l=0}^{n-1}(2l-1)!!(2n-3-2l)!! \tag{26}$$

diagrams in $\bar{P}^j_{1,2n}$.

$\bar{P}^j_{1,c}$: There are

$$\frac{(2j+1)!}{2^j j!} \tag{27}$$

diagrams in $\pi^j_{1,c}$, $j!$ of which belong to type B. Therefore, there are

$$\frac{(2j+1)!-(j!)^2 2^j}{2^j j!} \tag{28}$$

diagrams in $\bar{P}^j_{1,c}$.

From Eqs. (26) and (28) we get

$$\bar{g}_n\frac{(j+n-1)!}{(j+n-1)j!}\bar{h}_{j+n-1}\bar{P}^{j+n-1}_{1,c}(q^2;m,m') = \bar{P}^j_{1,2n}(q^2;m,m'), \tag{29}$$

where

$$\bar{g}_n = \frac{2^{n-1}n\sum_{l=0}^{n-1}(2l-1)!!(2n-3-2l)!!}{(2n-1)!} \tag{30}$$

and

$$\bar{h}_{j+n-1} = \frac{j+n-1}{1-[(j+n-1)!]^2 2^{j+n-1}/(2j+2n-1)!}\,. \tag{31}$$

Again multiplying Eq. (29) by y^j, summing on j, and letting $m,m'\to 0$ with fixed m/m' gives

$$\left.\frac{d^k\bar{H}_1}{dy^k}\right|_{y=\alpha} = 0 \quad (k\geq 0) \tag{32}$$

($k=n-2$); $\bar{H}_1$ is the coefficient of the logarithmically divergent part of $\bar{P}_{1,c}$ where

$$\bar{P}_{1,c}(q^2;m,y)-\bar{P}_{1,c}(q^2;m',y) = \sum_{j=1}^{\infty}y^{j-1}h_j\bar{P}^j_{1,c}(q^2;m,m')\,. \tag{33}$$

Thus both $\bar{H}_1$ and $y\bar{H}_1$ (see Fig. 3) have an infinite-order zero at α.

V. A LINEAR COMBINATION OF SERIES AND ITS LARGE-PHOTON-NUMBER LIMIT

Let us define the linear combination

$$S_1 = y\frac{dF_1}{dy} - y\bar{H}_1; \tag{34}$$

in other words, S_1 is the coefficient of the logarithmically divergent part of

$$y\frac{d\pi_{1,c}}{dy} - y\bar{P}_{1,c}\,. \tag{35}$$

In this section it is shown that the jth order term of S_1 includes, for large j, all diagrams of type B and no diagrams of other types.

First it is clear, from Eqs. (2) and (32), that S_1 vanishes with an essential singularity at $y=\alpha$; note that S_1 is not identically zero since ydF_1/dy and $y\bar{H}_1$ have a different series representation. We now write

$$y\frac{dF_1}{dy} = \sum_{j=1}^{\infty}y^j j(A_j+B_j+C_j), \tag{36}$$

$$y\bar{H}_1 = \sum_{j=1}^{\infty}y^j\bar{h}_j(A_j+C_j), \tag{37}$$

where A_j, B_j, and C_j denote the contributions of diagrams of types A, B, and C, respectively. The series representation of S_1 is

$$S_1 = \sum_{j=1}^{\infty}y^j[(j-\bar{h}_j)A_j + jB_j + (j-\bar{h}_j)C_j]\,. \tag{38}$$

From Eq. (31) we find

$$j-\bar{h}_j\to 0 \quad (j\to\infty), \tag{39}$$

which gives[15] with Eq. (38)

$$j\text{th term of } S_1\to jB_j \quad (j\to\infty)\,. \tag{40}$$

Equation (40) is the main result of this paper. We conclude from it that the essential singularity of F_1 can be located by considering only those diagrams in which all internal photons are straight (see Fig. 2). There are all together $j!$ such diagrams, although not all of them are topologically distinct.

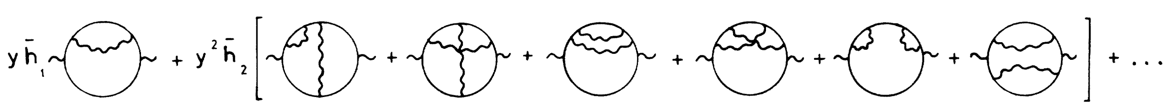

FIG. 3. Topologically distinct diagrams in yH_1; $\bar{h}_j$ are defined in Eq. (31).

VI. CONCLUSIONS

It was shown that if F_1, the coefficient of the logarithmically divergent part of the single-fermion-loop vacuum-polarization diagrams, vanishes with an infinite-order zero at α, the α can be located by considering a simple set of diagrams. It is enough to calculate (in any gauge chosen) only those diagrams with a large number of photons exchanged between the parts of the loop (Fig. 2). Diagrams with curved photons (self-energy insertions, vertex corrections, etc.) are not needed.

A calculation of these diagrams is highly desirable since it will either prove or disprove the conjectured finiteness of QED,[2] and if Adler's loopwise summation[4] holds one may have the extra bonus of having calculated α.

It is interesting that the set of diagrams considered is very similar to multiperipheral diagrams, assumed to be dominant in high-energy scattering. Such diagrams have in fact been discussed,[16] however, only in the limit appropriate for high-energy two-body fixed-angle scattering.

Note added in proof. Bad and good contractions are operations on T_1 if finite QED is free for $m \to 0$.[17] We thank R. Roskies for correspondence regarding this point.

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[5]For an alternative proof of Eq. (2) see J. Bernstein, Nucl. Phys. B95, 461 (1975).

[6]From here on we write α as the eigenvalue just for brevity, without commitment to the renormalized coupling constant.

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[11]Even if the series is divergent at $y=0$ [as conjectured in F. J. Dyson, Phys. Rev. 85, 631 (1952)] we can find $\bar{y}>0$ such that a power series exists around $(\bar{y}-\alpha)/2$. In any case, the large-order terms determine the location of the essential singularity.

[12]The factor $(2n-1)!$ is here for convenience. It accounts for permutation of external photons in T_1, and since it is j-independent is irrelevant for our discussion.

[13]Please note that a factor of 2^{j+n-1} in the second part of the denominator of h_{j+n-1} is erroneously missing from Eq. (35) in Ref. 10.

[14]For all the following numerical coefficients see Ref. 10.

[15]Note that $j-\bar{h}_j \to 0$ results in $(j-\bar{h}_j)(A_j+C_j)\to 0$ since $y\bar{H}_1$ has an essential singularity at α, and from Eq. (8) it is easy to see that the radius of convergence of $\sum_j y^j(j-\bar{h}_j)(A_j+C_j)$ is greater than α.

[16]See, for instance, H. Cheng and T. T. Wu, Phys. Rev. 186, 1611 (1969).

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