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Bhabha first-order wave equations. IV. Causality with minimal electromagnetic coupling*

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We demonstrate that the arbitrary-spin Bhabha fields with minimal electromagnetic coupling are causal in both the c-number and q-number theories. We first obtain the Klein-Gordon (KG) divisors in closed form in terms of the elementary symmetric functions. c-number causality is easily demonstrated for half-integer spin with the Velo-Zwanziger method and for integer spin by using Wightman's suggestion involving the KG divisors. For the q-number demonstration we set up an indefinite-metric second-quantized formalism, and use the above KG divisors to show causality in closed form for arbitrary spin. In both the c-number and q-number theories a special handling of the integer-spin subsidiary components is necessary. Our discussion focuses on the Bhabha indefinite metric and on the connection between the number of derivatives in a theory and the occurrence or nonoccurrence of causality.

I. INTRODUCTION

In this series of papers[1-3] we have been investigating the properties of the Bhabha first-order wave equations for arbitrary spin[4-6]

$$(\partial\cdot\alpha+\chi)\psi=0, \tag{1.1}$$

where the Bhabha matrices α_μ for representations up to maximum spin $\mathcal{S}$ are defined by the equations

$$[[\alpha_\mu,\alpha_\nu],\alpha_\lambda]=\alpha_\mu\delta_{\nu\lambda}-\alpha_\nu\delta_{\mu\lambda}, \tag{1.2}$$

$$\prod_{n=-\mathcal{S}}^{\mathcal{S}}(\alpha_\mu-nI)=0, \tag{1.3}$$

with unity I added by hand for integer-spin representations.[7] That is, the α_μ correspond to the $J_{\mu 5}$ generators of the algebra so(5).[2,4-9] It should be observed that in two papers obscured by World War II, Lubański actually preceded Bhabha in investigating the so(5) Eqs. (1.1)–(1.3). Also, work by Madhavarao at least partially did the same. (See Ref. 9 for details.)

As before, our α_μ matrices will be self-adjoint, we will use the metric $\delta_{\mu\nu}$ relating four-vector quantities $x_\mu=(\vec{x},ix_0)$, $\mu=1,2,3,4$, $\partial\cdot\alpha=\partial_\lambda\alpha_\lambda$, and α_4 will be diagonal with eigenvalues $+\mathcal{S}$ to $-\mathcal{S}$. For the volume-element integration in Sec. IV we will use $d^4x=dx_1dx_2dx_3dx_0$ to conform to standard usage.

Paper I (Ref. 1) discussed the C, P, and T transformations of these equations, paper II (Ref. 2) discussed the mass and spin compositions, the Hamiltonians, and the general Sakata-Taketani reductions of the equations, and in paper III (Ref. 3) the Poincaré generators were investigated.

At the end of paper III (Ref. 3) we reemphasized a point made by Jauch and Rohrlich[10] for the Dirac field Poincaré generators. Given that the uncoupled generators satisfy the associated Lie algebra, they observed that, since the interaction Lagrangian $ie\bar{\psi}\gamma_\lambda A_\lambda\psi$ contains no derivatives, simply by showing that the commutation relations of the second-quantized minimally coupled fields are preserved, one has also shown that the interacting field Poincaré generators satisfy the associated Lie algebra.

In principle this observation should go through for minimally coupled high-spin fields, such as the Bhabha field where we have already shown[3] that the free Bhabha generators in Eqs. (III1.5)–(III1.9) satisfy the associated Lie algebra. However, in practice deep problems have arisen, essentially from the constraint derivatives inherent in most high-spin equations. Johnson and Sudarshan[11] focused on the problem by showing that the spin-$\frac{3}{2}$ Rarita-Schwinger (RS) field[12] no longer obeys the proper equal-time field anticommutation relation (zero at spacelike points) when it is minimally coupled to an external electromagnetic field using the standard canonically conjugate formalism. Johnson and Sudarshan also demonstrated the same effect for a mixed spin-$\frac{1}{2}$ and spin-$\frac{3}{2}$ field invented by Bhabha.[13] However, as has been emphasized,[3,14-16] this field is not one of the fields we are talking about in this paper, it is another field. In fact, for the spin-$\frac{3}{2}$ piece the Lorentz-group content of this special field is that of the RS field.

The results of Johnson and Sudarshan were placed on a new foundation by the important discovery of Velo and Zwanziger[17,18] that the problem could be traced even to the c-number theory. By using the method of characteristics to study the form of the hyperbolic differential equations involved, they were able to determine the propagation properties of a number of interacting field theories. In particular, they found that when minimal electromagnetic coupling is introduced into the RS spin-$\frac{3}{2}$ and the tensor spin-2 theories, they become noncausal. That is, the propagation goes outside the forward light cone, no matter how small the interaction. The same was found to be the case for the spin-1 Proca field with electric quadrupole coupling.

Given this background a large literature has grown on the causality problem, and we cite a number of examples.[15-51] These references contain discussions on arbitrary- or multiple-spin fields,[15-33] as well as on specific spin-1 fields,[34-39] spin-$\frac{3}{2}$ fields,[40-47] and spin-2 fields,[48,49] among which there are a few disagreements.[38,40,41] We also specifically refer the reader to the general discussions of Wightman on causality and other problems in constructing interacting field theories.[23,24,50,51]

As of yet, no universally accepted demonstration has been given that there exists an interacting high-spin ($\geq \frac{3}{2}$) field theory devoid of all problems. In fact one of the motivations of this series of papers has been to see how devoid of problems the Bhabha system is. It is the specific purpose of this paper to demonstrate that noncausality from minimal electromagnetic interaction is not a problem it has.

We begin in Sec. II by using the theory of symmetric functions,[52-54] discussed in the Appendix, to obtain general closed expressions for the Klein-Gordon (KG) divisors of arbitrary-spin Bhabha fields. We will use these closed expressions for the divisors to help us in both our c-number and q-number demonstrations of causality.

Section III gives the easier c-number demonstration. We first review the prescriptions for determining causality or noncausality given for nonsingular field equation matrices by Velo and Zwanziger[17,18,20] and by Wightman,[23] and then for singular field equation matrices as discussed by Wightman[50] for Duffin-Kemmer-Petiau (DKP) fields, based on the work of Hörmander.[55,56] We then simplify the calculations of Nagpal[16,30] and others[32] on causality for the Bhabha c-number theory with minimal electromagnetic interaction. Nagpal's first calculation[16] actually only showed causality for the half-integer-spin case since he was assuming a nonsingular α_4. To show causality for the integer-spin case, where α_4 is singular, involves what is a special handling of the subsidiary components, something that will also be necessary in the q-number theory. Recently Nagpal[30] realized, as we did independently,[30] that his previous c-number calculation did not apply to the integer-spin case because α_4 is singular. He then used a manipulation of the multimass Klein-Gordon (KG) equation to explain how the integer-spin c-number fields are causal.[30] This method is essentially the method we will use., i.e., that discussed by Wightman.[50,56] However, our demonstration will be simpler, being based on the closed form for the KG divisors given in Sec. II, and the properties of the α_μ matrices we have previously discussed in this series. We will conclude this section with calculations on nonminimal coupling, renormalization, and a method of determining if there is causality in the case of constant coefficients.[23]

The causality of the minimally coupled spin-$\frac{3}{2}$ Bhabha q-number field was first shown by Baisya.[15,22] Nagpal later extended these results by an iteration method to show how the calculation goes through for arbitrary half-integer-spin Bhabha fields, but he also came to the conclusion that the integer-spin q-number fields were noncausal.[16] However, recently Nagpal[30] changed his previous conclusions[16] about the integer-spin fields being noncausal and, by case-by-case analyses of the systems of q-number equations, came to the causality conclusions we have independently[30] arrived at. The advantage of our method is that we will use a proper indefinite-metric quantization and will be able to do the general calculations in simple closed form for arbitrary spin by using the theory of symmetric functions[52-54] reviewed in the Appendix.

In Secs. IV and V we will present our demonstration that arbitrary-spin Bhabha fields with minimal electromagnetic coupling are causal in the q-number theory. Section IV presents the formalism. In this discussion we will emphasize a physically important variation of our method from that of Baisya and Nagpal. We will perform an indefinite-metric second quantization with an interaction Hamiltonian that is metric-(pseudo-)Hermitian (which, as discussed in Sec. II of paper III,[3] it properly should be). We also do this because one should come squarely face to face with the fact that there is a built-in indefinite metric[57-61] in the Bhabha system.[62-64]

Having presented the formalism, we then demonstrate in Sec. V that the coupled Heisenberg fields have the same form as the free auxiliary fields for the half-integer-spin case, and hence show that the field anticommutation relations are preserved. For the integer-spin case we will show that although the coupled Heisenberg fields contain new pieces, these new pieces are multiplied by the projection operators onto the subsidiary components $\mathcal{G}_0(\mathcal{S})$ defined in Eq. (II3.8). Thus, the physical particle-components fields have their commutation relations preserved and so are also causal. (This extra piece is well known to exist in the DKP case.)

In Sec. VI we will review our results and comment on a number of points. These include the connection of the extra subsidiary-components terms in the integer-spin q-number theory both to the c-number integer-spin subtlety and also to the effects of extra derivatives in couplings. We will also discuss the significance of the indefinite metric to the physical interpretation and mathematical consistency of the Bhabha theory. In paper V of this series[65] we will discuss the indefinite metric in more detail, and calculate the generalized Foldy-Wouthuysen transformations for Bhabha fields.

II. KLEIN-GORDON DIVISORS

The Klein-Gordon divisors for general Bhabha fields were discussed in principle by Umezawa and Visconti[66] (but see Glass[67] concerning errors with the Harish-Chandra[68-72] algebra). Special cases were calculated by Umezawa and Visconti,[66] Baisya,[15,22] and Nagpal,[16] who also gave a complicated expression for general integer spin.[30] In this section we will derive general, closed-form expressions for the KG divisors in terms of the elementary symmetric functions. These explicit expressions, to our knowledge, have never been given before. We will discuss the half-integer- and integer-spin cases separately, and then proceed in Secs. III and IV to perform the causality calculations.

A. Half-integer spin

For the half-integer-spin case the Klein-Gordon divisors $\mathfrak{D}(\partial,\ \mathcal{S}=n+\frac{1}{2})$ are the solutions to the equations

$$\begin{aligned}\mathfrak{D}(\partial,\ \mathcal{S}=n+\tfrac{1}{2})\Lambda &\equiv \mathfrak{D}(\partial,\ \mathcal{S}=n+\tfrac{1}{2})(\partial\cdot\alpha+\chi)\\ &=\prod_{j=1}^{\mathcal{S}+1/2}[\Box-\chi^2/(j-\tfrac{1}{2})^2]\\ &=\sum_{k=0}^{\mathcal{S}+1/2}(-1)^{\mathcal{S}+1/2-k}\Box^k(\chi^2)^{\mathcal{S}+1/2-k}\\ &\qquad\times\phi_{\mathcal{S}+1/2-k}(l_{\{\mathcal{S}+1/2\}}),\end{aligned}\tag{2.1}$$

where the elementary symmetric functions $\phi_{\mathcal{S}+1/2-k}$ of the $(\mathcal{S}+\frac{1}{2})$ quantities

$$l_k=\frac{1}{(k-\frac{1}{2})^2}\tag{2.2}$$

are described in the Appendix. For the reasons given in the next paragraph, we can write the KG divisors in the form

$$\begin{aligned}\mathfrak{D}(\partial,\ \mathcal{S}=n+\tfrac{1}{2})=&\sum_{i=0}^{\mathcal{S}-1/2}\chi^{2\mathcal{S}-2i}[1-(\partial\cdot\alpha)/\chi]\\ &\times\sum_{j=0}^{i}\Box^{i-j}(\partial\cdot\alpha)^{2j}c_{ij},\end{aligned}\tag{2.3}$$

where the c_{ij}, $i\geqslant j$, remain to be determined.

The factor $[1-(\partial\cdot\alpha)/\chi]$ in Eq. (2.3) comes about because, since

$$\chi[1-(\partial\cdot\alpha)/\chi]\Lambda=[\chi^2-(\partial\cdot\alpha)^2]\tag{2.4}$$

and the right-hand side of Eq. (2.4) is a factor in the characteristic KG equation for the α_μ, only then will one have the proper even powers of χ on the left-hand side of Eq. (2.1). In particular, rewriting (2.1) as

$$\begin{aligned}&\sum_{i=0}^{\mathcal{S}-1/2}\chi^{2\mathcal{S}-1-2i}[\chi^2-(\partial\cdot\alpha)^2]\sum_{j=0}^{i}\Box^{i-j}(\partial\cdot\alpha)^{2j}c_{ij}\\ &\qquad=\sum_{k=0}^{\mathcal{S}+1/2}(-1)^{\mathcal{S}+1/2-k}\Box^k(\chi^2)^{\mathcal{S}+1/2-k}\phi_{\mathcal{S}+1/2-k}(l_{\{\mathcal{S}+1/2\}}),\end{aligned}\tag{2.5}$$

one has just the powers $\chi^0, \chi^2, \ldots, \chi^{2\mathcal{S}+1}$.

Also, the second sum on the left-hand side of (2.5) is the sum of all the possible products $\Box^{i-j}(\partial\cdot\alpha)^{2j}$ such that each term of the outside sum has the proper dimension $(\text{mass})^{2\mathcal{S}+1}$. Our problem is now reduced to finding the quantities c_{ij}.

Starting with the $\chi^{2\mathcal{S}+1}$ term in Eq. (2.5) one sees by inspection that

$$c_{00}=(-1)^{\mathcal{S}+1/2}\phi_{\mathcal{S}+1/2}(l).\tag{2.6}$$

Next taking the $\chi^{2\mathcal{S}-1}$ term in Eq. (2.5), one has the equation

$$\Box c_{10}+(\partial\cdot\alpha)^2c_{11}-(\partial\cdot\alpha)^2c_{00}=(-1)^{\mathcal{S}-1/2}\Box\phi_{\mathcal{S}+1/2-1}(l),\tag{2.7}$$

which yields the solution

$$\begin{aligned}c_{11}&=c_{00},\\ c_{10}&=(-1)^{\mathcal{S}-1/2}\phi_{\mathcal{S}-1/2}(l).\end{aligned}\tag{2.8}$$

Similarly proceeding to the $\chi^{2\mathcal{S}-3}$ and $\chi^{2\mathcal{S}-5}$ terms yields the solutions

$$\begin{aligned}c_{33}&=c_{22}=c_{11}=c_{00}=(-1)^{\mathcal{S}+1/2}\phi_{\mathcal{S}+1/2}(l),\\ c_{32}&=c_{21}=c_{10}=(-1)^{\mathcal{S}+1/2-1}\phi_{\mathcal{S}+1/2-1}(l),\\ c_{31}&=c_{20}=(-1)^{\mathcal{S}+1/2-2}\phi_{\mathcal{S}+1/2-2}(l),\\ c_{30}&=(-1)^{\mathcal{S}+1/2-3}\phi_{\mathcal{S}+1/2-3}(l),\end{aligned}\tag{2.9}$$

suggesting that the general solution for the c_{ij} is

$$c_{ij}=(-1)^{\mathcal{S}+1/2-i+j}\phi_{\mathcal{S}+1/2-i+j}(l_{\{\mathcal{S}+1/2\}}).\tag{2.10}$$

The solution (2.10) can be demonstrated by induction. From Eq. (2.9), Eq. (2.10) is true for $(i,j)=0,1,2,3$. Now assume that Eq. (2.10) is true for $n=i\geqslant j$. Then by considering the case $(n+1)=i\geqslant j$ one finds that Eq. (2.10) still holds for $(\mathcal{S}-\frac{1}{2})\geqslant(n+1)$. Finally, by inspection the solution holds for the χ^0 term.

Thus, the KG divisors for half-integer-spin Bhabha fields are

$$\mathfrak{D}(\partial,\ \mathcal{S}=n+\tfrac{1}{2})=\sum_{i=0}^{\mathcal{S}-1/2}\chi^{2\mathcal{S}-2i}\left[1-\frac{(\partial\cdot\alpha)}{\chi}\right]\sum_{j=0}^{i}\Box^{i-j}(\partial\cdot\alpha)^{2j}(-1)^{\mathcal{S}+1/2-i+j}\phi_{\mathcal{S}+1/2-i-j}(l_{\{\mathcal{S}+1/2\}}),\tag{2.11}$$

$$\phi_0=1,\tag{2.12a}$$

$$\phi_{j\neq 0}(l_{\{\mathcal{S}+1/2\}})\equiv\left[\sum_{i(1)>i(2)>\cdots>i(j)=1}^{\mathcal{S}+1/2}(l_{i(1)}l_{i(2)}\cdots l_{i(j)})\right],\tag{2.12b}$$

$$l_k\equiv 1/(k-\tfrac{1}{2})^2.\tag{2.12c}$$

Special cases are

$$\mathfrak{D}(\partial,\ \mathcal{S}=\tfrac{1}{2})=-4\chi\left[1-\frac{(\partial\cdot\alpha)}{\chi}\right],\tag{2.13}$$

$$\mathfrak{D}(\partial,\ \mathcal{S}=\tfrac{3}{2})=\tfrac{16}{9}\chi^3\left[1-\frac{(\partial\cdot\alpha)}{\chi}\right]+\tfrac{16}{9}\chi\left[1-\frac{(\partial\cdot\alpha)}{\chi}\right][(\partial\cdot\alpha)^2-\tfrac{40}{16}\Box],\tag{2.14}$$

$$\mathfrak{D}(\partial,\ \mathcal{S}=\tfrac{5}{2})=-\tfrac{64}{225}\chi^5\left[1-\frac{(\partial\cdot\alpha)}{\chi}\right]\{1+\chi^{-2}[(\partial\cdot\alpha)^2-\tfrac{35}{4}\Box]+\chi^{-4}[(\partial\cdot\alpha)^4-\tfrac{35}{4}(\partial\cdot\alpha)^2\Box+\tfrac{259}{16}\Box^2]\}.\tag{2.15}$$

Equation (2.13) is the Dirac equation KG divisor, only in the Bhabha normalization instead of the usual γ-matrix normalization. Equation (2.14) for $\mathcal{S}=\frac{3}{2}$ agrees with the Baisya[15,22] and Nagpal[16] calculations, and Eq. (2.15) agrees[73] with Nagpal's[16] calculation. Also, one can easily demonstrate that the KG divisors in Eqs. (2.13)–(2.15) satisfy the original defining equation (2.1).

B. Integer spin

For the integer-spin case the Klein-Gordon divisors $\mathfrak{D}(\partial,\ \mathcal{S}=n)$ are the solutions to the equations

$$\begin{aligned}\mathfrak{D}(\partial,\ \mathcal{S}=n)\Lambda&\equiv\mathfrak{D}(\partial,\ \mathcal{S}=n)(\partial\cdot\alpha+\chi)\\&=\prod_{j=1}^{\mathcal{S}}(\Box-\chi^2/j^2)\\&=\sum_{k=0}^{\mathcal{S}}\Box^k(\chi^2)^{\mathcal{S}-k}(-1)^{\mathcal{S}-k}\phi_{\mathcal{S}-k}(\mathcal{L}_{\{\mathcal{S}\}}),\end{aligned}\tag{2.16}$$

where now we are dealing with the elementary symmetric functions $\phi_{\mathcal{S}-k}$ of the $\mathcal{S}$ quantities

$$\mathcal{L}_k=\frac{1}{k^2}.\tag{2.17}$$

The solution proceeds in the same manner as for the half-integer-spin case, and one finds that

$$\mathfrak{D}(\partial,\ \mathcal{S}=n)=\sum_{i=0}^{\mathcal{S}}\left(1-\frac{\partial\cdot\alpha}{\chi}\right)\chi^{2(\mathcal{S}-i)-1}\sum_{j=0}^{i}A_{ij}\tag{2.18a}$$

$$=\left[\sum_{i=0}^{\mathcal{S}-1}\left(1-\frac{\partial\cdot\alpha}{\chi}\right)\chi^{2(\mathcal{S}-i)-1}\sum_{j=0}^{i}A_{ij}\right]+\chi^{-1}\sum_{j=0}^{\mathcal{S}}A_{\mathcal{S}j},\tag{2.18b}$$

$$A_{ij}=\Box^{i-j}(\partial\cdot\alpha)^{2j}(-1)^{\mathcal{S}-i+j}\phi_{\mathcal{S}-i+j}(\mathcal{L}_{\{\mathcal{S}\}}),\tag{2.19}$$

$$\phi_0=1,\tag{2.20a}$$

$$\phi_{j\neq 0}(\mathcal{L}_{\{\mathcal{S}\}})=\left[\sum_{i(1)>i(2)>\cdots>i(j)=1}^{\mathcal{S}}(\mathcal{L}_{i(1)}\mathcal{L}_{i(2)}\cdots\mathcal{L}_{i(j)})\right],\tag{2.20b}$$

$$\mathcal{L}_k=\frac{1}{k^2}.\tag{2.20c}$$

The second form in Eq. (2.18) comes by taking the characteristic equation (1.3) for α_4, multiplying it by $(\partial_4)^{2\mathcal{S}+1}$ and then transforming it to an arbitrary system, yielding

$$(\partial\cdot\alpha)\sum_{j=0}^{\mathcal{S}}A_{\mathcal{S}j}=0.\tag{2.21}$$

The extra factor $(\partial\cdot\alpha)$, of course, comes from the subsidiary components (the 0 eigenvalues of α_4), and is not accounted for in the KG operator of Eq. (2.16). This will cause $\mathfrak{D}(\partial,\ \mathcal{S}=n)$ to yield extra pieces in the integer-spin Heisenberg interacting fields, as we will see in Sec. V.

Special cases of Eq. (2.18) are

$$\mathfrak{D}(\partial,\ \mathcal{S}=1)=\chi^{-1}[(\partial\cdot\alpha)(\chi-\partial\cdot\alpha)+(\Box-\chi^2)],\tag{2.22}$$

$$\mathfrak{D}(\partial,\ \mathcal{S}=2)=\tfrac{1}{4}\chi\left(1-\frac{\partial\cdot\alpha}{\chi}\right)\{\chi^2+[-5\Box+(\partial\cdot\alpha)^2]\}+\tfrac{1}{4}\chi^{-1}[4\Box^2-5\Box(\partial\cdot\alpha)^2+(\partial\cdot\alpha)^4],\tag{2.23}$$

$$\mathfrak{D}(\partial,\ \mathcal{S}=3)=\frac{-1}{36}\chi^5\left(1-\frac{\partial\cdot\alpha}{\chi}\right)+\frac{\chi^3}{36}\left(1-\frac{\partial\cdot\alpha}{\chi}\right)[14\Box-(\partial\cdot\alpha)^2]+\frac{\chi}{36}\left(1-\frac{\partial\cdot\alpha}{\chi}\right)[-49\Box^2+14\Box(\partial\cdot\alpha)^2-(\partial\cdot\alpha)^4]$$
$$+\frac{\chi^{-1}}{36}[36\Box^3-49\Box^2(\partial\cdot\alpha)^2+14\Box(\partial\cdot\alpha)^4-(\partial\cdot\alpha)^6]. \quad (2.24)$$

Equation (2.22) is the DKP Klein-Gordon divisor. The special cases (2.23) and (2.24) agree with the calculations of Nagpal,[16] as well as the $q=0$, $n=2, 3$ special cases of Wightman's[23] Eq. (2.35) generalized Harish-Chandra[68-72] KG divisors. Again one can easily demonstrate that the special cases (2.22)–(2.24) satisfy the defining Eq. (2.16).

III. CAUSALITY OF THE c-NUMBER THEORY

The method used by Velo and Zwanziger[17,18] to discuss causality in the c-number theory consists of first writing the wave equation in the form

$$\Lambda(\partial,\chi,\mathfrak{C})\psi=(\partial\cdot\alpha+\chi+\mathfrak{C})\psi=0, \quad (3.1)$$

where $\mathfrak{C}$ is a matrix describing the interaction, and then solving the determinant equation

$$\mathrm{Det}\hat{\Lambda}(\partial_\lambda\to in_\lambda,\chi,\mathfrak{C}(\partial_\lambda\to in_\lambda))=0 \quad (3.2)$$

for n_λ, where $\hat{\Lambda}$ is that matrix piece of Λ which will yield the highest powers of ∂_λ in the determinant. The n_λ are the normals to the characteristic surfaces, so for the propagation to be causal one wants solutions of n_λ from (3.2) which are never timelike. This method is applicable if the matrix $\hat{\Lambda}$ is nonsingular.

For Bhabha fields, with the minimal substitution

$$\partial_\lambda\to\partial_\lambda^-\equiv\partial_\lambda-ieA_\lambda,\quad p_\lambda^-\equiv p_\lambda-eA_\lambda, \quad (3.3)$$

the interaction $-ieA_\lambda$ enters into the matrix elements of Λ only as ∂_λ^-. Thus, the determinant of the highest derivatives will have only the highest products of ∂_λ without any A_λ involved, so that

$$\mathrm{Det}\hat{\Lambda}(in_\lambda)=\mathrm{Det}(in\cdot\alpha)=0. \quad (3.4)$$

To rule out timelike n_λ one can show a contradiction. (At this stage it is an assumption that such solutions do or do not exist.) Suppose that a timelike n_λ existed as a solution to Eq. (3.4). Then one could perform a Lorentz transformation to a frame where

$$0=\mathrm{Det}\hat{\Lambda}=\mathrm{Det}(in_4\alpha_4). \quad (3.5)$$

But then taking the diagonal representation for α_4 in the notation of paper II,

$$\alpha_4=\text{block diagonal}[\mathcal{S}\mathcal{I}_{\mathcal{S}}^+,(\mathcal{S}-1)\mathcal{I}_{\mathcal{S}-1}^+,\ldots,-(\mathcal{S}-1)\mathcal{I}_{\mathcal{S}-1}^-,-\mathcal{S}\mathcal{I}_{\mathcal{S}}^-], \quad (3.6)$$

one would have

$$0=\mathrm{Det}\hat{\Lambda}=(in_0)^{[d_5(\mathcal{S},S)-a(0)]}0^{a(0)}\prod_{j=(1/2,1)}^{\mathcal{S}}(\mathcal{S}-j)^{2a(j)}, \quad (3.7)$$

where the $a(j)$ are the dimensionalities of the j eigenvalue block of α_4, and the dimensionality of the matrices of an arbitrary Bhabha algebra $(\mathcal{S}, S)$ is

$$d_5(\mathcal{S},S)=\tfrac{1}{6}(2\mathcal{S}+3)(2S+1)\times[(\mathcal{S}+1)(\mathcal{S}+2)-S(S+1)] \quad (3.8)$$

$$\equiv a(0)+\sum_{j=(1/2,1)}^{\mathcal{S}}2a(j). \quad (3.9)$$

For half-integer-spin representations, there is no $a(0)$, so (3.7) implies the contradiction $n_0=0$, ruling out timelike solutions. (The same argument can be used to rule out spacelike solutions for the n_λ. In fact, the solutions are lightlike; i.e., the propagation is bounded by the light cone.)

Because of the factor $0^{a(0)}$ in Eqs. (3.7) for integer-spin Bhabha fields (the α_λ are singular), the above method is not applicable there. Then one has to use the method Wightman[50,56,74] attributed to Svensson[74] and discussed in Ref. 50 for Dirac and DKP fields. This method states that given the KG divisor, if one can show that

$$\mathfrak{D}(\partial^-,\mathcal{S})\Lambda(\partial^-)=\prod_{j=(1/2,1)}^{\mathcal{S}}(\partial^-\cdot\partial^--\chi^2/j^2)+K, \quad (3.10)$$

where K is a matrix whose principal part involves derivatives of order less than those in the KG multimass operator on the right of (3.10); i.e., less than order $(2\mathcal{S}+1)$ for half-integer spin and less than order $(2\mathcal{S})$ for integer spin, then the fields are causal whether or not the α_λ matrices are singular.

From the general closed forms for the KG divisors given in Sec. II, this can easily be shown.

First one realizes that the term K is nonzero only because the ∂_λ^- do not commute,

$$[\partial_\mu^-,\partial_\nu^-]=-ieF_{\mu\nu}. \quad (3.11)$$

So, the maximum power of ∂_λ in K comes from that piece involving a single commutation relation of the type (3.11). Since the maximum power of ∂_λ in the half-integer-spin KG divisor (2.11) is $(2\mathcal{S})$, the product of $\mathfrak{D}(\partial^-)$ with $\Lambda(\partial^-)$ will yield a maximum power of ∂_λ of $(2\mathcal{S}+1)$. Therefore, the maximum derivative in K will be of order $(2\mathcal{S}-1)$, two orders less than that in the KG operator on the right of (3.11), implying causality.

For integer-spin fields, one should start with the form of the KG divisor given in Eq. (2.18b). Then, in a similar manner, the KG divisor is of order $(2\mathcal{S})$, meaning the product $\mathfrak{D}\Lambda$ will have derivatives of order $(2\mathcal{S}+1)$, so that the maximum power derivative in the matrix K will be of order $(2\mathcal{S}-1)$. This is one order less than the $(2\mathcal{S})$ maximum power in the KG operator on the right of (3.10), so one has shown causality for Bhabha minimally coupled integer-spin fields.

For reference, using the standard notation instead of the Bhabha notation, the Dirac and DKP explicit results are

$$\mathfrak{D}(\partial^-, \mathcal{S}=\tfrac{1}{2})\Lambda(\partial^-) = (\partial^-\cdot\partial^- - m^2) + eF_{\mu\nu}[\gamma_\mu, \gamma_\nu]/(4i), \tag{3.12}$$

$$\mathfrak{D}(\partial^-, \mathcal{S}=1)\Lambda(\partial^-) = (\partial^-\cdot\partial^- - m^2) + \frac{ie}{2m}[(\partial^-\cdot\beta)F_{\mu\nu}\beta_\mu\beta_\nu + F_{\mu\nu}\beta_\mu(\partial^-\cdot\beta)\beta_\nu + \partial^-_\lambda F_{\mu\nu}\beta_\mu\beta_\nu\beta_\lambda - \partial^-_\mu F_{\mu\nu}\beta_\nu - F_{\mu\nu}\partial^-_\mu\beta_\nu]. \tag{3.13}$$

Finally we mention a method for investigating causality which is potentially extremely powerful, but for now is limited to interaction matrices with constant coefficients. Wightman[23] used this method to show noncausality for certain higher-multipole interactions in the DKP spin-0 and spin-1 so(5) representations (1, 0) and (1, 1). Specifically, Wightman found noncausality for the couplings $F_{\mu\nu}[\beta_\mu, \beta_\nu]$ and $G_{\mu\nu}[\{\beta_\mu, \beta_\nu\} + \frac{1}{2}\delta_{\mu\nu}\beta\cdot\beta]$ in the spin-0 case [$F_{\mu\nu}$ ($G_{\mu\nu}$) is antisymmetric (symmetric) in $\mu\nu$], and noncausality for the electric quadrupole coupling $\partial_\lambda F_{\mu\nu}(5+2\beta\cdot\beta)\beta_\mu\beta_\nu\beta_\lambda$ in the spin-1 case. This last agrees with the Velo-Zwanziger[18] Proca field result, and lends further interest to the calculation of Peaslee[75] that (owing to the higher order of the algebra and the associated derivatives) the spin-1 DKP current has new dipole and quadrupole moment divergences in second order beyond the usual first-order charge and mass renormalizations.[75-79] (These divergences do not exist in the DKP spin-0 case, and perhaps emphasize the fact that, since $[\beta_\mu, \beta_\nu]$ is a spin operator, it would be curious to have such a coupling for a spin-0 particle, even if it were causal.)

This method[23,80] states that the principal part of the determinant of the entire matrix Λ will test causality, *versus* the determinant of the principal part $\hat{\Lambda}$, as in Eq. (3.2). Thus, it is only necessary that Λ be nonsingular, even if $\hat{\Lambda}$ is not. The problem is that this method has so far only been shown to be a theorem for constant coefficients[23,80] [$\mathfrak{C}$ in Eq. (3.1) is a matrix of constant coefficients.] However, it still has two useful applications for us. The first comes from observing that if we were to assume this method is valid for nonconstant coefficients, we would, exactly as in Eqs. (3.2)–(3.9), come to the same conclusion of the causality of Bhabha fields with minimal interaction. This is a further[23,80] indication that this method may be able to be shown to hold for nonconstant coefficients, which would be a very important result, indeed.

The second application is in demonstrating the difference between the Dirac and DKP cases for $F_{\mu\nu}$ coupling. Assume constant $F_{\mu\nu}$ and A_ν in p^-_ν. Further, note that since both $F_{\mu\nu}$ and A_ν are constant: (i) They should not be considered to be mathematically related as in Eq. (3.11) but rather both taken as fundamental fields; (ii) in taking the determinant below, all matrix elements commute. Then we have for Dirac and spin-0 DKP

$$\mathrm{Det}\Lambda^D = \mathrm{Det}(ip^-\cdot\gamma + m - \kappa F_{\mu\nu}[\gamma_\mu, \gamma_\nu]/4i) \tag{3.14a}$$

$$\begin{aligned} &= [(p_0^-)^2 - (\vec{\mathrm{p}}^-)^2 - m^2]^2 \\ &\quad + \kappa^2\{8p_0^-\vec{\mathrm{p}}^-\cdot(\vec{\mathrm{E}}\times\vec{\mathrm{B}}) + 4[(\vec{\mathrm{E}}\cdot\vec{\mathrm{p}}^-)^2 + (\vec{\mathrm{B}}\cdot\vec{\mathrm{p}}^-)^2] - 2[(p_0^-)^2 + (\vec{\mathrm{p}}^-)^2](\vec{\mathrm{E}}^2 + \vec{\mathrm{B}}^2) + 2m^2(\vec{\mathrm{E}}^2 - \vec{\mathrm{B}}^2) + (\vec{\mathrm{E}}^2 - \vec{\mathrm{B}}^2)^2 + 4(\vec{\mathrm{E}}\cdot\vec{\mathrm{B}})^2\}, \end{aligned} \tag{3.14b}$$

$$\mathrm{Det}\Lambda^{\mathrm{DKP}} = \mathrm{Det}(ip^-\cdot\beta + m - \kappa F_{\mu\nu}[\beta_\mu, \beta_\nu]/i) \tag{3.15a}$$

$$\begin{aligned} &= (-m)\{m^2[(p_0^-)^2 - (\vec{\mathrm{p}}^-)^2 - m^2 + \kappa^2(\vec{\mathrm{E}}^2 - \vec{\mathrm{B}}^2)] \\ &\quad + \kappa^2[2p_0^-\vec{\mathrm{p}}^-\cdot(\vec{\mathrm{E}}\times\vec{\mathrm{B}}) + (\vec{\mathrm{E}}\cdot\vec{\mathrm{p}}^-)^2 + (\vec{\mathrm{B}}\cdot\vec{\mathrm{p}}^-)^2 + (\vec{\mathrm{E}}\cdot\vec{\mathrm{B}})^2 - (p_0^-)^2\vec{\mathrm{B}}^2 - (\vec{\mathrm{p}}^-)^2\vec{\mathrm{E}}^2]\}. \end{aligned} \tag{3.15b}$$

For Dirac the principal part being zero means

$$0 = \mathrm{P}(\mathrm{Det}\Lambda^{D}) = [p_0{}^2 - (\vec{\mathrm{p}})^2]^2, \tag{3.16}$$

which implies causal solutions for these constant interaction fields. However, for DKP

$$0 = \mathrm{P}(\mathrm{Det}\Lambda^{\mathrm{DKP}}) = -m\{m^2(p_0{}^2 - \vec{\mathrm{p}}^2) + \kappa^2[2p_0\vec{\mathrm{p}}\cdot\vec{\mathrm{E}}\times\vec{\mathrm{B}} + (\vec{\mathrm{E}}\cdot\vec{\mathrm{p}})^2 + (\vec{\mathrm{B}}\cdot\vec{\mathrm{p}})^2 - p_0{}^2\vec{\mathrm{B}}^2 - \vec{\mathrm{p}}^2\vec{\mathrm{E}}^2]\}, \tag{3.17}$$

which implies noncausal propagation for nonzero spin-0 DKP coupling of the form $F_{\mu\nu}[\beta_\mu, \beta_\nu]$. For example, take $\vec{\mathrm{B}}$ parallel to $\vec{\mathrm{p}}$. Then Eq. (3.15) means

$$0 = (p_0{}^2 - \vec{\mathrm{p}}^2)(1 - \kappa^2\vec{\mathrm{B}}^2/m^2), \tag{3.18}$$

which allows noncausal propagation in a frame where

$$1 = \kappa^2\vec{\mathrm{B}}^2/m^2. \tag{3.19}$$

This is the same type of mathematical solution that Velo and Zwanziger[17] found in their Eq. (2.17) for minimally coupled RS fields.

IV. q-NUMBER FORMALISM

In this section we will perform an indefinite-metric second quantization of the Bhabha fields. (For reference we list a few[10,81-90] of the standard works which describe second-quantized formalisms.) Our method is similar in procedure to the positive-norm multimass method of Baisya[15,22] and Nagpal,[16] which in turn is a generalization of the single-mass techniques of Katayama,[83] Takahashi,[85-88] and Umezawa.[84-86] (The clear second-quantized calculations of Akhiezer and Berestetskii[90] for minimally coupled Dirac and DKP fields are also very useful to consult.) Besides properly taking account of the indefinite metric, we will also discuss the renormalizations involved in interacting second-quantized field theories. Ultimately, this section is devoted to deriving the critical Eq. (4.53), which will be evaluated in Sec. V to show causality.

A. Free fields

The second-quantized Bhabha free fields and adjoint fields can be written as

$$\psi(x) = \sum_{j,s}\int \frac{d^3p}{(2\pi)^{3/2}}[a(\vec{\mathrm{p}},j,s)u(\vec{\mathrm{p}},j,s)e^{ip\cdot x} + b^\dagger(\vec{\mathrm{p}},j,s)v(\vec{\mathrm{p}},j,s)e^{-ip\cdot x}], \tag{4.1}$$

$$\bar{\psi}(x) = \psi^\dagger(x)\eta_4 = \sum_{j,s}\int \frac{d^3p}{(2\pi)^{3/2}}[a^\dagger(\vec{\mathrm{p}},j,s)\bar{u}(\vec{\mathrm{p}},j,s)e^{-ip\cdot x} + b(\vec{\mathrm{p}},j,s)\bar{v}(\vec{\mathrm{p}},j,s)e^{ip\cdot x}], \tag{4.2}$$

where η_4 is the adjoint operator defined in Eq. (I3.47), and $a(\vec{\mathrm{p}},j,s)$ and $a^\dagger(\vec{\mathrm{p}},j,s)$ ($b(\vec{\mathrm{p}},j,s)$ and $b^\dagger(\vec{\mathrm{p}},j,s)$) are the annihilation and creation operators for particles (antiparticles) of momentum $\vec{\mathrm{p}}$, mass state χ/j, and spin state s.

The annihilation and creation operators satisfy the relations

$$[a(\vec{\mathrm{p}},j,s), a^\dagger(\vec{\mathrm{p}}',j',s')]_\pm = [b(\vec{\mathrm{p}},j,s), b^\dagger(\vec{\mathrm{p}}',j',s')]_\pm = \tau_{jj}\delta_{jj'}\delta_{ss'}\delta^3(\vec{\mathrm{p}}-\vec{\mathrm{p}}'), \tag{4.3a}$$

$$[a,a]_\pm = [b,b]_\pm = [a,b^\dagger]_\pm = \cdots(\text{all other relations})\cdots = 0, \tag{4.3b}$$

where anticommutation relations

$$[a,a^\dagger]_+ \equiv \{a,a^\dagger\} \equiv aa^\dagger + a^\dagger a \tag{4.4a}$$

are to be used for half-integer-spin fields, and commutation relations

$$[a,a^\dagger]_- \equiv [a,a^\dagger] \equiv aa^\dagger - a^\dagger a \tag{4.4b}$$

are to be used for integer-spin fields. (We will always take the upper sign for half-integer-spin fields, and the lower sign for integer-spin fields.) If one accepts the standard arguments about the connection of spin and statistics,[91-95] along with the requirement of a positive-definite metric and energy density, then one desires

$$\tau_{jj'} \to \hat{\tau}_{jj'} \equiv \delta_{jj'} \quad \text{(standard requirements)}. \tag{4.5}$$

However, because of their indefinite metric,[62-64] Eq. (4.5) will not hold for the general Bhabha fields, as we will explicitly show.

Proceeding with the formalism, Eqs. (4.1)–(4.3) imply that

$$[\psi_\alpha(x), \bar{\psi}_\beta(x')]_\pm = \sum_j \int \frac{d^3p}{(2\pi)^3}[F^+_{\alpha\beta}(x, x', \vec{p}, j) \pm F^-_{\alpha\beta}(x, x', \vec{p}, j)], \tag{4.6}$$

$$F^+_{\alpha\beta}(x, x', \vec{p}, j) \equiv \tau_{jj} \sum_s u_\alpha(\vec{p}, j, s) \times \bar{u}_\beta(\vec{p}, j, s)e^{ip\cdot(x-x')}, \tag{4.7}$$

$$F^-_{\alpha\beta}(x, x', \vec{p}, j) \equiv \tau_{jj} \sum_s v_\alpha(\vec{p}, j, s) \times \bar{v}_\beta(\vec{p}, j, s)e^{-ip\cdot(x-x')}. \tag{4.8}$$

For the well-known Dirac case ($2\chi = m,\ 2\alpha_\mu = \gamma_\mu$),

$$j \equiv \tfrac{1}{2}, \quad \tau_{\frac{1}{2}\frac{1}{2}} = 1, \tag{4.9}$$

$$F^+_{\alpha\beta}(x, x', \vec{p}, \tfrac{1}{2}) = -\frac{1}{2E}\mathfrak{D}_{\alpha\beta}(ip,\ \mathcal{S} = \tfrac{1}{2})e^{ip\cdot(x-x')}, \tag{4.10}$$

$$F^-_{\alpha\beta}(x, x', \vec{p}, \tfrac{1}{2}) = +\frac{1}{2E}\mathfrak{D}_{\alpha\beta}(-ip,\ \mathcal{S} = \tfrac{1}{2})e^{-ip\cdot(x-x')}, \tag{4.11}$$

with $\mathfrak{D}_{\alpha\beta}(\partial,\ \mathcal{S} = \frac{1}{2})$ given by Eq. (2.13), so that

$$\{\psi_\alpha(x), \bar{\psi}_\beta(x')\}^D = -i\mathfrak{D}_{\alpha\beta}(\partial,\ \mathcal{S} = \tfrac{1}{2})[\Delta^+(x - x', 2\chi) + \Delta^-(x - x', 2\chi)] = -i\mathfrak{D}_{\alpha\beta}(\partial,\ \mathcal{S} = \tfrac{1}{2})\Delta(x - x', 2\chi), \tag{4.12}$$

where $\Delta(x - x', \chi/j)$ is the single-particle invariant Δ function of mass χ/j. For the DKP case,[90] one has ($\chi \equiv m,\ \alpha_\mu \equiv \beta_\mu$)

$$j \equiv 1, \quad \tau_{11} = 1, \tag{4.13}$$

$$F^+_{\alpha\beta}(x, x', \vec{p}, 1) = \frac{ip\cdot\beta}{2mE}(ip\cdot\beta - m)e^{ip\cdot(x-x')}, \tag{4.14}$$

$$F^-_{\alpha\beta}(x, x', \vec{p}, 1) = \frac{ip\cdot\beta}{2mE}(ip\cdot\beta + m)e^{-ip(x-x')}, \tag{4.15}$$

so that

$$[\psi_\alpha(x), \bar{\psi}_\beta(x')]^{\mathrm{DKP}} = -i\left\{\frac{(\partial\cdot\beta)[m - (\partial\cdot\beta)]}{m}\right\} \times [\Delta^+(x - x', m) + \Delta^-(x - x', m)]. \tag{4.16}$$

Thus, since

$$\left(\Box - \frac{\chi^2}{j^2}\right)\Delta(x - x', \chi/j) = 0, \tag{4.17}$$

where

$$\Delta(x - x', \chi/j) = \Delta^+(x - x', \chi/j) + \Delta^-(x - x', \chi/j), \tag{4.18}$$

then from the form $\mathfrak{D}(\partial,\ \mathcal{S} = 1)$ given in Eq. (2.22) combined with Eq. (4.16) we obtain

$$[\psi_\alpha(x), \bar{\psi}_\beta(x')]^{\mathrm{DKP}} = -i\mathfrak{D}_{\alpha\beta}(\partial,\ \mathcal{S} = 1)\Delta(x - x', m). \tag{4.19}$$

In obtaining the results (4.12) and (4.19) for the Dirac and DKP cases, the crucial signs come out correctly because of the normalizations for the quantities

$$\sum_s \bar{u}(\vec{p}, s)\gamma_4 u(\vec{p}, s) = \sum_s u^\dagger(\vec{p}, s)u(\vec{p}, s) = +1, \tag{4.20}$$

$$\sum_s \bar{v}(\vec{p}, s)\gamma_4 v(\vec{p}, s) = \sum_s v^\dagger(\vec{p}, s)v(\vec{p}, s) = +1 \tag{4.21}$$

for Dirac, and

$$\bar{u}(\vec{p})\beta_4 u(\vec{p}) = u^\dagger(\vec{p})\eta_4\beta_4 u(\vec{p}) = +1, \tag{4.22}$$

$$\bar{v}(\vec{p})\beta_4 v(\vec{p}) = v^\dagger(\vec{p})\eta_4\beta_4 v(\vec{p}) = -1 \tag{4.23}$$

for DKP. These signs are due to the general Bhabha metric

$$M = \eta_4\alpha_4 \tag{4.24}$$

which, from the defining equations (IA18) for η_4 and the characteristic equation (I2.31) for α_4 with α_4 diagonal, has the form

$$M(\mathcal{S} = n + \tfrac{1}{2}) = \text{block diagonal}[+\mathcal{S}\mathcal{G}^+_{\mathcal{S}}, -(\mathcal{S} - 1)\mathcal{G}^+_{\mathcal{S}-1}, +(\mathcal{S} - 2)\mathcal{G}_{\mathcal{S}-2}, \ldots, (-1)^{\mathcal{S}-1/2}\tfrac{1}{2}\mathcal{G}^+_{1/2}, (-1)^{\mathcal{S}-1/2}\tfrac{1}{2}\mathcal{G}^-_{1/2}, \ldots, -(\mathcal{S} - 1)\mathcal{G}^-_{\mathcal{S}-1}, +\mathcal{S}\mathcal{G}^-_{\mathcal{S}}], \tag{4.25}$$

$$M(\mathcal{S} = n) = \text{block diagonal}[+\mathcal{S}\mathcal{G}^+_{\mathcal{S}}, -(\mathcal{S} - 1)\mathcal{G}^+_{\mathcal{S}-1}, +(\mathcal{S} - 2)\mathcal{G}_{\mathcal{S}-2}, \ldots, (-1)^{\mathcal{S}+1}\mathcal{G}^+_1, 0\mathcal{G}_0, -(-1)^{\mathcal{S}+1}\mathcal{G}^-_1, \ldots, +(\mathcal{S} - 1)\mathcal{G}^-_{\mathcal{S}-1}, -\mathcal{S}\mathcal{G}^-_{\mathcal{S}}]. \tag{4.26}$$

The $\mathcal{G}^\pm_j(\mathcal{S})$ are the mass state $\pm\chi/j$ projection operators and $\mathcal{G}_0(\mathcal{S})$ is the subsidiary-components projection operator discussed in Secs. III B and III C of paper II (Ref. 2) and also in Eqs. (5.10)–(5.22) below. [The matrix M in (4.24) is also the $\lambda = 4$ case of the matrices $\xi_\lambda = \eta_\lambda\alpha_\lambda$ (no sum) defined by Madhavarao, Thiruvenkatachar, and Venkatachaliegar.[96]]

For half-integer spin, Eq. (4.25) shows that the particle and antiparticle ground states $\pm\chi/\mathcal{S}$ have positive norms, the first excited states $\pm\chi/(\mathcal{S}-1)$ have negative norms, and so on. The simplest case ($\mathcal{S}=\frac{1}{2}$) is the Dirac equation which has only positive-norm states. For integer spin, Eq. (4.26), there is an extra minus sign. From Eq. (IA18) this is due to the extra minus sign in η_4 caused by the extra subsidiary-components block projected by $\mathcal{G}_0$.

Thus, for DKP where $\mathcal{S}=1$, although the particle state has a positive norm, the antiparticle state has a negative norm [this being the origin of the sign difference between Eqs. (4.21) and (4.23)]. The standard resolution of this most simple negative norm is the Pauli-Weisskopf[97] device of saying this is a charge probability density instead of the particle probability density for the Dirac case. But for $\mathcal{S}>1$, such a resolution is no longer possible for either integer or half-integer spin, since the first excited state of a particle necessarily has the opposite norm, oscillating back and forth with each further excitation, with an added minus sign for the antiparticle normalization of integer-spin particles.

Going back to Eq. (4.3a), we can now see that $\tau_{jj'}$ is just the sign of the metric (except for zero in the integer-spin subsidiary-components block), or, as a matrix,

$$\begin{aligned}\tau &= \text{sign}\,(M)\\ &= \text{block diagonal}[+\mathcal{G}^+_{\mathcal{S}}, -\mathcal{G}^+_{\mathcal{S}-1}, +\mathcal{G}^+_{\mathcal{S}-2}, \ldots,\\ &\qquad \pm\mathcal{G}^-_{\mathcal{S}-2}, \mp\mathcal{G}^-_{\mathcal{S}-1}, \pm\mathcal{G}^-_{\mathcal{S}}].\end{aligned} \tag{4.27}$$

What Eqs. (4.3a) and (4.27) tell us is that we are quantizing the Bhabha fields with a metric similar to the Dirac[57] indefinite metric used by Gupta,[98,99] Bleuler,[100,101] and Heitler[101] for the photon field in quantum electrodynamics, but without the fortunate happenstance of quantum electrodynamics that the negative-normed states are eliminated by a certain subset of the positive-normed states. At this stage the negative-normed states remain in the Bhabha theory, but we can proceed with the general quantization.

Returning to Eqs. (4.7) and (4.8), the normalization methods of Takahashi[88] and Baisya[15,22] now straightforwardly lead to

$$\sum_s u_\alpha(\vec{p}, j, s)\bar{u}_\beta(\vec{p}, j, s) = \frac{-\tau_{jj}\mathfrak{D}_{\alpha\beta}(ip)}{2E_j X(\mathcal{S}, j)}, \tag{4.28}$$

$$\sum_s v_\alpha(\vec{p}, j, s)\bar{v}_\beta(\vec{p}, j, s) = \frac{\pm\tau_{jj}\mathfrak{D}_{\alpha\beta}(-ip)}{2E_j X(\mathcal{S}, j)}, \tag{4.29}$$

$$X(\mathcal{S}, j) \equiv \prod_{\substack{k=(1/2,1)\\ k\neq j}}^{\mathcal{S}} \left(\frac{\chi^2}{j^2} - \frac{\chi^2}{k^2}\right). \tag{4.30}$$

Putting these into Eqs. (4.7) and (4.8) we see that the τ_{jj} quantities are all squared, so that Eq. (4.6) will read

$$\begin{aligned}[\psi_\alpha(x), \bar{\psi}_\beta(x')]_\pm &= -i\mathfrak{D}_{\alpha\beta}(\partial) \sum_j^{\mathcal{S}} \int \frac{d^3p}{(2\pi)^3 2E_j X(\mathcal{S}, j)} \\ &\qquad \times (-ie^{ip\cdot x} + ie^{-ip\cdot x}) \\ &= -i\mathfrak{D}_{\alpha\beta}(\partial) \sum_j^{\mathcal{S}} \frac{\Delta(x-x', \chi/j)}{\prod\limits_{\substack{k=(1/2,1)\\ k\neq j}}^{\mathcal{S}} (\chi^2/j^2 - \chi^2/k^2)}, \\ &\equiv -i\mathfrak{D}_{\alpha\beta}(\partial, \mathcal{S})\Delta_{\mathcal{S}}(x-x'),\end{aligned} \tag{4.31}$$

which is to be combined with the other free-field commutation relations

$$[\psi_\alpha(x), \psi_\beta(x')]_\pm = [\bar{\psi}_\alpha(x), \bar{\psi}_\beta(x')]_\pm = 0. \tag{4.32}$$

A crucial point to be emphasized is that if we had not used the indefinite-metric quantization procedure in Eq. (4.3), then the second equality in Eq. (4.30) would have had an alternating minus sign within the sum, i.e., a τ_{jj}. Such a propagator,

$$\hat{\Delta}_{\mathcal{S}}(x-x') = \sum_j^{\mathcal{S}} \frac{\tau_{jj}\Delta(x-x', \chi/j)}{\prod\limits_{\substack{k=(1/2,1)\\ k\neq j}}^{\mathcal{S}} (\chi^2/j^2 - \chi^2/k^2)}, \tag{4.33}$$

as we shall see in Sec. V, would not yield causal fields with minimal coupling. The extra sign is important.

We note that Baisya[15,22] and Nagpal (at least at first[16]) avoided $\hat{\Delta}_{\mathcal{S}}$ in another way. They kept the ordinary positive-normed quantization, but then considered not the fields ψ and $\bar{\psi}$, but rather ψ and $\hat{\psi}$, $\hat{\psi}$ being

$$\hat{\psi} \equiv \bar{\psi}\epsilon(j), \tag{4.34}$$

where $\epsilon(j)$ was an undefined signed quantity, which turns out to be our τ_{jj}. However, with this method the Hamiltonian is not metric-(pseudo-)Hermitian. We prefer to have the Hamiltonian metric-Hermitian and use the indefinite-metric quantization procedure. We feel this is the correct method. Observe, however, that in his recent paper[30] Nagpal considered the commutators $[\psi(x), \bar{\psi}(x')]_\pm$ without stating that he had used an indefinite-metric second quantization instead of the positive-normed quantization of his[16] and Baisya's[15,22] previous works. Strictly speaking, Nagpal's[30] use of Eq. (4.31) is only proper if, unstated, he were using our indefinite-metric quantization. [Of course he could still use the fields ψ and $\hat{\psi}$ with positive-normed quantization to obtain the invariant function $\Delta_{\mathcal{S}}(x-x')$ and not $\hat{\Delta}_{\mathcal{S}}(x-x')$.] Once $\Delta_{\mathcal{S}}(x-x')$ is obtained, though, the calculation proceeds as follows.

B. Interacting fields

Having the free-field commutation relations (4.31) and (4.32), we can now proceed to the interacting-field case. Here we are dealing with the coupled field equations

$$\Box A_\mu(x) = + ie\overline{\Psi}(\chi)\alpha_\mu \Psi(x) \equiv -j_\mu(x), \tag{4.35}$$

$$\begin{aligned}\Lambda(\partial, \chi_{\text{bare}})\Psi(x) &= (\partial \cdot \alpha + \chi_{\text{bare}})\Psi(x)\\ &= ieA_\mu(x)\alpha_\mu \Psi(x) \equiv j(x),\end{aligned} \tag{4.36}$$

where $A_\mu(x)$ and $\Psi(x)$ denote the interacting Heisenberg photon and Bhabha fields.

Following, as we do in this subsection, the discussion of Chap. 8 of Ref. 88, χ_{bare} refers to the fact that with the standard mass renormalization, Eq. (4.36) with its bare mass should be changed to

$$\begin{aligned}\Lambda(\partial, \chi_{\text{bare}} + \delta\chi)\Psi(x) &= \Lambda(\partial, \chi_{\text{obs}})\Psi(x)\\ &= [(\Delta\chi)\Psi(x) + ieA_\mu\alpha_\mu\Psi(x)]\\ &\equiv \hat{j}(x),\end{aligned} \tag{4.37}$$

where with our simple first-order equation, $\delta\chi = \Delta\chi$. Equation (4.37) (one hopes) allows the relation

$$\langle 0|\hat{j}(x)|\text{one-particle state}\rangle = 0 \tag{4.38}$$

to be satisfied so that in turn one can have

$$\lim_{t\to\pm\infty} \langle a|\hat{j}(x)|b\rangle = 0, \tag{4.39}$$

allowing necessary integrals to converge in the Källén[102] and Yang and Feldman[103] discussion of the Heisenberg field operators which will follow. In our case, since both $j(x)$ and $\hat{j}(x)$ involve no derivatives and are linear functionals of $\Psi(x)$, it will turn out that the existence of a $\delta\chi \neq 0$ will have no bearing on the resulting final form of the Heisenberg field commutation relations for $\Psi(x)$. Therefore we will take

$$\begin{aligned}&\delta\chi \equiv \Delta\chi = 0,\\ &\chi_{\text{bare}} = \chi_{\text{obs}} = \chi,\\ &\hat{j}(x) = j(x),\end{aligned} \tag{4.40}$$

with the knowledge that in our final results Eqs. (5.12) and (5.20) below, one could resubstitute $\hat{j}(x)$ for $j(x)$, etc., and still obtain the same causal physics.

Continuing with the Källén-Yang-Feldman procedure, one defines the Heisenberg photon and Bhabha fields as

$$\begin{aligned}A_\mu(x) = Z_3^{1/2}\mathcal{A}_\mu(x)_{\text{in}}\\ + \int_{-\infty}^{\infty} d^4x' D^{(\text{ret})}(x-x')j_\mu(x'),\end{aligned} \tag{4.41}$$

$$\Psi(x) = Z_2^{1/2}\psi(x)_{\text{in}} - \int_{-\infty}^{\infty} d^4x' S^{(\text{ret})}(x-x')j(x'). \tag{4.42}$$

One also defines associated auxiliary fields as a function of a spacelike surface σ,

$$\mathcal{A}_\mu(x,\sigma) = Z_3^{1/2}\mathcal{A}_\mu(x)_{\text{in}} + \int_{-\infty}^{\sigma} d^4x' D(x-x')j_\mu(x'), \tag{4.43}$$

$$\psi(x,\sigma) = Z_2^{1/2}\psi(x)_{\text{in}} - \int_{-\infty}^{\sigma} d^4x' S(x-x')j(x'). \tag{4.44}$$

In the above, Z_3 and Z_2 are the usual charge and wave-function renormalization constants. The auxiliary fields $\mathcal{A}_\mu(x,\sigma)$ and $\psi(x,\sigma)$ are photon and Bhabha fields which satisfy the free wave equations and commutation relations, and which become the renormalized *in* fields as $\sigma \to -\infty$. The propagator Green's functions are

$$D(x-x') = \Delta(x-x', m=0), \tag{4.45}$$

$$D^{(\text{ret})}(x-x') = \Theta(x_0 - x_0')D(x-x'), \tag{4.46}$$

$$S(x-x') = \mathfrak{D}(\partial, \mathcal{S})\Delta_{\mathcal{S}}(x-x'), \tag{4.47}$$

$$S^{(\text{ret})}(x-x') = \mathfrak{D}(\partial, \mathcal{S})\Theta(x_0-x_0')\Delta_{\mathcal{S}}(x-x'), \tag{4.48}$$

where $\Theta(x)$ is the usual unit step function.

Denoting by (x/σ) when x is on the surface σ, Eqs. (4.43) and (4.44) yield

$$\begin{aligned}\mathcal{A}_\mu(x/\sigma) = Z_3^{1/2}\mathcal{A}_\mu(x)_{\text{in}}\\ + \int_{-\infty}^{\infty} d^4x' \Theta^\sigma(x_0-x_0')D^\sigma(x-x')j_\mu(x'),\end{aligned} \tag{4.49}$$

$$\begin{aligned}\psi(x/\sigma) = Z_2^{1/2}\psi(x)_{\text{in}}\\ - \int_{-\infty}^{\infty} d^4x' \Theta^\sigma(x_0-x_0')S^\sigma(x-x')j(x'),\end{aligned} \tag{4.50}$$

where the superscripts σ's mean that x is understood to be on σ in the argument. [We will now drop these superscripts since the restriction (x/σ) in the integrands will not affect the rest of the calculation.] Equations (4.41) and (4.49) trivially imply, as in ordinary Dirac quantum electrodynamics, that

$$A_\mu(x) = \mathcal{A}_\mu(x/\sigma), \tag{4.51}$$

so that the photon commutation relations

$$[A_\mu(x), A_\nu(x')] = i\delta_{\mu\nu}D(x-x') \tag{4.52}$$

are preserved. Equations (4.42) and (4.50) yield

$$\begin{aligned}\Psi(x) &= \psi(x/\sigma) + \int_{-\infty}^{\infty} d^4x'[\Theta(x_0-x_0'), \mathfrak{D}(\partial,\mathcal{S})]\\ &\qquad\times \Delta_{\mathcal{S}}(x-x')j(x')\\ &= \psi(x/\sigma) + \tfrac{1}{2}\int_{-\infty}^{\infty} d^4x'[\epsilon(x_0-x_0'), \mathfrak{D}(\partial,\mathcal{S})]\\ &\qquad\times \Delta_{\mathcal{S}}(x-x')j(x'),\end{aligned} \tag{4.53}$$

where

$$\epsilon(x)=\Theta(x)-\Theta(-x). \tag{4.54}$$

Equation (4.53) is the critical one. We want to show that

$$\Psi(x)\to\psi(x/\sigma) \tag{4.55}$$

so that the Bhabha free-field commutation relations (4.31) and (4.32) are preserved, meaning the Bhabha minimally coupled fields are causal.

V. CAUSALITY OF THE q-NUMBER THEORY

A. Half-integer spin

From Eq. (4.53), the quantity we need to calculate is

$$Y\equiv\tfrac{1}{2}[\epsilon(x_0-x_0'),\mathfrak{D}(\partial,\mathcal{S})]\Delta_{\mathcal{S}}(x-x') \tag{5.1}$$

$$=\sum_{i=0}^{\mathcal{S}-1/2}(\chi^{2\mathcal{S}-2i})\sum_{j=0}^{i}(-1)^{\mathcal{S}+1/2-i+j}\phi_{\mathcal{S}+1/2-i+j}(l_{\{\mathcal{S}+1/2\}})\tfrac{1}{2}\left[\epsilon(x_0-x_0'),\left(1-\frac{\partial\cdot\alpha}{\chi}\right)\Box^{i-j}(\partial\cdot\alpha)^{2j}\right]\sum_{k=1/2}^{\mathcal{S}}\frac{\Delta(x-x',\chi/k)}{X(\mathcal{S},k)}, \tag{5.2}$$

where in (5.2) we have used Eqs. (2.11), (4.30), and (4.31). To evaluate the commutator in (5.2) we recall the Katayama[83,104] result

$$\tfrac{1}{2}[\epsilon(x_0-x_0'),\partial_{\mu(1)}\partial_{\mu(2)}\cdots\partial_{\mu}(r)]\Delta(x-x',m)$$
$$=\left(\frac{-1}{\Box-m^2}\right)\sum_{\substack{q=0\\ \text{cyclic}}}^{r}(-1)^q\hat{n}_{\mu(1)}\cdots\hat{n}_{\mu(q)}\partial^s_{\mu(q+1)}\cdots\partial^s_{\mu(r)}$$
$$\times\left((\hat{n}\cdot\partial)^q-\frac{(\Delta-m^2)^{q/2}}{2}\left\{[1+(-1)^q]+[1-(-1)^q]\frac{(\hat{n}\cdot\partial)}{(\Delta-m^2)^{1/2}}\right\}\right)\delta^4(x-x'), \tag{5.3}$$

where $\hat{n}$ is a timelike unit vector and the superscript s signifies the space part, i.e.,

$$\hat{n}_\mu=(0,0,0,i), \tag{5.4}$$

$$\partial^s_\mu=\partial_\mu+\hat{n}_\mu(\hat{n}\cdot\partial), \tag{5.5}$$

and cyclic means all interchanges of the type $\hat{n}_\nu\partial_\lambda\to\hat{n}_\lambda\partial_\nu$.

Since in the commutator of (5.2) $\Box^{i-j}(\partial\cdot\alpha)^{2j}$ is a sum of products of an even number of derivatives, take r to be even in (5.3). The first two terms in the sum of (5.3) ($q=0$ and $q=1$) are zero. The factor in heavy parentheses of the $q=2$ term is just $[-(\Box-m^2)]$, which cancels the factor in large parentheses in front of the sum. The factor in heavy parentheses of the $q=3$ term is $[-(\partial\cdot n)(\Box-m^2)]$. In general the $q=2n=$ (even) factors in the heavy parentheses are of the form

$$[(\hat{n}\cdot\partial)^{2n}-(\Delta-m^2)^n]$$
$$=[(-\partial_4{}^2)^n-(\Delta-m^2)^n]$$
$$=-(\Box-m^2)\sum_{u=0}^{n-1}(-\partial_4{}^2)^{n-1-u}(\Delta-m^2)^u$$
$$=-(\Box-m^2)\sum_{u=0}^{n-1}(-\partial_4{}^2)^{n-1-u}$$
$$\times\sum_{v=0}^{u}(-1)^v\binom{u}{v}\Delta^{u-v}(m^2)^v, \tag{5.6}$$

so that the $-1/(\Box-m^2)$ is cancelled, the power series in (m^2) goes to $(m^2)^{n+1}$, and the highest power occurs when $2n=r$. The next term, $q=2n+1$ $=$(odd), has a factor of the form

$$(\hat{n}\cdot\partial)[(\hat{n}\cdot\partial)^{2n}-(\Delta-m^2)^n], \tag{5.7}$$

so it has the same powers of (m^2) as Eq. (5.6).

Now consider the second term in the commutator of (5.2), the one which contains the $[-(\partial\cdot\alpha)/\chi]$ in front of the $\Box^{i-j}(\partial\cdot\alpha)^{2j}$. In the relation (5.3) this would add to any particular $q=2n=$ (even) heavy parentheses factor, like (5.6), a factor of the form (5.7), and to any $q=2n+1=$ (odd) heavy parentheses factor, like (5.7), a factor of the form (5.6) with $n\to(n+1)$. Further, the highest-power term would be when $q=r+1=$ (odd), so that one would have the same highest power of (m^2), i.e., a factor like

$$-\frac{(\hat{n}\cdot\partial)}{\chi}[(\hat{n}\cdot\partial)^r-(\Delta-m^2)^{r/2}]. \tag{5.8}$$

Putting all these factors together means that in (5.2) the commutator times an individual particle propagator $\Delta(x,-x',\chi/k)$ can be written as [remember, $\Box$ in (5.2) is a derivative to the second power]

$$\frac{1}{2}\left[\epsilon(x_0-x_0'),\left(1-\frac{\partial\cdot\alpha}{\chi}\right)\Box^{i-j}(\partial\cdot\alpha)^{2j}\right]\Delta(x-x',\chi/k)=\sum_{n=0}^{i}\mathfrak{M}(\hat{n},\alpha,\partial,i,j,n)\sum_{u=0}^{n-1}(-\partial_4{}^2)^{n-1-u}\sum_{v=0}^{u}(-1)^v\binom{u}{v}\Delta^{u-v}\left(\frac{\chi^2}{k^2}\right)^v. \tag{5.9}$$

The matrices $\mathfrak{M}(\hat{n},\alpha,\partial,i,j,n)$ can be explicitly determined from (5.2), (5.3), and (5.9). However, except for one case in Sec. V B, it will not be necessary for us to calculate the $\mathfrak{M}$'s. All that is necessary for us is to know that they are *not* functions of the mass (χ/k). Putting (5.9) into (5.1) gives

$$Y=\sum_{i=0}^{S-1/2}(\chi^{2S-2i})\sum_{j=0}^{i}(-1)^{S+1/2-i+j}\phi_{S+1/2-i+j}(l_{\{S+1/2\}})\times\sum_{n=0}^{i}\mathfrak{M}(\hat{n},\alpha,\partial,i,j,n)\sum_{u=0}^{n-1}(-\partial_4{}^2)^{n-1-u}\sum_{v=0}^{u}(-1)^v\binom{u}{v}\Delta^{u-v}\left[\sum_{k=1/2}^{S}\frac{(\chi^2/k^2)^v}{\prod_{l=1/2,\,l\neq k}^{S}(\chi^2/k^2-\chi^2/l^2)}\right]\delta^4(x-x'). \tag{5.10}$$

But the square-bracket terms in (5.10) are just the homogeneous symmetric functions h_a of Eqs. (A3) and (A4), which here reduce to

$$h_{v-S+1/2}\left(\left(\frac{\chi^2}{k^2}\right)_{\{S\}}\right)=0,$$

$$\text{since } 0\le v\le u\le n-1\le i-1\le S-\tfrac{3}{2}. \tag{5.11}$$

Therefore, Eq. (5.1) for Y is zero, meaning that the second term on the right in (4.53) is zero. Thus,

$$\Psi(x)=\psi(x/\sigma),\quad S \text{ a half-integer}, \tag{5.12}$$

which in turn yields the desired result that the field anticommutation relations are preserved for the minimally coupled half-integer-spin Bhabha fields, and so they are causal. (The well-known Dirac field is, of course, a special case of our result.)

Two final points: If we had not used the indefinite-metric quantization, meaning that instead of the propagator $\Delta_S(x-x')$ we would have had the propagator $\hat{\Delta}_S(x-x')$ of (4.33), the term in square brackets in (5.10) would have had the extra factor τ_{kk}. Then the quantities in the square brackets would not have been the homogeneous symmetric functions, so (5.12) would not have held, and noncausality would have ensued. The indefinite metric is necessary for causality. Also, because (5.12) does hold, our result would have been the same if we had used the renormalized masses in the KG divisors and the propagators and also had used the renormalized current $\hat{j}$ of (4.37) in the second term on the right-hand side of (4.53).

B. Integer spin

For integer spin Y is of the form

$$Y\equiv\tfrac{1}{2}[\epsilon(x_0-x_0'),\mathfrak{D}(\partial,S)]\Delta_S(x-x') \tag{5.13}$$

$$=\sum_{i=0}^{S}\chi^{2(S-i)-1}\sum_{j=0}^{i}(-1)^{S-i+j}\phi_{S-i+j}(\mathcal{L}_{\{S\}})\tfrac{1}{2}\left[\epsilon(x_0-x_0'),\left(1-\frac{\partial\cdot\alpha}{\chi}\right)\Box^{i-j}(\partial\cdot\alpha)^{2j}\right]\sum_{k=1}^{S}\frac{\Delta(x-x',\chi/k)}{X(S,k)}, \tag{5.14}$$

where this time Eqs. (2.18), (4,30), and (4.31) have been used. The discussion proceeds exactly as for half-integer spin, up to the point where one has obtained

$$Y=\sum_{i=0}^{S}\chi^{2(S-i)-1}\sum_{j=0}^{i}(-1)^{S-i+j}\phi_{S-i+j}(\mathcal{L}_{\{S\}})\times\sum_{n=0}^{i}\mathfrak{M}(\hat{n},\alpha,\partial,i,j,n)\sum_{u=0}^{n-1}(-\partial_4{}^2)^{n-1-u}\sum_{v=0}^{u}(-1)^v\binom{u}{v}\Delta^{u-v}\left[\sum_{k=1}^{S}\frac{(\chi^2/k^2)^v}{\prod_{l=1,\,l\neq k}^{S}(\chi^2/k^2-\chi^2/l^2)}\right]\delta^4(x-x'). \tag{5.15}$$

But now the square brackets contain the homogeneous symmetric functions

$$h_{v-\mathcal{S}+1}\left(\left(\frac{\chi^2}{k^2}\right)_{\{\mathcal{S}\}}\right) = \begin{cases} 0 & \text{for all } 0 \leq v \leq u \leq n-1 \leq i-1 < \mathcal{S}-1, \\ 1 & \text{for } v = u = n-1 = i-1 = \mathcal{S}-1, \end{cases} \tag{5.16}$$

so that we have one piece remaining in Y,

$$Y = \chi^{-1} \sum_{j=0}^{\mathcal{S}} (-1)^j \phi_j(\mathcal{L}_{\{\mathcal{S}\}}) \mathfrak{M}(\hat{n}, \alpha, \partial, \mathcal{S}, j, \mathcal{S}) \times (-1)^{\mathcal{S}-1} \delta^4(x - x'). \tag{5.17}$$

Now we need to evaluate $\mathfrak{M}(\hat{n}, \alpha, \partial, \mathcal{S}, j, \mathcal{S})$. Observing that $\mathfrak{M}(\hat{n}, \alpha, \partial, \mathcal{S}, j, \mathcal{S})$ comes from the $A_{\mathcal{S}j}$ term in the KG divisor of Eq. (2.18), we use the second form, Eq. (2.18b), which eliminates the term in the commutator (5.14) containing the $(-\partial \cdot \alpha/\chi)$ in front of $\Box^{\mathcal{S}-j}(\partial \cdot \alpha)^j$. Then using Eq. (5.3), one can see that

$$\begin{aligned} \mathfrak{M}(\hat{n}, \alpha, \partial, \mathcal{S}, j, \mathcal{S}) &= (-1)^{2\mathcal{S}}(\hat{n} \cdot \hat{n})^{\mathcal{S}-j}(\hat{n} \cdot \alpha)^{2j} \\ &= (-1)^{\mathcal{S}-j}(-1)^j(\alpha_4)^{2j} \\ &= (-1)^{\mathcal{S}}(\alpha_4{}^2)^j. \end{aligned} \tag{5.18}$$

This result obtains because, since [in the notation of (5.3)] $q = r = 2\mathcal{S}$, only $(\hat{n}_\mu)$'s and no (∂_μ^s)'s are obtained on the right in (5.3). Of these $(\hat{n}_\mu)$'s, $(\mathcal{S} - j)$ pairs of them are dotted into one another because of the $(\mathcal{S} - j)$ powers of $\Box$ in the commutator. The $2j$ remaining $(\hat{n}_\mu)$'s are dotted into α's because the remaining original derivatives in the commutator were.

Thus, putting (5.18) into (5.17) we have

$$Y = -\chi^{-1}\left[\sum_{j=0}^{\mathcal{S}} (-1)^j (\alpha_4{}^2)^j \phi_j(\mathcal{L}_{\{\mathcal{S}\}})\right] \delta^4(x - x'). \tag{5.19}$$

But from (A1) or (2.16) the quantity in the square brackets of (5.19) can be written as

$$\left[\prod_{j=1}^{\mathcal{S}} \left(1 - \frac{\alpha_4{}^2}{j^2}\right)\right] = \frac{\prod_{j=1}^{\mathcal{S}} (\alpha_4{}^2 - j^2)}{\prod_{k=1}^{\mathcal{S}} (-k^2)} \equiv \mathcal{G}_0(\mathcal{S}). \tag{5.20}$$

The $\mathcal{G}_0(\mathcal{S})$ are exactly the projection operators onto the subsidiary components for arbitrary integer spin defined in (II3.8); i.e., they are the generalizations of the special DKP case operator

$$\mathcal{G}_0(1) \equiv (1 - \mathcal{G}), \tag{5.21}$$

which Sakata and Taketani[105-109] first used for the $\mathcal{S} = 1$ DKP system to decouple the particle-components fields and Hamiltonian from the subsidiary components quantities. They have the idempotent properties necessary for projection operators

$$\begin{aligned} [\mathcal{G}_0(\mathcal{S})]^2 &= \mathcal{G}_0(\mathcal{S}), \\ \mathcal{G}_0(\mathcal{S})[1 - \mathcal{G}_0(\mathcal{S})] &= 0. \end{aligned} \tag{5.22}$$

Thus, (5.19) and (5.20) mean that

$$Y = -\chi^{-1}\mathcal{G}_0(\mathcal{S})\delta^4(x - x'), \tag{5.23}$$

which from (4.53) means that

$$\Psi(x) = \psi(x/\sigma) - \mathcal{G}_0(\mathcal{S})\chi^{-1} j(x) \tag{5.24}$$

$$= \psi(x/\sigma) - \mathcal{G}_0(\mathcal{S})\chi^{-1}[ieA_\mu(x)\alpha_\mu\Psi(x)]. \tag{5.25}$$

It is an informative and simple exercise to explicitly verify Eq. (5.25) for the cases $\mathcal{S} = 1$ and $\mathcal{S} = 2$. One combines the KG divisors for $\mathcal{S} = 1$ and $\mathcal{S} = 2$ given in Eqs. (3.22) and (3.23) with the $\mathcal{S} = 1$ and $\mathcal{S} = 2$ invariant functions $\Delta_{\mathcal{S}}(x - x')$ of Eq. (4.31), and then uses the corrected[104,110] special case commutation formulas for up to four derivatives of Katayama[83] and Takahashi[88] to end up with the result (5.25) with the correct projection operators of Eq. (II3.11),

$$\begin{aligned} \mathcal{G}_0(1) &= 1 - \alpha_4{}^2, \\ \mathcal{G}_0(2) &= \tfrac{1}{4}(\alpha_4{}^2 - 4)(\alpha_4{}^2 - 1). \end{aligned} \tag{5.26}$$

What (5.25) tells us is that the commutation relations of the physical particle components of the fields,

$$(1 - \mathcal{G}_0)\Psi(x) = (1 - \mathcal{G}_0)\psi(x/\sigma), \tag{5.27}$$

are preserved. That is, the physical fields are causal. The extra piece in the subsidiary-components equation,

$$\mathcal{G}_0(\mathcal{S})\Psi(x) = \mathcal{G}_0(\mathcal{S})\psi(x/\sigma) - \mathcal{G}_0(\mathcal{S})\chi^{-1} j(x), \tag{5.28}$$

is the q-number analog of the $0^{a(0)}$ factor obtained in Eq. (3.7) if one tries to do the c-number problem for integer spin using the Velo-Zwanziger method.

These two apparent noncausality problems are resolved when one realizes that it is only the nonphysical subsidiary components that are involved, and then handles the calculation accordingly.

Finally, the comments made at the end of Sec. V A pertain here, too. The calculation needs the indefinite-metric quantization of Eq. (4.3a) to succeed. Also, if the renormalized mass χ_{obs} had been used, Eq. (5.24) would still have been obtained, only with $j(x) \to \hat{j}(x)$, meaning the same causal physics for the particle-components fields would have resulted.

VI. DISCUSSION

In this paper we have demonstrated, in a simple closed form, the causality of both the c-number and the q-number arbitrary-spin Bhabha fields with minimal electromagnetic coupling. The c-

number demonstration was done with the Wightman[23] modification of the Velo-Zwanziger[17,18] method, as this modification allows the integer-spin subsidiary components to be handled without further complications in the calculations.

The q-number demonstration was done starting with an indefinite-metric second-quantization technique, necessary because the entire Bhabha system has negative normed states built into it. Then, with the aid of the symmetric functions to keep track of the many terms, causality could be demonstrated in closed form for arbitrary-spin Bhabha fields. For integer spin there is an extra piece in the Heisenberg fields which at first glance can appear to imply noncausality. But this extra piece turns out to be entirely composed of subsidiary components, so that the physical-particle-components fields are causal.

In both the c-number and the q-number causality demonstrations, the subsidiary components required a slightly special handling. (This, of course, has been true in the calculations throughout this series of papers.) Here the need for special handling can be technically traced to the extra derivative in the integer-spin defining algebra.

For half-integer spin both the KG operator of (3.1) and the defining Bhabha algebra [see the characteristic equation (1.3) and its relation to the entire "half-integer-spin KG equation" (I2.35)] are of order $(2\mathcal{S}+1)$. This means that the KG divisor (3.11) has terms with derivatives up to a maximum power of $(2\mathcal{S}) = (\text{odd})$. This number is one less than the number of physical mass states.

For integer spin, the KG operator of (2.16) used to obtain the KG divisor is only of order $2\mathcal{S}$, even though the defining Bhabha algebra [see (1.3) and now the entire "integer-spin KG equation" (I2.34)] is still of order $(2\mathcal{S}+1)$. Thus in the KG divisor (2.18b) there appear extra nonzero terms with derivatives of maximum order not of $(2\mathcal{S}-1)$, but of order $(2\mathcal{S}) = (\text{even})$. This number is now equal to the number of physical mass states. These terms are the $\Box^{\mathcal{S}-j}(\partial \cdot \alpha)^{2j}$ in the $A_{\mathcal{S}j}$, and are the terms which lead to the additional pieces in the subsidiary components of the Heisenberg fields. The effects of the relative extra derivative were limited here to the subsidiary-components Heisenberg fields because those were the ones where the extra derivative came from to begin with [the extra derivative in (I2.34) vs the KG equation (2.16)].

However, one can thus technically, as well as physically, see what higher derivatives will do; they will eventually lead to noncausality. For example, one could find in place of Eqs. (5.11) and (5.16), homogeneous symmetric functions of the type $h_{a>0} \neq (0,1)$. Another example is in the standard discussion[11] of the RS (Ref. 12) spin-$\frac{3}{2}$ field, where noncausality is ultimately due to the imposition of external constraint conditions which involve derivatives.[17] A related example, as we mentioned in Sec. III, is Wightman's[23] demonstration for the DKP $\mathcal{S}=1$ case that $[\beta_\mu, \beta_\nu]F_{\mu\nu}$ coupling introduces noncausality in the spin-0 representation. Here the problem is involved with the order and eigenvalues of the algebra.

Given all this, it is the minimal derivative nature (first-order wave equation with no external derivative constraints) and the particular algebra of the Bhabha system which leads to causality with minimal electromagnetic interactions.

Thus, so far we have seen that the Bhabha system is CPT and Lorentz invariant with a well-defined multimass and multispin spectrum, and is causal with minimal electromagnetic interaction. However, at this point, as was discussed around Eq. (4.24), one has an indefinite metric. Since (for convenience in our representation of the α_λ matrices) the different normed states are coupled by $\vec{\alpha}$ matrices, this metric implies that the standard quantum-mechanical probability (or charge) density interpretation is in doubt, for in principle unitarity can be violated by a nonconservation of probability (or charge). This is despite the conserved-current condition

$$\partial_\lambda j_\lambda = \partial_\lambda [ie\bar{\psi}\alpha_\lambda \psi] = 0, \tag{6.1}$$

which trivially follows from the free-field and adjoint-field equations (I2.24) and (I2.45).

We will discuss this further in paper V, where we will concentrate on the indefinite metric and its meaning, as well as its relation to generalized Foldy-Wouthuysen transformations[111-113] for various-spin fields,[114-123] and the Bhabha system in particular.

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APPENDIX: SYMMETRIC FUNCTIONS

The elementary symmetric functions $\phi_r(x_{\{n\}})$ of the n quantities x_j are well known.[52-54] They can be obtained from the product formula

$$0=\prod_{j=1}^{n}(A-x_j)\equiv\sum_{k=0}^{n}A^k(-1)^{n-k}\phi_{n-k}(x_{\{n\}}), \tag{A1}$$

and thus are given by

$$\phi_0=1, \tag{A2a}$$

$$\phi_{r\neq 0}(x_{\{n\}})=\left\{\sum_{i(1)>i(2)>\cdots>i(r)=1}^{n}[x_{i(1)}x_{i(2)}\cdots x_{i(r)}]\right\}. \tag{A2b}$$

That is, the function ϕ_r "is the sum of the $\binom{n}{r}$ products of r different quantities" x_i.[53]

We also deal with the functions h_a, where a can be a positive or negative integer, or zero. These functions are defined by

$$h_{q-n+1}(y_{\{n\}})=\sum_{k=1}^{n}\frac{y_k^q}{\prod_{\substack{j=1\\ j\neq k}}^{n}(y_k-y_j)}, \tag{A3}$$

and have the properties[54]

$$\begin{aligned}
&h_{-|a|}=0,\\
&h_0=1,\\
&h_1=\phi_1,\\
&h_2=h_1\phi_1-h_0\phi_2,\\
&\quad\vdots\\
&h_n=-\sum_{b=0}^{n-1}(-1)^{n-b}\phi_{n-b}h_b.
\end{aligned} \tag{A4}$$

For $a\geq 0$, these functions equal the "homogeneous symmetric functions" $h_{|a|}$ of Littlewood,[53] so we call the entire set (A3) and (A4) this name. (Louck and Biedenharn[54] denote these h_a by β_a, a notation we avoid, so as not to confuse them with the DKP matrices β_λ.)

The result quoted in Eq. (4.31) for the form of the multimass invariant Δ function comes from considering a special case of Eqs. (A3) and (A4),

$$h_{-n}(y_{\{n+1\}})=0=\sum_{k=1}^{n+1}\frac{1}{\prod_{\substack{j=1\\ j\neq k}}^{n+1}(y_k-y_j)}. \tag{A5}$$

Separate the $(n+1)$st term in the sum, to give

$$\frac{-1}{\prod_{j=1}^{n}(y_{n+1}-y_j)}=\sum_{k=1}^{n}\frac{1}{(y_k-y_{n+1})\prod_{\substack{j=1\\ j\neq k}}^{n}(y_k-y_j)}. \tag{A6}$$

Now let $y_{n+1}=-p^2$ and $y_{j\neq n+1}=\chi^2/j^2$, and you have

$$\frac{1}{\prod_{j=1}^{n}(p^2+\chi^2/j^2)}=\sum_{k=1}^{n}\frac{1}{(p^2+\chi^2/k^2)\prod_{\substack{j=1\\ j\neq k}}^{n}(\chi^2/k^2-\chi^2/j^2)}. \tag{A7}$$

But $(p^2+\chi^2/k^2)^{-1}$ is just the momentum-space representation of the standard invariant $\Delta(x-x', \chi/k)$ function for a single mass χ/k, and the left-hand side of Eq. (A7) is the momentum-space representation of the multimass invariant Δ function. Thus, integrating around the appropriate contour[81] yields the desired expression for $\Delta_s(x-x')$ in Eq. (4.31),

$$\Delta_s(x-x')=\sum_{k=(1/2,1)}^{s}\left[\frac{\Delta(x-x',\chi/k)}{\prod_{\substack{j=(1/2,1)\\ j\neq k}}^{s}(\chi^2/k^2-\chi^2/j^2)}\right]. \tag{A8}$$

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