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Dynamical derivation of vacuum operator-product expansion in Euclidean conformal quantum field theory

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An expansion of the type

$$\langle\varphi(x_1)\cdots\varphi(x_n)\rangle_0=\langle\varphi(x_1)\varphi(x_2)\rangle_0\langle\varphi(x_3)\cdots\varphi(x_n)\rangle_0$$
$$+\sum_{\chi_l}C^2(\chi_l)\int(dp)Q^{\tilde{\chi}_l}(x_1,x_2;-p)w_{\chi_l}(p)Q^{\tilde{\chi}_l}(p;x_1,\ldots,x_4)$$

is derived, where $\chi_l=[l,c_l]$ are labels for infinite-dimensional symmetric tensor representations of the Euclidean conformal group $O^\dagger(2h+1,1)$, $\tilde{\chi}_l=[l,-c_l]$, the constants $C(\chi_l)$ are real, and $Q^{\tilde{\chi}}$ and w_χ have the properties of vacuum expectation values of field products. The starting point is an infinite set of coupled nonlinear integral equations for Euclidean Green's functions in $2h$ space-time dimensions of the type written some 15 years ago by Fradkin and Symanzik. The Green's functions of the corresponding Gell-Mann–Low limit theory are expanded in conformal partial waves. The dynamical equations imply the existence of poles and factorization of residues in the partial waves as functions of the representation parameters. In proving the validity of the expansion we use some differential relations between partially equivalent exceptional representations of $O^\dagger(2h+1,1)$, established in an earlier paper. This work completes the group-theoretical derivation of the vacuum operator-product expansion undertaken by Mack in 1973.

INTRODUCTION

An effort was made in the last few years to exploit the conformal invariance of the Gell-Mann–Low limit theory for some Yukawa-type interactions in order to obtain nonperturbative information for Green's functions and operator products at short distances (see, e.g., Refs. 1–19). In particular, Mack[10] showed for a model of a self-interacting scalar field that the conformal partial-wave expansion of Euclidean Green's functions allows one to diagonalize and solve the set of renormalized dynamical equations[20,21] for that model. It was noted[7,9,10] that the so-called bootstrap equations for the 3-point functions imply the existence of real poles in the conformal partial waves as functions of the dimension. The remaining integral equations of the model lead to some factorization properties for the residues at these poles.[10]

The aim of the present paper is to derive a discrete expansion for Euclidean Green's functions and Wightman functions which corresponds to a vacuum operator-product expansion in the terminology of Ref. 17 [i.e., an expansion of the vector distribution $\varphi(x_2)\varphi(x_1)|0\rangle$]. The derivation is based on the above results on conformal partial-wave analysis and on our previous study[11] of the Clebsch-Gordan expansion for the pseudo-orthogonal group. This approach always involves a conjecture about the analyticity (and the asymptotic behavior) of conformal partial waves, which is partly justified by the analysis of the skeleton diagram expansion.[2] The identities among Clebsch-Gordan kernels at exceptional integer points in the representation space, which were derived in Ref. 11, are crucial for cancelling fake singularities coming from kinematical factors (Sec. II B 3). As a by-product we verify a positivity condition for the 4-point Wightman function, which was established in a different manner in Ref. 13.

We attempt to make the exposition reasonably self-contained and review (chiefly in Sec. 1) a number of results of Refs. 10 and 11. This introductory material also contains some new points: One example is the discussion of the $(\varphi^*\varphi)^2$ model in Secs. I A 4 and II A 1. We mention also the explicit expression for a general "basic field" which enters the operator-product expansion of a pair of free fields (Sec. II A 3). The main new results are contained in Sec. II B (and the related Appendix *B*). We would like to stress the role of recently established relations between partially equivalent representations of the Euclidean conformal group[11(b), 11(c)] in the derivation of the vacuum expansion. Sec. II C contains a discussion—but no ultimate solution—of the rather difficult problem of incorporating crossing symmetry in the present scheme. It closes by a general summary of results (Sec. II C 3).

I. DYNAMICAL EQUATIONS AND CONFORMAL PARTIAL-WAVE EXPANSION FOR EUCLIDEAN GREEN'S FUNCTIONS

A. Renormalizable models of self-interacting scalar fields

1. A six-dimensional model. Euclidean Green's functions. Generating functionals

The simplest model of a renormalizable self-interacting field $\varphi(x)$ is given by the interaction Lagrangian $L_I(x) = -(g/3!){:}\varphi^3(x){:}$ in six space-time dimensions. Although this model is unrealistic (since the corresponding classical Hamiltonian is not positive definite) it can (and does) serve as a testing ground for various quantum field-theoretic techniques (apart from its role in the work[21,10] which we are going to review, it presents the simplest example of a theory with asymptotic freedom—see, e.g., Ref. 22). We shall indicate at the end of this section how one might modify the model, in order to eliminate its obvious deficiency.

Having in mind models in different numbers of dimensions, we shall work in a general framework of $2h$-dimensional space-time ($2h = 2, 3, 4, \ldots$).

In what follows we shall mostly deal with Euclidean Green's functions (also called Schwinger functions; cf. Ref. 23):

$$s(\vec{x}_1\sigma_1, \ldots, \vec{x}_n\sigma_n) = \tau(i\sigma_1\vec{x}_1, \ldots, i\sigma_n\vec{x}_n), \tag{1.1}$$

where $\tau(x_1, \ldots, x_n) = \langle T\varphi(x_1)\cdots\varphi(x_n)\rangle_0$ (φ is the interacting Heisenberg field).

One can define connected, one-particle irreducible (1PI), etc., Green's functions without recourse to perturbation theory. The most compact way to do that is in terms of generating functionals (see, e.g., Refs. 21 and 24).

Let $J(x)$ be s scalar external source and let $\mathcal{S}(J)$ be the generating functional for the s functions

$$\begin{aligned}\mathcal{S}(J) &= 1 + \sum_{n=1}^{\infty}\frac{1}{n!}\int\cdots\int dx_1\cdots dx_n\, s(x_1, \ldots, x_n)J(x_1)\cdots J(x_n)\\ &\equiv \left\langle \exp\left[\int J(x)\phi(x)dx\right]\right\rangle_0 \quad (dx = d^{2h}x)\end{aligned} \tag{1.2}$$

[$\phi(x)$ is by definition the Euclidean field]. The generating functional $\mathcal{G}(J)$ of the connected (Euclidean) Green's functions $G(x_1, \ldots, x_n)$ is defined by

$$\mathcal{S}(J) = e^{\mathcal{G}(J)}. \tag{1.3}$$

The source $J(x)$ is associated with a classical (Euclidean) field $\phi_c(x)$ by

$$\phi_c(x) = \frac{\delta\mathcal{G}}{\delta J(x)} = \mathcal{S}^{-1}(J)\frac{\delta\mathcal{S}}{\delta J(x)}. \tag{1.4}$$

The generating functional for the 1PI Green's functions (or *proper vertex functions*) $\Gamma(x_1, \ldots, x_n)$ is given by the Legendre transformation[24]

$$\begin{aligned}\Gamma(\phi_c) &= \mathcal{G}(J) - \int dx\, J(x)\phi_c(x)\\ &= \sum_{n=2}^{\infty}\frac{1}{n!}\int\cdots\int dx_1\cdots dx_n\,\Gamma(x_1, \ldots, x_n)\phi_c(x_1)\cdots\phi_c(x_n).\end{aligned} \tag{1.5}$$

To obtain the right-hand side of (1.5) we express $J(x)$ in terms of $\phi_c(x)$ from (1.4).

2. Graphical notation. 1PI *and* 2PI *kernels*

In order to write down the (renormalized) equations for the model under consideration we shall need some additional auxiliary notions (cf. Refs. 20, 21, 10, and 18).

We introduce the *amputated* (connected) Green's functions

$$A(x_1, \ldots, x_n) = \int\cdots\int dy_1\cdots dy_n\, G^{-1}(x_1 - y_1)\cdots G^{-1}(x_n - y_n)G(y_1\cdots y_n), \tag{1.6}$$

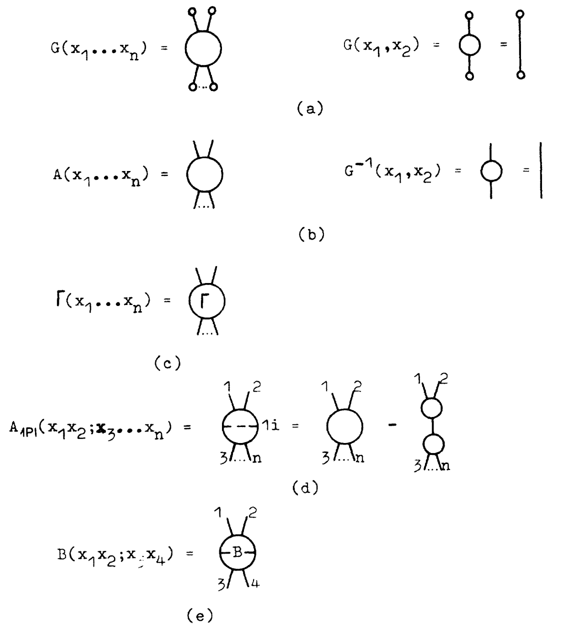

FIG. 1. Graphical notation for the connected Green's function G, the amputated amplitude A, the proper vertex function Γ, the 1PI-amplitude A_{1PI}, the BS kernel B.

where

$$G^{-1}*G(x_1-x_2)\equiv\int dx\,G^{-1}(x_1-x)G(x-x_2)=\delta(x_1-x_2). \tag{1.7}$$

[We use alternately the notation $G(x_1, x_2)$ and $G(x_1-x_2)$ for the 2-point function; that is legitimate, because of translation invariance.] We have

$$\begin{aligned}A(x_1,x_2)&=-\Gamma(x_1,x_2)\\&=G^{-1}(x_1,x_2),\\A(x_1,x_2,x_3)&=\Gamma(x_1,x_2,x_3).\end{aligned} \tag{1.8}$$

We shall use the graphical notation of Figs. 1(a)–1(c) for G, A, and Γ. The 1PI amplitude for the channel $12\to 3\cdots n$ ($n\geq 4$) is defined by

$$\begin{aligned}A_{1PI}&(x_1,x_2;x_3,\ldots,x_n)\\&=A(x_1,\ldots,x_n)\\&\quad-\int dy_1\int dy_2\,A(x_1,x_2,y_1)G(y_1,y_2)A(y_2,x_3,\ldots,x_n),\end{aligned} \tag{1.9}$$

or graphically by Fig. 1(d).

We *define* the Bethe-Salpeter (BS) kernel [see Fig. 1(e)] as the solution of the (integral) BS equation in Fig. 2(a). (The factor $\frac{1}{2}$ in the right-hand side is necessary because of the symmetry of the theory of a single neutral scalar field.) Finally, we introduce the 2-particle irreducible (2PI) kernel for the channel $12\to 3\cdots n$ ($n\geq 5$) by induction in n as shown in Figs. 2(b) and 2(c). The first sum in the right-hand side of the Equation in Fig. 2(c) [and in Fig. 2(b)] includes all $2^{n-3}-1$ partitions of the set of external lines $3\cdots n$ into two nonempty subsets. The second sum involves all splittings of these lines into k nonempty subsets ($k=3,\ldots,n-3$).

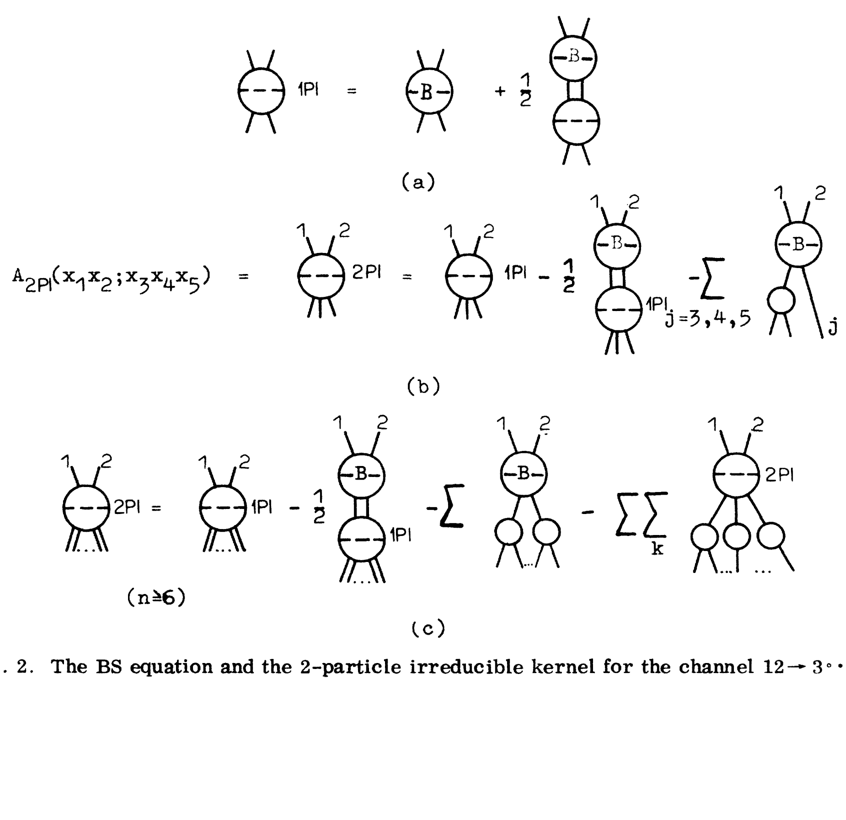

FIG. 2. The BS equation and the 2-particle irreducible kernel for the channel $12\to 3\cdots n$.

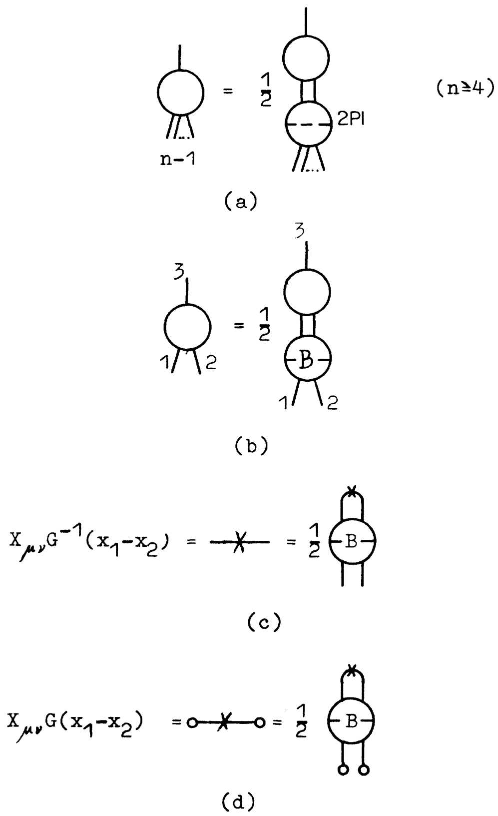

FIG. 3. The dynamical equations for the Green's functions. [External lines are attached to the BS kernel in the right-hand side of (d).]

3 Dynamical equations. Stress-energy tensor, Ward identities

The dynamical equations can be written either in terms of the connected Green's functions G (cf. Refs. 21 and 10) or in terms of the proper vertex functions (cf. Refs. 20 and 18), the two forms being equivalent. We shall adopt here the latter form, writing, however, the equation for the 2-point function in a way suggested in Refs. 6 and 10.

In the Gell-Mann–Low limit (in which the renormalization constant $Z_1=0$—see Ref. 25) the dynamical equations for the vertex functions have the form shown in Fig. 3. In Figs. 3(c) and 3(d) we have used the notation

$$X_{\mu\nu}f(x)=(x_\nu\nabla_\mu - x_\mu\nabla_\nu)f(x). \qquad (1.10)$$

Note that the equation in Fig. 3(c) is equivalent to that in Fig. 3(d).

The bootstrap form of the equations in Figs. 3(b)–3(d) is peculiar to the Gell-Mann–Low limit theory. In general (away from that limit), there is an inhomogeneous term in the right-hand side of the equation in Fig. 3(b) and all three equations require subtractions in momentum space or multiplication by $(x_1-x_2)_\mu$ in coordinate space. Masses and coupling constants appear in such an approach as initial conditions (cf. Ref. 10).

It is convenient to use an alternative form of the equations in Figs. 3(c) and 3(d) involving the stress-energy tensor $\Theta_{\mu\nu}(x)$ (see Ref. 6). Let $G_{\mu\nu}(x_1,x_2;x_3)$ be the Euclidean region continuation of $\langle T\varphi(x_1)\varphi(x_2)\times\Theta_{\mu\nu}(x_3)\rangle_0$ and let

$$\Gamma_{\mu\nu}(x_1,x_2;x_3)=\int dy_1\int dy_2 G^{-1}(x_1-y_1)G^{-1}(x_2-y_2)\times G_{\mu\nu}(y_1,y_2;x_3)$$

be the corresponding vertex function. It is assumed the $G_{\mu\nu}$ and $\Gamma_{\mu\nu}$ satisfy the following (equivalent between each other) Ward-Takahashi identities:

$$\nabla_3^\lambda G_{\lambda\mu}(x_1,x_2;x_3)=\sum_{i=1}^{2}\left[\delta(x_3-x_i)\nabla_{i\mu}G(x_1-x_2)-aG(x_1-x_2)\nabla_{i\mu}\delta(x_3-x_i)\right], \qquad (1.11a)$$

$$\nabla_3^\lambda \Gamma_{\lambda\mu}(x_1,x_2;x_3)=\sum_{i=1}^{2}\left[\delta(x_3-x_i)\nabla_{i\mu}G^{-1}(x_1-x_2)+(1+a)G^{-1}(x_1-x_2)\nabla_{i\mu}\delta(x_3-x_i)\right]. \qquad (1.11b)$$

The last (Schwinger) term can be eliminated by multiplying both sides of each of the equations (1.11) by $(x_1-x_3)_\nu$ $(\nu\neq\mu)$ and integrating over x_3. After antisymmetrization in (μ,ν) we find

$$\int dx\,\nabla^\lambda G_{\lambda[\mu}(x_1,x_2;x)(x_1-x)_{\nu]}=-X_{\mu\nu}G(x_1-x_2), \qquad (1.12a)$$

$$\int dx\,\nabla^\lambda \Gamma_{\lambda[\mu}(x_1,x_2;x)(x_1-x)_{\nu]}=X_{\mu\nu}G^{-1}(x_1-x_2), \qquad (1.12b)$$

where $X_{\mu\nu}$ is the rotation generator (1.10). This form of the identity has the advantage of being independent of the arbitrary constant a.

The set of equations in Fig. 3 is equivalent to the set obtained when Figs. 3(c) and 3(d) are replaced by Figs. 4(a) and 4(b) and the Ward identity (1.11) or (1.12) is assumed to hold. As a consequence we obtain an infinite set of additional integral equations for the $(n+1)$-point functions $\Gamma_{\mu\nu}(x_1,\ldots,x_n;x)$ displayed in Fig. 4(c). They also satisfy Ward identities of the type (1.11). To be consistent with the scale invariance of the Gell-Mann–Low limit theory, we have to require that $\Theta_{\mu\nu}$ is traceless, so that

$$\Gamma_{\mu\mu}(x_1, \ldots, x_n; x) = 0 = G_{\mu\mu}(x_1, \ldots, x_n; x), \quad n = 2, 3, \ldots. \tag{1.13}$$

4. A more realistic model

Although the above dynamical equations are derived from a renormalizable Lagrangian in six space-time dimensions, their final form displayed in Figs. 3 and 4 and in Eqs. (1.11) and (1.12) makes sense for an arbitrary number $2h$ of dimensions.

Here we shall indicate how one can fit the more realistic model of a charged (pseudo) scalar field with a quartic interaction in four space-time dimensions,

$$\mathcal{L}(\varphi(x), \varphi^*(x)) = :\nabla_\mu \varphi^* \nabla^\mu \varphi: - \frac{\lambda}{2} :(\varphi^* \varphi)^2:, \tag{1.14}$$

into the above framework.

The clue lies in the observation (made, e.g., by Symanzik) that the model given by (1.14) can equivalently be described by the Lagrangian

$$\mathcal{L}(\varphi(x), \varphi^*(x); B(x)) = :\nabla_\mu \varphi^* \nabla^\mu \varphi: + \tfrac{1}{2}:B^2: - \sqrt{\lambda}\, :\varphi^* \varphi B: \tag{1.15}$$

of a system of two fields, φ and B, with a cubic interaction. Indeed, varying $\mathcal{L}(\varphi, \varphi^*; B)$ with respect to B, we find the algebraic "equation of motion" $B = \sqrt{\lambda}\, :\varphi^* \varphi:$ which reduces (1.15) to (1.14). In a canonical perturbation theory the propagator corresponding to the field B would be a δ function. On the other hand, the topological structure of Feynman diagrams in this model is the same as in a theory with Yukawa coupling of a charged field φ and a neutral scalar field B (which will be represented graphically by a dashed line).

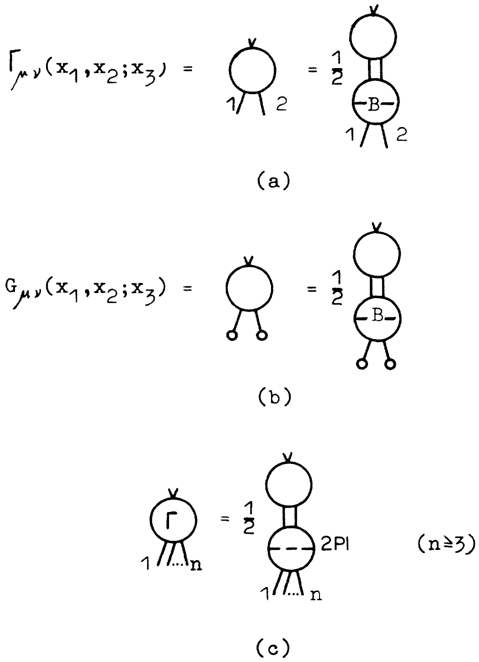

FIG. 4. Dynamical equations for Green's functions involving the stress-energy tensor.

Without going into the details of the Green's function formulation of this model, we notice that it will involve (*a priori*) four types of vertex functions shown in Fig. 5. The bootstrap equation for the charged propagator can be equivalently obtained[6] from the corresponding equation for the current-field 3-point function

$$\langle T\varphi(x_1)\varphi^*(x_2) j_\mu(x_3)\rangle_0 \leftrightarrow G_\mu(x_1, x_2; x_3), \tag{1.16}$$

which satisfies the Ward identity

$$\nabla_3^\mu G_\mu(x_1, x_2; x_3) = eG(x_1 - x_2)[\delta(x_1 - x_3) - \delta(x_2 - x_3)] \tag{1.17}$$

[e being the electric charge carried by $\varphi^*(x)$].

B. Invariance and invariant solutions of the dynamical equations. Conformal expansions.

1. Euclidean conformal invariance of the equations

As already noted, all equations of Sec. I A only relate Green's functions among themselves. They involve no parameters (in particular, no dimensional parameters) and are manifestly scale in-

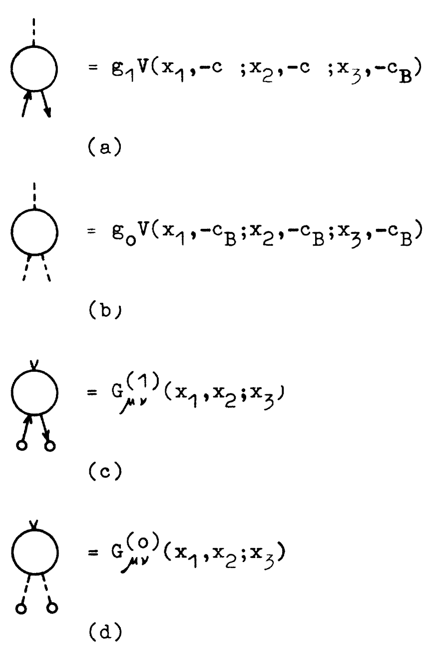

FIG. 5. The Green's functions of the (φ, φ^*, B) model.

variant. Indeed, if we ascribe to the field φ in $2h$ dimensions a scale dimension $d_\varphi = h + c$ (c real), then the Green's functions G and Γ would have the following transformation properties under dilation:

$$G(x_1, \ldots, x_n) \to \rho^{n(h+c)} G(\rho x_1, \ldots, \rho x_n), \tag{1.18a}$$

$$\Gamma(x_1, \ldots, x_n) \to \rho^{n(h-c)} \Gamma(\rho x_1, \ldots, \rho x_n) \quad \rho > 0; \tag{1.18b}$$

in particular,

$$G^{-1}(x_1 - x_2) = -\Gamma(x_1, x_2) \to \rho^{2(h-c)} G^{-1}(\rho x_1 - \rho x_2). \tag{1.18c}$$

(The canonical dimension for a spinless field φ is obtained for $c = -1$.) The equation in Fig. 3 are obviously invariant under the substitution (1.18). The Ward identities (1.11) and (1.17) imply that the scale dimensions of the stress-energy tensor $\Theta_{\mu\nu}$ and of the conserved current j_μ are

$$d_\Theta = 2h, \quad d_j = 2h - 1, \tag{1.19}$$

respectively. The dimension d_B of the field B in the model, considered in Sec. IA 4, can be ascribed independently.

It turns out that the dynamical equations are invariant under the Euclidean conformal group $O^\dagger(2h+1, 1)$ which can be generated by (Euclidean) Poincaré transformations, dilations, and the conformal inversion.

$$Rx = -\frac{x}{x^2} \quad (x^2 = x_1^2 + \cdots + x_{2h}^2). \tag{1.20}$$

The transformation law of Green's functions under the inversion (1.20) is summarized by the following rules for Euclidean fields:

$$U(R)\phi(x)U(R)^{-1} = (x^2)^{-h-c}\phi(Rx), \tag{1.21a}$$

$$U(R)J_\mu(x)U(R)^{-1} = (x^2)^{-2h+1} r_{\mu\nu}(x) J_\nu(Rx), \tag{1.21b}$$

$$U(R)\Theta_{\mu\nu}(x)U(R)^{-1} = (x^2)^{-2h} r_{\mu\mu'}(x) r_{\nu\nu'}(x) \Theta_{\mu'\nu'}(x), \tag{1.21c}$$

where

$$r_{\mu\nu}(x) = x^2 \nabla_\mu (Rx)_\nu = \frac{2x_\mu x_\nu}{x^2} - \delta_{\mu\nu}. \tag{1.22}$$

The so-called special conformal transformations are given by a translation, sandwiched between two inversions. The R invariance of the dynamical equations follows from the covariance law for the volume element

$$\begin{aligned} dRx &= (x^2)^{-2h} \det(r_{\mu\nu}) dx \\ &= (x^2)^{-2h} dx. \end{aligned} \tag{1.23}$$

2. Conformal-invariant 2- and 3-point functions

We shall study in the rest of the paper conformal-invariant solutions of the dynamical equations for the models described in Sec. IA. If the Gell-Mann–Low limit is ultraviolet stable (as is usually assumed in this approach) then the conformal-invariant solution would provide the small-distance behavior of Green's functions in a more realistic theory with positive-mass particles.

The invariance property of the solution allows one to determine the 2- and 3-point functions up to a constant factor without actually solving the equations. Before writing down the corresponding expressions, we shall make a remark about the freedom in the choice of normalization.

In a canonical field theory the field operators are normalized in such a way that the residue in the pole of the 2-point function is one. In a scale-invariant theory with anomalous dimensions the 2-point function has no pole and there is no unique choice of field normalization. A multiplicative change $\varphi(x) \to \kappa\varphi(x)$ in the field [where $\kappa = \kappa(c)$ is some function of the dimension] leads to the following transformation law for Green's functions:

$$\begin{aligned} &G(x_1, \ldots, x_n) \to \kappa^n G(x_1, \ldots, x_n), \\ &\Gamma(x_1, \ldots, x_n) \to \kappa^{-n} \Gamma(x_1, \ldots, x_n), \\ &G_{\mu\nu}(x_1, \ldots, x_n; x) \to \kappa^n G_{\mu\nu}(x_1, \ldots, x_n; x). \end{aligned} \tag{1.24}$$

[The normalization of $\Theta_{\mu\nu}$ is fixed by the Ward identity (1.11).] Thus we can choose the normalization of the 2-point function of φ (and B) according to convenience; only the relative normalization of the 2- and 3-point functions will have a physical significance.

We shall choose the normalized invariant 2-point function for a fundamental field φ to be

$$G(x_1 - x_2) = (2\pi)^{-h} \frac{\Gamma(c+h)}{\Gamma(-c)} \left(\frac{2}{x_{12}^2}\right)^{h+c}, \quad x_{12} = x_1 - x_2. \tag{1.25}$$

With this normalization the Fourier transform of G is

$$G(p) = \int e^{-ipx} G(x) dx = (\tfrac{1}{2}p^2)^c, \tag{1.26}$$

so that the inverse Green's function $G^{-1}(x_{12})$ is obtained from (2.8) by changing the sign of c. We shall say that the field φ is *fundamental* if its dimension parameter $c = c_\varphi$ satisfies the inequalities

$$-1 \leq c_\varphi < 0. \tag{1.27}$$

For a fundamental field φ the 2-point function G corresponds to a positive-definite Wightman function

$$w(p) = -i\theta(p_0)[G(\vec{p}, -ip_0+0) - G(\vec{p}, -ip_0-0)]$$
$$= -2\theta_+(p)\sin\pi c(\tfrac{1}{2}p_M^2)^c, \tag{1.28}$$

where $\theta_+(p) = \theta(p_0)\theta(p_M^2)$, $p_M^2 = p_0^2 - \vec{p}^2$. (The subscript M stands for real Minkowski-space vectors and scalar products.) If we multiply φ by $\kappa(c) = [2^c\Gamma(-c)]^{1/2}$ we will obtain a positive-definite Wightman function for all $c \geq -1$ which coincides with the 2-point function of a free zero-mass field for $c = -1$.

We shall also need in what follows the conformal-invariant 2-point function of an arbitrary rank-l symmetric traceless tensor field $O_{\mu_1\cdots\mu_l}(x)$ of dimension $h+c$, $c = c(l)$. We shall use the notation

$$\chi = [l, c] \tag{1.29}$$

of Ref. 11 for the corresponding elementary representation of the Euclidean conformal group. In order to write down the 2-point function of the field $O_{\mu_1\cdots\mu_l}$ we shall use the homogeneous polynomial formalism (cf. Refs. 11, 26, and 27) in which every symmetric traceless tensor $f_{\mu_1\cdots\mu_l}$ is replaced by the generating polynomial

$$f(z) = f_{\mu_1\cdots\mu_l} z_{\mu_1}\cdots z_{\mu_l}$$

on the complex light cone $K_{2h} = \{z \in C^{2h};\ z^2 = z_1^2 + \cdots + z_{2h}^2 = 0\}$.

The convolution of a rank-l tensor f and a rank-k tensor g is given by

$$(f*g)(z) = \frac{(k-l)!}{k!\,(h+k-l-1)_l} f(D)g(z),$$

where

$(a)_s = \Gamma(a+s)/\Gamma(a)$.

$$D_\mu = (h-1+z\cdot\partial)\partial_\mu - \tfrac{1}{2}z_\mu\Delta_z \quad \left(\partial_\mu = \frac{\partial}{\partial z_\mu},\quad \Delta_z = \sum \partial_\mu^2\right) \tag{1.30a}$$

is an interior differentiation on the cone K_{2h}, such that

$$[D_\mu, z_\nu] = (h-1+z\cdot\partial)\delta_{\mu\nu} + Z_{\mu\nu},\quad Z_{\mu\nu} = z_\nu\partial_\mu - z_\mu\partial_\nu, \tag{1.30b}$$

$$D^2 = D_\mu D_\mu = 0 \quad (=z^2). \tag{1.30c}$$

We shall use the following normalized 2-point function for the field $O_{\mu_1\cdots\mu_l}$ (see Ref. 11):

$$G_\chi(x_{12}; z_1, z_2) = \frac{n(\chi)}{(2\pi)^h}\left(\frac{2}{x_{12}^2}\right)^{h+c}[-z_{1\mu}r_{\mu\nu}(x_{12})z_{2\nu}]^l, \tag{1.31}$$

where $r_{\mu\nu}(x)$ is given by (1.22) and

$$n(\chi) = \frac{\Gamma(h+c+l)\Gamma(h-c-1)}{\Gamma(-c)\Gamma(h+l-c-1)} \tag{1.32}$$

(see Ref. 28). With this normalization the momentum-space 2-point function can be written in the form[11]

$$G_\chi(p) = (\tfrac{1}{2}p^2)^c\sum_{s=0}^{l}\alpha_s(c)\Pi^{ls}(p), \tag{1.33a}$$

where

$$\Pi^{ls}(p; z, D) = \frac{(s+h-1)_{l-s}}{(2h+2s-2)_{l-s}}\binom{l}{s}\left(2\frac{p\cdot z\,p\cdot D}{p^2}\right)^{l-s}\Pi^{ss}(p; z, D), \tag{1.33b}$$

$$\Pi^{ss}(p; z, D) = \frac{s!\,(h-1)_s}{(2h-3)_{2s}}\left(-2\frac{p\cdot z\,p\cdot D}{p^2}\right)^s C_s^{h-3/2}\left(1-\frac{p^2 z\cdot D}{p\cdot z\,p\cdot D}\right)$$

are the $SO(2h-1)_p$ projection operators, C_s^ν is the Gegenbauer polynomial, and

$$\alpha_s(c) = \frac{(h+c-1)_s}{(h-c-1)_s}. \tag{1.34}$$

The projection operators $\Pi^{ls}(p)$ are characterized by the following algebraic properties:

$$\Pi^{ls}(p)\Pi^{ls'}(p) \equiv \frac{1}{l!\,(h-1)_l}\Pi^{ls}(p; z, D')\Pi^{ls'}(p; z', D) = \delta_{ss'}\Pi^{ls}(p), \tag{1.35a}$$

$$\sum_{s=0}^{l}\Pi^{ls}(p) = 1, \tag{1.35b}$$

$$(p\cdot D_1)^{l-s+1}\Pi^{ls}(p; z_1, D_2) = 0, \tag{1.36}$$

$$p\cdot z_1\Pi^{ls}(p; z_1, D_2)p\cdot D_2 = a(l,s)p^2\Pi^{l+1s}(p; z_1, D_2), \tag{1.37a}$$

$$(l+1)^{-2}(h+l-1)^{-2}p\cdot D_1\Pi^{l+1s}(p; z_1, D_2)p\cdot z_2 = a(l,s)p^2\Pi^{ls}(p; z_1, D_2) \tag{1.37b}$$

(if both sides are applied to a homogeneous polynomial of z_2 of degree l), where

$$a(l,s) = \frac{(l-s+1)(2h+l+s-2)}{(l+1)(2h+2l-2)}. \tag{1.37c}$$

It follows from (1.33)–(1.35) that the inverse 2-point function is given by

$$G_\chi^{-1}(x_{12}) = G_{\tilde{\chi}}(x_{12}) \quad \text{where } \tilde{\chi} = [l, -c] \tag{1.38}$$

(for $\chi=[l,c]$).

In order to be able to handle the most general situation (including the model described in Sec. I A 4) we shall write down the 3-point functions for three different spinless fields with dimension parameters c_1, c_2, and c_3. They are

$$G(x_1,x_2,x_3)=gV(x_1,c_1;x_2,c_2;x_3,c_3), \tag{1.39a}$$

$$\Gamma(x_1,x_2,x_3)=gV(x_1,-c_1;x_2,-c_2;x_3,-c_3), \tag{1.39b}$$

where the function V can be associated with an "infraparticle" triangular diagram (see Refs. 1 and 2) with scale-invariant propagators $(\frac{1}{2}x_{ik}^2)^{-d_{ik}}$. The parameters d_{ik} satisfy the conservation of dimension law in each vertex of the diagram:

$$d_{ik}+d_{jk}=d_k=h+c_k,\quad (i,j,k)=\text{permutation}(1,2,3) \tag{1.40}$$

[cf. Eq. (1.42) below].

We shall also need the 3-point functions for two scalar (or pseudoscalar) fields of dimension parameters c_1 and c_2, and a rank-l tensor field $O(x,z)$ of dimension $h+c$. According to Refs. 8 and 11 they are given by

$$G(x_1,x_2;x_3,z)=g_l V(x_1,c_1;x_2,c_2;x_3,z,\chi), \tag{1.41a}$$

$$\Gamma(x_1,x_2;x_3,z)=g_l V(x_1,-c_1;x_2,-c_2;x_3,z,\chi), \tag{1.41b}$$

where

$$V(x_1,c_1;x_2,c_2;x_3,z,\chi) = \frac{N_l}{(2\pi)^h}\left(\frac{2}{x_{12}^2}\right)^{h-\delta_\chi+c_+}\left(\frac{2}{x_{13}^2}\right)^{\delta_\chi+c_-}\left(\frac{2}{x_{23}^2}\right)^{\delta_\chi-c_-}(\lambda z)^l, \tag{1.42a}$$

$$c_\pm=\tfrac{1}{2}(c_1\pm c_2),\quad \delta_\chi=\tfrac{1}{2}(h+c-l), \tag{1.42b}$$

$$\lambda=\nabla_3 \ln\frac{x_{23}^2}{x_{13}^2}=2\left(\frac{x_{13}}{x_{13}^2}-\frac{x_{23}}{x_{23}^2}\right). \tag{1.43}$$

(We do not consider amputation of the external line associated with the tensor field O.) The normalization factor $N_l=N_l(c_+,c_-;c)$ is restricted by the symmetry conditions

$$V(x_1\chi_1,x_2\chi_2,x_3\chi_3)=V(x_i\chi_i,x_j\chi_j,x_k\chi_k), \tag{1.44}$$

where (i,j,k) is any permutation of the indices 1, 2, 3, and

$$V(x_1\chi_1,x_2\chi_2,x_3\tilde{\chi}_3)=\int dx\,V(x_1\chi_1,x_2\chi_2,x\chi_3)G_{\tilde{\chi}_3}(x-x_3). \tag{1.45}$$

It will be determined in Sec. IB 4 by adding an orthonormalization property of the V's.

The 3-point function of the stress-energy tensor involves no such normalization arbitrariness, because of the Ward identity (1.11). We have $G(x_1x_2;x_3,z)=0$ for $c_1\neq c_2$ and

$$G(x_1,x_2;x_3,z)=\frac{\Gamma(h+1)}{2(2\pi)^{2h}}\frac{\Gamma(h+c+1)}{(2h-1)\Gamma(-c)}\left(\frac{2}{x_{12}^2}\right)^{c+1}\left(\frac{4}{x_{13}^2x_{23}^2}\right)^{h-1}(\lambda z)^2 \tag{1.46a}$$

$$=\frac{\Gamma(h-1)}{2(2\pi)^{2h}}\frac{\Gamma(h+c+1)}{(2h-1)\Gamma(-c)}\left(\frac{2}{x_{12}^2}\right)^{c+1}\left[(z\cdot\nabla_1)^2-\frac{2h}{h-1}(z\cdot\nabla_1)(z\cdot\nabla_2)+(z\cdot\nabla_2)^2\right]\left(\frac{4}{x_{13}^2x_{23}^2}\right)^{h-1} \tag{1.46b}$$

for $c_1=c_2=c$.

With this G the coefficient to the Schwinger term in (1.11) is $a=-(c+h)/2h$. In verifying (1.11) we have used the relations

$$(2h-1)^{-1}(D\cdot\nabla_3)\left[(z\cdot\nabla_1)^2-\frac{2h}{h-1}(z\cdot\nabla_1)(z\cdot\nabla_2)+(z\cdot\nabla_2)^2\right]f(x_{13},x_{23})$$
$$=-\left[\nabla_1^2\left(z\cdot\nabla_1-\frac{h+1}{h-1}z\cdot\nabla_2\right)+\nabla_2^2\left(z\cdot\nabla_2-\frac{h+1}{h-1}z\cdot\nabla_1\right)\right]f(x_{13},x_{23}),$$

$$-\tfrac{1}{2}(2\pi)^{-h}\Gamma(h-1)\nabla_1^2(2x_{13}^{-2})^{h-1}=\delta(x_{13}).$$

3. *Skeleton diagram expansion*

Having constructed the physical propagator and (3-point) vertex function one can expand the n-point functions $\Gamma(x_1,\ldots,x_n)$ ($n\geq 4$) in terms of skeleton diagrams.[29] It is important to know for the self-consistency of conformal invariance that the skeleton diagrams, as well as the graphs encountered in the bootstrap equations in Figs. 3(b), 4(a), and 4(b), have no ultraviolet (or momentum-independent infrared) divergences. Indeed, such divergences would have destroyed even the scale invariance of the Green's functions. It was demonstrated in Ref. 2 that for a certain range of the parameters c the boson-fermion Yukawa interaction in 4 dimensions is divergence-free (for the

Gell-Mann–Low limit theory under consideration). The analysis of Ref. 2 is trivially extended to the $2h$-dimensional models considered in Sec. 1A. For the simplest φ^3 model the most stringent restriction on the scaling parameter $c=c_\varphi$ comes from the requirement that the 3-point functions $G(x_1,x_2,x_3)$ and $\Gamma(x_1,x_2,x_3)$ are both given by ordinary convergent integrals in momentum space. That leads to

$$-\tfrac{1}{3}h<c<\tfrac{1}{3}h. \tag{1.47}$$

The 3-point function $G_{\mu\nu}$ [see (1.46)] can be expressed in terms of a convergent p-space integral if $c<0$, i.e., if the field is fundamental [see (1.27)]. However, it can be continued analytically in $c>0$ to cover the range (1.47), only the point $c=0$ being excluded.

Coming to the more realistic model envisaged in Sec. IA4 we see that the existence of the 3-point functions in Figs. 5(a) and 5(b) (complete and amputated on either leg) as convergent integrals in momentum space gives

$$2|c_\varphi|+|c_B|<h \tag{1.48}$$

and

$$-\tfrac{1}{3}h<c_B<\tfrac{1}{3}h, \tag{1.49}$$

respectively. The convergence condition for the skeleton diagram of the $\varphi\varphi$-scattering amplitude with a two-B-line exchange leads to

$$-\frac{h}{2}<c_\varphi<\frac{h}{2}. \tag{1.50}$$

Assuming that φ is a fundamental field while B is a composite one so that $c_\varphi<0$, $c_B>0$, we end up with the following complete set of inequalities

$$\begin{aligned}&-\tfrac{1}{2}h<c_\varphi<0,\\ &0<c_B<\tfrac{1}{3}h,\\ &0<c_B-2c_\varphi<h,\end{aligned} \tag{1.51}$$

which guarantee absence of divergences.

The skeleton expansion does not satisfy, however the dynamical equations for all values of g and c. It turns out[2,6] that the entire infinite set of equations presented in Figs. 3 and 4 and in (1.11) and (1.12) will be satisfied provided that the two bootstrap equations in Figs. 3(b) and 3(c) [or Figs. 3(b), 4(a), and Eq. (1.12)] are satisfied. Since both sides of the equation in Figs. 3(b) and 4(a) are conformal invariant they have to be proportional to the 3-point functions (1.42) with $l=0$ and 2, respectively. Thus, these bootstrap equations lead to coordinate-independent transcendental equations for the two parameters g and c of the theory. As could have been predicted, these equations turn out to be equivalent to the Gell-Mann–Low equation for the coupling constant. That was verified by the ϵ-expansion method for the φ^3 theory in $6+\epsilon$ dimensions in Ref. 22. Unfortunately, this new version of the self-consistency equations does not seem any easier to handle. That is one reason that a new approach to the whole problem was attempted in Refs. 10 and 11 and is going to be pursued in what follows.

4. *Conformal partial-wave expansion*

The equation in Fig. 3 can be regarded as generalized (off-shell) unitarity equations. It is well known that in terms of the ordinary partial waves the (elastic) unitarity condition becomes an algebraic equation. It was demonstrated in Ref. 10 that the conformal extension of the partial-wave analysis allows one to solve the infinite set of dynamical equations for the φ^3 model.

Ordinary partial-wave expansion is nothing else but the tensor product expansion of two irreducible (positive-energy) representations of the Poincaré group. Conformal partial-wave expansion is by definition the tensor-product expansion of two irreducible representations of the Euclidean conformal group (see Ref. 11).

The proper vertex function $\Gamma(x_1,x_2,\ldots,x_n)$ ($n\geq4$) considered as a function of the first two coordinates and integrated over the remaining $n-2$ coordinates with a "nice" test function f satisfies the square integrability condition

$$\int dx_1\int dx_2\int dy_1\int dy_2\bar{\Gamma}_f^{(n)}(x_1,x_2)G_{\chi_1}(x_1-y_1)G_{\chi_2}(x_2-y_2)\Gamma_f^{(n)}(y_1,y_2)<\infty, \tag{1.52}$$

where

$$\chi_1=[0,c_1],\quad \chi_2=[0,c_2],$$

$$\Gamma_f^{(n)}(x_1,x_2)=\int\cdots\int dx_3\cdots dx_n\,\Gamma(x_1,\ldots,x_n)\,f(x_3,\ldots,x_n) \tag{1.53}$$

in any order of the skeleton perturbation theory.[2] [This is, however, not true for the 1-particle reducible diagrams, appearing in the right-hand side of the equation in Fig. 1(d).] The integral in (1.52) is nothing else but the scalar product in the representation space of the tensor product of two irreducible complementary series representations χ_1 and χ_2 [as long as $-h<c_i<h$ $(i=1,2)$, which is certainly true if either (1.47) or (1.48)–(1.51) take place]. If we assume in addition that

$$|c_1|+|c_2|<h, \tag{1.54}$$

which is always true if the convergence conditions (1.47)–(1.51) are satisfied, then we can use the tensor-product expansion formula of Ref. 11 which gives

$$\Gamma(x_1,\ldots,x_n)=\sum\int d\chi\int dx\, V(x_1,-c_1;x_2,-c_2;x,\tilde{\chi})\Gamma_\chi(x;x_3,\ldots,x_n), \tag{1.55}$$

where V is the invariant 3-point function (1.42) and

$$\sum\int d\chi=\sum_{l=0}^{\infty}\int_{-i\infty}^{i\infty}\frac{dc}{2\pi i}\rho_l(c), \tag{1.56a}$$

$$\rho_l(c)=\frac{\Gamma(l+h)}{2(2\pi)^h l!}\frac{\Gamma(h-1+c)\Gamma(h-1-c)}{\Gamma(c)\Gamma(-c)}[(h+l-1)^2-c^2]=\frac{\Gamma(l+h)}{2(2\pi)^h l!}n(\chi)n(\tilde{\chi}). \tag{1.56b}$$

Here $d\chi=\rho_l(c)(dc/2\pi i)$ is called the Plancherel measure; for $c_1=c_2$ $(c_-=0)$ and Γ symmetric with respect to (x_1,x_2) the sum in (1.55) and (1.56) is over even values of l, only. The conformal partial wave Γ_χ is, conversely, expressed in terms of $\Gamma(x_1,\ldots,x_n)$ by

$$\Gamma_\chi(x;x_3,\ldots,x_n)=\frac{1}{2}\int dx_1\int dx_2\, V(x_1,c_1;x_2,c_2;x,\chi)\Gamma(x_1,\ldots,x_n). \tag{1.57}$$

The compatibility of (1.55) and (1.57) implies the following orthonormalization condition for the "Clebsch-Gordan kernels" V:

$$\int dx_1\int dx_2\, V(x_1,-c_1;x_2,-c_2;x,\tilde{\chi})V(x_1,c_1;x_2,c_2;x',\chi')=\delta(x-x')1\delta(\chi,\chi')+G_{\chi'}(x-x')\delta(\tilde{\chi},\chi'), \tag{1.58}$$

where

$$\delta(\chi,\chi')=\frac{2\pi i}{\rho_l(c)}\delta(c-c')\delta_{ll'}. \tag{1.59}$$

In writing (1.58) we have assumed that for pure imaginary c (and real c_1 and c_2) V satisfies the reality property

$$\bar{V}(x_1,-c_1;x_2,-c_2;x,\chi)=V(x_1,-c_1;x_2,-c_2;x,\tilde{\chi})$$

(which puts another restriction on N_l). Equation (1.58) along with the symmetry conditions (1.44) and (1.45) determine the normalization factor N_l up to a sign (see Ref. 11):

$$N_l(c_+,c_-;c)=\left[\frac{\Gamma(h-\delta_\chi+c_+)\Gamma(h-\delta_{\tilde{\chi}}+c_+)\Gamma(h-\delta_{\tilde{\chi}}-c_-)\Gamma(h-\delta_{\tilde{\chi}}+c_-)}{\Gamma(h-\delta_\chi-c_+)\Gamma(h-\delta_{\tilde{\chi}}-c_+)\Gamma(h-\delta_\chi-c_-)\Gamma(h-\delta_\chi+c_-)}\right]^{1/2}. \tag{1.60}$$

One can define a unique branch of the square root (1.60) by demanding that it is a single-valued analytic function in the complex c plane with cuts along the intervals of the real c axis where the expression under the square root is negative, which takes positive values for $|c_1|+|c_2|<h$, $|c_1|+|c_2|-h-l<c<h+l-|c_1|-|c_2|$.

In the special case of the 4-point function it follows from conformal invariance that the partial wave $\Gamma_\chi(x;x_3,x_4)$ is again proportional to V:

$$\Gamma_\chi(x;x_3,x_4)=\gamma(\chi)V(x_3,-c_3;x_4,-c_+;x,\chi). \tag{1.61}$$

The conformal Fourier transform of the 4-point function $\gamma(\chi)$ also depends on the dimension parameters c_i of the underlying fields, but not on the x's.

The entire space-time dependence of Γ is given by a standard known function of the x's (the integral in x of the product of two V's). Conversely, using (1.57) we can express the conformal partial wave $\gamma(\chi)$ in terms of Γ. We assume at this point that

$$c_1-c_2=c_3-c_4=2c_-. \tag{1.62}$$

According to Sec. 2 of Appendix A the result is

$$\gamma(\chi)=\frac{l!\,b_l(-c_{12},-c_{34},c)}{2(2h-2)_l(\frac{1}{2}x_{34}^2)^{-c_{34}}}\int dx_1\int dx_2\left(\frac{x_{34}^2}{x_{12}^2}\right)^{h-\delta_{\tilde{\chi}}}\left(\frac{2}{x_{12}^2}\right)^{-c_{12}}C_l^{h-1}\left(\frac{x_{12}x_{34}}{(x_{12}^2x_{34}^2)^{1/2}}\right)\Gamma(x_1,x_2,x_3,x_4), \tag{1.63}$$

where $c_{ik}=\frac{1}{2}(c_i+c_k)$ and the factor b_l is given by Eq. (A8) of Appendix A.

A similar formula can be obtained by exchanging the roles of (x_1,x_2) and (x_3,x_4). The two expressions are consistent between each other because of the symmetry of $\Gamma(x_1,x_2,x_3,x_4)$ with respect to the substitution $(x_1,c_1;x_2,c_2)\rightleftarrows(x_3,c_3;x_4,c_4)$.

The symmetry property (1.45) of the Clebsch-Gordan kernels implies the following relations between conformal partial waves:

$$\Gamma_{\tilde{\chi}}(x;x_3,\ldots,x_n)=\int dy\,G_{\tilde{\chi}}(x-y)\Gamma_\chi(y;x_3,\ldots,x_n), \tag{1.64a}$$

$$\gamma(\tilde{\chi})=\gamma(\chi). \tag{1.64b}$$

II. POLE STRUCTURE OF CONFORMAL PARTIAL WAVES. VACUUM OPERATOR-PRODUCT EXPANSION.

A. Implications of the dynamical equations

1. *Poles in the conformal partial waves implied by the vertex bootstrap equations*

We shall start with a brief review of the solution[10] of the BS equation for the simple φ^3 model, and will then extend the results to the more realistic model of Sec. 1A4.

The 1PI amplitudes A_{1PI} and the BS kernel B satisfy the same covariance and square integrability conditions (with respect to the arguments x_1, x_2) as the proper vertex functions Γ. We can therefore apply Eqs. (1.55) and (1.61) to these functions:

$$A_{1PI}(x_1x_2;x_3x_4)=\sum\int d\chi\,a(\chi)F_\chi(x_1,x_2;x_3,x_4), \tag{2.1}$$

$$B(x_1x_2;x_3x_4)=\sum\int d\chi\,b(\chi)F_\chi(x_1,x_2;x_3,x_4), \tag{2.2}$$

where

$$F_\chi(x_1,x_2;x_3,x_4)=\int dx\,V(x_1,-c_1;x_2,-c_2;x,\tilde{\chi})\times V(x_3,-c_3;x_4,-c_4;x,\chi) \tag{2.3}$$

and a (χ), $b(\chi)$, and F_χ also depend on the dimension parameters c of the fields.[30] Using the orthonormalization condition (1.58), we reduce the BS equation for the φ^3 model of Sec. IA to the simple algebraic equation

$$a(\chi)=b(\chi)+b(\chi)a(\chi) \tag{2.4}$$

for the conformal partial waves. It implies that the partial-wave amplitude

$$a(\chi)=\frac{b(\chi)}{1-b(\chi)} \tag{2.5}$$

has a pole for $\chi=\chi_l$ for which $b(\chi_l)=1$. Using the relation

$$a(\chi)=\gamma(\chi)+a_1(\chi), \tag{2.6}$$

where $a_1(\chi)$ is the partial wave of the sum of 1-particle reducible diagrams displayed in Fig. 6, which has no singularities in the (12) channel, we conclude that $\gamma(\chi)$ has the same poles as $a(\chi)$.

On the other hand, the bootstrap equation in Fig. 3(b) for the vertex function (1.39b) and the analytic continuation to real c of the orthonormality relation (1.58) imply

$$gV(x_1,-c;x_2,-c;x_3,-c)=gb(\chi_0)V(x_1,-c;x_2,-c;x_3,-c)\quad\text{for } c=c_\varphi. \tag{2.7}$$

A similar relation follows from the equation in Fig. 4(a). Thus,

$$b(\chi_0)=1\ \text{for}\ \chi_0=[0,-c]\quad(c=c_\varphi), \tag{2.8a}$$

$$b(\chi_2)=1\ \text{for}\ \chi_2=[2,h], \tag{2.8b}$$

so that $a(\chi)$ and $\gamma(\chi)$ do have poles for $\chi=\chi_0$ and $\chi=\chi_2$.

Now we shall demonstrate that the same mechanism also works in the more complicated model of Sec. IA4.

Let us consider the set of Green's functions A_{1PI} with total charge zero in the channel (1,2). We shall use the following shorthand notation for the corresponding partial waves:

$$A_{1PI}(\varphi\varphi^*;\varphi\varphi^*)\to a_{\varphi\varphi}\equiv a(\varphi\varphi^*\to\varphi\varphi^*;\chi), \tag{2.9a}$$

$$A_{1PI}(BB;\varphi\varphi^*)\to a_{\varphi B}=a_{B\varphi}\equiv a(BB\to\varphi\varphi^*;\chi), \tag{2.9b}$$

$$A_{1PI}(BB;BB)\to a_{BB}\equiv a(BB\to BB;\chi), \tag{2.9c}$$

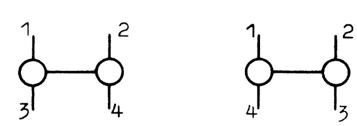

FIG. 6. 1-particle reducible diagrams, nonsingular in the (12) channel.

and similarly for the BS amplitudes. The BS equations are reduced to a system of algebraic equations whose solution is given by

$$a_{\varphi\varphi}=\frac{1-b_{BB}}{\Delta}-1,$$

$$a_{B\varphi}=\frac{b_{B\varphi}}{\Delta}, \tag{2.10}$$

$$a_{BB}=\frac{1-b_{\varphi\varphi}}{\Delta}-1,$$

$$\Delta=\Delta(\chi)=(1-b_{BB})(1-b_{\varphi\varphi})-b_{B\varphi}{}^2. \tag{2.11}$$

On the other hand, the bootstrap equations for the vertex functions in Figs. 5(a) and 5(b) give

$$g_1=b_{B\varphi}(\chi_B)g_0+b_{\varphi\varphi}(\chi_B)g_1,$$
$$g_0=b_{BB}(\chi_B)g_0+b_{B\varphi}(\chi_B)g_1. \tag{2.12}$$

The condition that the systems (2.12) has a nontrivial solution with respect to the "coupling constants" g_0 and g_1 leads to the equation

$$\Delta(\chi_B)=0 \tag{2.13}$$

and thus implies the existence of a pole for $\chi=\chi_B$ of the amplitudes (2.10).

Similarly, starting from the equations for the 3-point functions in Figs. 5(c) and 5(d) which involve the stress-energy tensor, we obtain

$$\Delta(\chi_2)=0 \tag{2.14}$$

for χ_2 given by (2.8b). Finally, the bootstrap equation for the current-field 3-point function (1.16) and the vanishing of $\langle TB(x_1)B(x_2)j_\mu(x_3)\rangle_0$ give

$$b_{B\varphi}(\chi_1)=0,\quad b_{\varphi\varphi}(\chi_1)=1\Rightarrow\Delta(\chi_1)=0 \tag{2.15a}$$

for

$$\chi_1=[1,h-1]. \tag{2.15b}$$

2. Pole structure of the n-point partial waves. Expression for the residues

In this subsection we shall spell out the implications of the dynamical equations for the simplest (φ^3) model only. The extension of the results to the (φ,φ^*,B) model of Sec. IA 4 [which uses (2.10)–(2.15)] is quite straightforward.

First of all we shall demonstrate that the poles of $\gamma(\chi)$ [and a (χ)], corresponding to the points (2.8), are also poles of the n-point partial waves $\Gamma_\chi(x;x_3,\ldots,x_n)$ (1.57) for all $n\ge 4$. We deduce this statement in two steps. It is true, if we replace Γ_χ by the 1PI partial wave

$$A^\chi_{1\text{PI}}(x;x_3,\ldots,x_n)=\frac{1}{2}\int dx_1\int dx_2V(x_1,c_\varphi;x_2,c_\varphi;x_2,c_\varphi;x,\chi)A_{1\text{PI}}(x_1,x_2;x_3,\ldots,x_n). \tag{2.16}$$

Indeed, taking the conformal Fourier transform of the equations in Figs. 2(b) and 2(c) we obtain

$$\begin{aligned}[1-b(\chi)]A^\chi_{1\text{PI}}(x;x_3,\ldots,x_n)&\\ =A^\chi_{2\text{PI}}(x;x_3,\ldots,x_n)&+b(\chi)\sum\int dy_1\int dy_2V(y_1,c_\varphi;y_2,c_\varphi;x,\chi)A(y_1,x_3,\ldots)A(y_2,\ldots,x_{i_n})\\ &+\sum_{k=3}^{n}\sum\int\cdots\int dy_1\cdots dy_kA^\chi_{2\text{PI}}(x;y_1,\ldots,y_k)G(y_1-y_1')\cdots G(y_k-y_k')A(y_1',x_3,\ldots)\cdots\\ &\times A(y_k',x_{i_k}\ldots).\end{aligned} \tag{2.17}$$

It follows that for each $\chi=\chi_l$ for which

$$b(\chi_l)=1 \tag{2.18}$$

$$b_l=\left.\frac{db(\chi)}{dc}\right|_{\chi=\chi_l}\neq 0. \tag{2.19}$$

[and the right-hand side of (2.17) does not vanish] the partial wave $A^\chi_{1\text{PI}}$ must have a pole. This is true, in particular, for $\chi=\chi_0$ and $\chi=\chi_2$, because of (2.8). It remains to show (as a second step) that the conformal Fourier transforms Γ_χ of the proper vertex functions also have poles in these points. That follows from the observation that the difference between $A^\chi_{1\text{PI}}$ and Γ_χ is given by a convergent skeleton diagram, provided that c_φ satisfies (1.47) (see Sec. IB 3 and Ref. 3).

We shall assume at this point that we are dealing with simple poles only, so that

The conformal expansion of the 2PI kernel in the dynamical equations in Fig. 3(a) leads to the relation

$$\Gamma(x,x_3,\ldots,x_n)=gA^{\chi_0}_{2\text{PI}}(x;x_3,\ldots,x_n). \tag{2.20}$$

Noting the relation between the amputated Green's function A and the proper vertex functions Γ and combining (2.17), (2.8a), (2.19), the equation in Figs. 2(b) and 2(c), and (2.20) we can express the residue of $A^\chi_{1\text{PI}}$ (or Γ_χ) at the pole $\chi=\chi_0$ in terms of the amputated Green's function A:

$$-gb_0 \operatorname{Res}_{\chi=\chi_0} A^{\chi}_{1PI}(x; x_3, \ldots, x_n)$$
$$= -gb_0 \operatorname{Res}_{\chi=\chi_0} \Gamma_\chi(x; x_3, \ldots, x_n) = A(x, x_3, \ldots, x_n). \quad (2.21)$$

Similarly, the residue of A^{χ}_{1PI} (or Γ_χ) at $\chi = \chi_2$ can be expressed in terms of the amputated $(n-1)$-point function $A_{\mu\nu}(x; x_3, \ldots, x_n)$, which involves the stress-energy tensor $\Theta_{\mu\nu}(x)$.

3. Basic conformal covariant tensor fields. Analyticity assumption

The preceding argument can be generalized as follows.

Let $O_l(x,z)$ be a conformal covariant rank-l tensor field of dimension $h+c_l$ for which the 3-point function

$$\langle T\varphi(x_1)\varphi^*(x_2)O_l(x,z)\rangle_0 \leftarrow G_l(x_1,x_2;x,z) \quad (2.22)$$

does not vanish. [In writing $O_l(x,z)$ in Minkowski space, we can regard z as a real lightlike vector—cf. Ref. 26.] Then the conformal partial waves $\Gamma_x(x; x_3, \ldots, x_n)$ have a pole for $\chi = \chi_l = [l, c_l]$. The argument is the same as above.

Let $\varphi(x)$ be a free zero-mass field; in other words, let $\varphi(x)$ satisfy the D'Alambert equation

$$\nabla^2\varphi(x) \equiv \Box\varphi(x) = 0, \quad (2.23)$$

and the canonical commutation relations. Consider the bilocal operator

$$O_l(x_1,x_2;z) = \kappa_l i^l D_l(h-1, z\cdot\nabla_1; h-1, z\cdot\nabla_2)\varphi^*(x_1)\varphi(x_2), \quad (2.24a)$$

where $D_l(a,\alpha;b,\beta)$ is the polynomial (A2) of Appendix A, and κ_l is a normalization constant ($z\cdot\nabla = z^0\nabla^0 - \vec{z}\cdot\vec{\nabla}$). In the simple case at hand (in which $a=b=h-1$) D_l can be expressed in terms of a Gegenbauer polynomial:

$$D_l(h-1,\alpha;h-1,\beta)$$
$$= l!\,(\alpha+\beta)^l P_l^{(h-2,\,h-2)}\left(\frac{\alpha-\beta}{\alpha+\beta}\right)$$
$$= l!\,\frac{(h-1)_l}{(2h-3)_l}(\alpha+\beta)^l C_l^{h-3/2}\left(\frac{\alpha-\beta}{\alpha+\beta}\right) \quad (2.24b)$$

[see, e.g., Eqs. 8.932 and 8.962.4 of Ref. 31; we have again used the shorthand notation $(a)_l$ of (1.34)]. We note that for the physically interesting case of 4-dimensional space-time $h-1=2h-3=1$ and $C_l^{1/2}$ coincides with the Legendre polynomial. The relevant property of the polynomials D_l for our present purposes is given by the following differential relations, valid for $\nabla_1^2 = 0 = \nabla_2^2$ [which can be assumed because of (2.23)]:

$$(\nabla_1\cdot D)D_l(h-1,z\cdot\nabla_1;h-1,z\cdot\nabla_2) = l(l+h-2)^2 D_{l-1}(h-1,z\cdot\nabla_1;h-1,z\cdot\nabla_2)\nabla_1\cdot\nabla_2$$
$$= -(\nabla_2\cdot D)D_l(h-1,z\cdot\nabla_1;h-1,z\cdot\nabla_2), \quad (2.25)$$

where D is the interior differentiation (1.30) on the light cone. Equation (2.25) implies that the local operator

$$O_l(x,z) = {:}O_l(x,x;z){:} = \lim_{x_1,x_2\to x}[O_l(x_1,x_2;z) - \langle O_l(x_1,x_2;z)\rangle_0] \quad (2.26)$$

is a conserved tensor current:

$$(D\cdot\nabla)O_l(x,z) = 0. \quad (2.27)$$

Moreover, it is a (weakly[5]) conformal-covariant *basic tensor* (in the sense of Ref. 8). We recall that for a basic tensor $O_l(x)$ the infinitesimal generators of special conformal transformations vanish at $x=0$. This means that the Euclidean counterpart of O_l transforms under an elementary representation of type χ_l of $O^\dagger(2h+1,1)$. In fact, O_l transforms according to the irreducible part of the "canonical representation"

$$\chi_l^{\text{can}} = [l, h+l-2]. \quad (2.28)$$

(It belongs to the space D_{l-11} in the notation of Ref. 11(b). We caution the reader that the gradient of a basic tensor is in general *not* a basic tensor—cf. Ref. 8.) If $\varphi(x)$ is a complex field ($\varphi \neq \varphi^*$) then the 3-point function (2.22) does not vanish for any of the operators (2.26). Choosing the normalization constant $\kappa_1 = e/(h-1)$, where e is the charge carried by φ^*, we can identify O_1 with the electromagnetic current:

$$O_1(x,z) = z^\mu j_\mu(x),$$
$$j_\mu(x) = ie\{\varphi^*(x)[\nabla_\mu\varphi(x)] - \varphi(x)[\nabla_\mu\varphi^*(x)]\}. \quad (2.29)$$

If φ is a neutral field ($\varphi = \varphi^*$), then the fields $O_l(x,z)$ with l odd vanish. Setting $2h(2h-1)\kappa_2 = 1$ we obtain as a special case the (traceless) stress-energy tensor for $l=2$:

$$O_2(x,z) = z^\mu z^\nu \Theta_{\mu\nu}(x), \quad (2.30a)$$

$$\Theta_{\mu\nu}(x) = :\nabla_\mu\varphi(x)\nabla_\nu\varphi(x): - \tfrac{1}{2}g_{\mu\nu}:\nabla^\lambda\varphi(x)\nabla_\lambda\varphi(x): + \frac{1}{2}\frac{h-1}{2h-1}(\Box g_{\mu\nu} - \nabla_\mu\nabla_\nu):\varphi^2(x):. \quad (2.30b)$$

We shall take the case of a free field as a guide concerning the set of basic tensor fields coupled to φ in general. We shall assume, in particular, in the case of the (neutral) φ^3 model of Sec. I A, that for each even l there exists at least one basic "composite" field $O_l(x,z)$ for which the 3-point function (2.22) with φ does not vanish. There is no reason to believe that for $l \geq 4$ the dimension of the field O_l—in a nontrivial, interacting theory—is canonical [i.e., that χ_l is given by (2.28)]. However, positivity of the 2-point Wightman function of O_l implies that if $\chi_l = [l, c_l]$ is the $O^\dagger(2h+1,1)$-representation label for O_l, then

$$c_l \geq h + l - 2 \quad (2.31)$$

[see Refs. 32, 11(a), and 11(b)].

Thus, the dynamical equations and our assumption about the set of composite basic fields imply the existence of a denumerable infinity of poles $\chi \equiv \chi_l$ in the conformal partial waves, satisfying (2.31). It is natural to conjecture that these are the only singularities of Γ_χ in the right half plane c. More precisely, we shall postulate that $\gamma(\chi)$ and $\Gamma_\chi(x; x_3, \ldots, x_n)$ are meromorphic functions of c in the right half plane with simple poles, restricted to the real c axis. We remark that unlike a similarly sounding ansatz about the singularity structure of scattering amplitudes in the complex angular momentum plane, this postulate is not in conflict with (off-shell) unitarity, since the dynamical equations are taken exactly into account.

B. Derivation of an operator-product expansion for vacuum expectation values

1. Splitting of conformal partial waves and Clebsch-Gordan kernels into Q kernels. Asymptotic behavior for Re$c \to \infty$.

In order to exploit the postulate about the meromorphic structure of Γ_χ it would be natural to try to close the integration path in the representation (1.55) in the right half plane c and then apply the residue theorem. In doing that, however, one encounters the problem of the asymptotic behavior in c of the integrand: Both V and Γ_χ increase exponentially for Re$c \to \infty$. In order to circumvent this difficulty, we shall derive an alternative form of the conformal expansion formula which exploits the spectral condition and the symmetry property (1.45) and (1.64).

As a first step it is convenient to replace the x integration in (1.55) by integration in p of the corresponding Fourier transforms. In other words, we write

$$\Gamma(x_1, \ldots, x_n) = \sum \int d\chi \int (dp) V_-^{\tilde{\chi}}(x_1, x_2; -p)\Gamma_\chi(p; x_3, \ldots, x_n), \; (dp) = (2\pi)^{-2h} d^{2h}p \quad (2.32)$$

where

$$V_\pm^{\tilde{\chi}}(x_1, x_2; -p) = \int V(x_1, \pm c_1; x_2, \pm c_2; x_3\tilde{\chi}) e^{ipx_3} dx_3. \quad (2.33)$$

One can show by a direct computation (see Appendix A) that

$$V_-^{\tilde{\chi}}(x_1, x_2; -p) = \frac{\pi}{\sin\pi(l+c)}[Q_-^{\tilde{\chi}}(x_1, x_2; -p) - G_{\tilde{\chi}}(-p)Q_-^{\chi}(x_1, x_2; -p)] \quad (2.34)$$

where

$$Q_-^{\tilde{\chi}}(x_1, x_2; -p) = -\frac{N_l(-c_+, -c_-; c)}{\Gamma(\delta_\chi + l - c_-)\Gamma(\delta_\chi + l + c_-)}\left(\frac{2}{x_{12}^2}\right)^{h-\delta_{\tilde{\chi}}-c_+} D_l(\delta_{\tilde{\chi}} - c_-, z\cdot\nabla_1; \delta_{\tilde{\chi}} + c_-, z\cdot\nabla_2)\left(\frac{x_{12}^2}{p^2}\right)^{(l+c)/2}$$
$$\times \int_0^1 du[u(1-u)]^{(h/2-1)}\left(\frac{1-u}{u}\right)^{c_-} e^{ip(ux_{12}+x_2)} I_{l+c}\{[u(1-u)x_{12}^2p^2]^{1/2}\}. \quad (2.35)$$

D_l is the lth-order differential operator (A2). The integral (2.35) is absolutely convergent for

$$l + \mathrm{Re}\,c + h > 2|c_-| = |c_1 - c_2|. \quad (2.36)$$

The decomposition (2.34) is reminiscent to the splitting of Legendre functions into two second-kind functions,

$$P_{-1/2+\nu}(z) = \frac{1}{\pi}\tan\pi\nu[Q_{-1/2-\nu}(z) - Q_{-1/2+\nu}(z)],$$

used in Regge theory. It displays, in particular, the symmetry relation

$$V_{\pm}^{\tilde{\chi}}(x_1, x_2; p) = G_{\tilde{\chi}}(p) V_{\pm}^{\chi}(x_1, x_2; p), \tag{2.37}$$

which follows from (1.45). The function $Q_-^{\chi}(x_1, x_2; -p, z)$ has the following characteristic properties which will be used in the sequel.

(i) $Q^{\tilde{\chi}}(x_1, x_2; -p, z)$ is an entire analytic function of p [provided that the integral in u converges, which is certainly true in the range (2.36)].

(ii) For Minkowski timelike vectors p, $Q^{\tilde{\chi}}$ decreases exponentially for $\text{Re}\, c \to \infty$. [That property, which will enable us to close the integration path in c, follows from the known asymptotic behavior of the Bessel function $J_\nu(x)$ for $\nu \to \infty$—see, e.g., Eq. 8.452.1 of Ref. 31.]

(iii) For small x_{12}, $Q_-^{\tilde{\chi}}$ is given by

$$Q_-^{\tilde{\chi}}(x_1, x_2; -p, z) \underset{x_{12} \to 0}{\approx} N_l(-c_+, -c_-; -c) \frac{\Gamma(-c)(h-c-1)_l}{\Gamma(-c-l)\Gamma(h+c+l)} e^{ipx_2} \left(\frac{2}{x_{12}^2}\right)^{h-\delta_\chi - c_+} (z \cdot x_{12})^l \tag{2.38}$$

[cf. (A6)].

Since the conformal partial wave Γ_χ exhibits the same symmetry property (1.64a) as the Clebsch-Gordan kernel V, it is natural to assume that it can be also decomposed in a similar way:

$$\Gamma_\chi(p; x_3, \ldots, x_n) = \frac{\pi}{\sin\pi(l-c)} [Q_\Gamma^{\chi}(p; x_3, \ldots, x_n) - G_\chi(p) Q_\Gamma^{\tilde{\chi}}(p; x_3, \ldots, x_n)], \tag{2.39}$$

where $Q_\Gamma^{\tilde{\chi}}$ satisfies conditions (i) and (ii) above. This assumption can be justified in the framework of the skeleton perturbation theory from what we already know about the 3-point function. [For $n=4$, it follows directly from Eq. (1.61)—see Appendix B.]

2. Another form of the conformal expansion, involving a Minkowski momentum-space integral

Our next step will be to deform the integration path in the complex p_{2h} plane to a contour C around the negative imaginary semi-axis (see Fig. 7).

To do that, we consider $\Gamma(x_1, \ldots, x_n)$ in the following domain of Euclidean time components:

$$\sigma_1 < 0, \quad \sigma_2 < 0; \quad \sigma_3 > 0, \ldots, \sigma_n > 0, \tag{2.40}$$

where $\sigma_j = (x_j)_{2h}$ [cf. (1.1)] and it is assumed, as usual, that $x_j \neq x_k$ for $j \neq k$. In this domain the exponential factor in the right-hand side of Eq. (2.35) decreases for $p_{2h} \to -i\infty$, and the same is true for $\Gamma_\chi(p; x_3, \ldots, x_n)$ because of the spectral condition. Hence, the deformation of the complex-energy integration path indicated above is indeed legitimate for such x's, and we are led to the evaluation of the discontinuity

$$\theta(p_0) [V_-^{\tilde{\chi}}(x_1, x_2; -\vec{p}, ip_0 - 0) \Gamma_\chi(\vec{p}, -ip_0 + 0; x_3, \ldots, x_n) - V_-^{\tilde{\chi}}(x_1, x_2; -\vec{p}, ip_0 + 0) \Gamma_\chi(\vec{p}, -ip_0 - 0; x_3, \ldots, x_n)]. \tag{2.41}$$

Inserting the decompositions (2.34) and (2.39) and using property (i) of the Q kernel (see Sec. II B 1) we see that only the 2-point Green's functions $G_\chi(p)$ and $G_{\tilde{\chi}}(p)$ give a nonvanishing contribution to the discontinuity (2.41). To evaluate the latter, we use the relations

$$(p^2)^c \to (p_M{}^2)^c e^{\mp i\pi c\theta(p_M{}^2)} \text{ for } p_{2h} \to -ip_0 \pm 0, p_M{}^2 = p_0{}^2 - \vec{p}^2, \tag{2.42a}$$

$$\Pi^{ls}(p; \xi, \eta) \to (-1)^l \Pi^{ls}(p_M; \vec{\xi}, -i\xi_{2h}, \vec{\eta}, -i\eta_{2h}). \tag{2.42b}$$

According to (1.33) the result is

$$\theta(p_0)[G_{[l,\pm c]}(\vec{p}, -ip_0 + 0) - G_{[l,\pm c]}(\vec{p}, -ip_0 - 0)] = \frac{i}{\pi} \sin\pi(l \mp c) w_{[l,\pm c]}(p_M), \tag{2.43}$$

where (omitting from now on the index M of p)

$$w_\chi(p) = 2\pi\theta_+(p)(\tfrac{1}{2}p^2) \sum_{s=0}^{l} \alpha_s(c) \Pi^{ls}(p), \quad \theta_+(p) = \theta(p_0)\theta(p^2). \tag{2.44}$$

That leads to the following expression for the discontinuity (2.41):

$$\frac{-i\pi}{\sin\pi(l+c)} [Q_-^{\tilde{\chi}}(x_1, x_2; -p) w_\chi(p) Q_\Gamma^{\tilde{\chi}}(p; x_3, \ldots, x_n) - Q_-^{\chi}(x_1, x_2; -p) w_{\tilde{\chi}}(p) Q_\Gamma^{\chi}(p; x_3, \ldots, x_n)]. \tag{2.45}$$

Since the range of c integration in (2.32) and (1.57) is symmetric with respect to the change $c \leftrightarrow -c$, we can drop the second term in (2.45), multiplying the result by 2. Thus, we reduce the conformal partial-wave expansion of Γ to the following form:

$$\Gamma(x_1, \ldots, x_n) = -2\pi \sum \int d\chi \frac{1}{\sin\pi(l+c)} \int (dp) Q_-^{\tilde{\chi}}(x_1, x_2; -p) w_\chi(p) Q_\Gamma^{\tilde{\chi}}(p; x_3, \ldots, x_n), \tag{2.46}$$

where the Minkowski p-space expression for $Q_-^{\tilde{\chi}}$ is obtained from (2.35) by replacing the p-dependent part of the integrand by

$$e^{i\vec{p}\cdot(u\vec{x}_{12}+\vec{x}_2)+p_0[u\sigma_1+(1-u)\sigma_2]} J_{l+c}([u(1-u)x_{12}{}^2p^2]^{1/2}) \tag{2.47}$$

and we have used $(dp) \to -i(dp_M)$ for $p_{2h} \to -ip_0$.

3. The vacuum operator-product expansion

Now it is legitimate to close the contour of c integration in (2.46) in the right half plane. However, transforming the integral over $V^{\tilde{\chi}}$ into an intergral over $Q^{\tilde{\chi}}$ (which vanishes for $\mathrm{Re}\, c \to \infty$) we have paid a certain price: the appearance of the factor $[\sin\pi(l+c)]^{-1}$ which introduces new "kinematical" poles. The main purpose of this subsection is to demonstrate that these poles are actually canceled out.

First of all, we note that a finite number of poles coming from the sine factor are canceled by the zero of the Plancherel weight (1.56b) for

$$c = 0, \ldots, h-2, h+l-1. \tag{2.48}$$

(At this point we assume, for the sake of simplicity, that h is a positive integer, which includes the cases $h=2$ and $h=3$ we are primarily interested in. The argument—and the result—can also be extended to the case of half odd integer h.) There remain two (infinite) sequences of "kinematical" poles to be dealt with; they correspond to the elementary representations $[l, c]$ with labels

$$c = h-1+l+\nu, \quad l = 0, 1, 2, \ldots, \quad \nu = 1, 2, \ldots \tag{2.49a}$$

and

$$c' = h-1+l, \quad l' = l+\nu \quad (l = 0, 1, \ldots, \quad \nu = 1, 2, \ldots) \tag{2.49b}$$

which satisfy

$$c' + l' = c + l = h - 1 + 2l + \nu. \tag{2.50}$$

The clue to the cancellation problem lies in the partial equivalence of the representations

$$\chi_{l\nu}^+ = [l, c] \equiv [l, h-1+l+\nu] \text{ and } \chi_{l\nu}'^+ = [l', c'] \equiv [l+\nu, h-1+l] \tag{2.51}$$

exhibited in Refs. 11(b) and 11(c)]. It leads, in particular, to the following identities among Q kernels and conformal partial waves (see Appendix B):

$$(ip\cdot z)^\nu Q_-^{\chi_{l\nu}^-}(x_1, x_2; -p.z) = \mathrm{sgn}\left[\left(\frac{1-\nu}{2}+c_-\right)_\nu\right] Q_-^{\chi_{l\nu}'^-}(x_1, x_2; -p, z), \tag{2.52}$$

$$(-ip\cdot z)^\nu Q_\Gamma^{\chi_{l\nu}^-}(p, z; x_3, \ldots, x_n) = \mathrm{sgn}\left[\left(\frac{1-\nu}{2}-c_-\right)_\nu\right] Q_\Gamma^{\chi_{l\nu}'^-}(p, z; x_3, \ldots, x_n), \tag{2.53}$$

where

$$\chi_{l\nu}^{(\prime)-} = \tilde{\chi}_{l\nu}^{(\prime)+} = [l^{(\prime)}, -c^{(\prime)}]. \tag{2.51$'$}$$

Furthermore, we shall use the relation

$$[(l+1)_\nu]^{-2}(p\cdot D_1)^\nu w_{\chi_{l\nu}'^+}(p; z_1, D_2)(p\cdot z_2)^\nu = (-1)^\nu (h+l-1)_\nu (2h+l-2)_\nu w_{\chi_{l\nu}^+}(p; z_1 D_2), \tag{2.54}$$

which follows from (1.37) and (2.44) [see also Proposition 6.5 of Ref. 11(b)], and the identity

$$\rho(\chi_{l\nu}'^+) \equiv \rho_{l+\nu}(h-1+l) = -\frac{(h+l-1)_\nu}{(2h+l-2)_\nu} \rho_l(h-1+l+\nu), \tag{2.55}$$

which is a direct consequence of the definition (1.56b) of the Plancherel weight.

Now we are ready to prove that the sum of the residues in the kinematical poles $\chi_{l\nu}^+$ and $\chi_{l\nu}'^+$ vanishes. Indeed, due to (2.50), for both these poles

$$\pi \operatorname{Res}[\sin\pi(l+c)]^{-1} = \pi \operatorname{Res}[\sin\pi(l'+c')]^{-1} = (-1)^{h-1+\nu} \tag{2.56}$$

and we have [according to (2.52)–(2.55)]

$$[(l+\nu)!\,(h-1)_{l+\nu}]^{-2}\rho(\chi'^{+}_{l\nu})Q_{-}^{\chi'_{l\nu}^{-}}(x_1,x_2;-p,D_1)w_{\chi'^{+}_{l\nu}}(p;z_1,D_2)Q_{\Gamma}^{\chi'_{l\nu}^{-}}(p,z_2;x_3,\ldots,x_n)$$
$$= -[l!\,(h-1)_l]^{-2}\rho(\chi^{+}_{l\nu})Q_{-}^{\chi_{l\nu}^{-}}(x_1,x_2;-p,D_1)w_{\chi^{+}_{l\nu}}(p;z_1,D_2)Q^{\chi_{l\nu}^{+}}(p,z_2;x_3,\ldots,x_n). \tag{2.57}$$

This proves the cancellation of the kinematical poles coming from $[\sin\pi(l+c)]^{-1}$.

Thus, closing the contour of integration in (2.46) in the right half plane c we obtain a representation of the proper vertex function as a sum over dynamical poles and poles coming from the normalization factor of $Q_{-}^{\tilde{\chi}}$ only:

$$\Gamma(x_1,\ldots,x_n) = 2\pi\sum_{\chi_l}\operatorname*{Res}_{\chi=\chi_l}\left(\frac{\rho_l(c)}{\sin\pi(l+c)}\int(dp)Q_{-}^{\tilde{\chi}}(x_1,x_2;-p)w_{\chi}(p)Q_{\Gamma}^{\tilde{\chi}}(p;x_3,\ldots,x_n)\right). \tag{2.58}$$

A similar expansion can be deduced from here for the full Green's function (see subsection II B 4 below). It can be regarded as the result of inserting a conformal-covariant operator-product expansion (of the type considered in Refs. 14–16) in the Wightman functions (which is subsequently continued analytically in the Euclidean region). It is, however, important for our derivation that the operator product $\varphi(x_1)\varphi(x_2)$ (which is effectively decomposed) acts directly on the vacuum. [That was used in exploiting the inequalities (2.40) for the Euclidean time components.] It is indicated in Ref. 17 that the general (global) operator-product expansion is more complicated. That is why we adopt the term "*vacuum expansion*" (of Ref. 17) for the situation envisaged here.

Remark. It follows from (2.52), (1.61), and (2.53) (for $n=4$) that

$$\left(\frac{1-\nu}{2}+c_-\right)_\nu \gamma(\chi^{+}_{l\nu}) = \left(\frac{1-\nu}{2}-c_-\right)_\nu \gamma(\gamma'^{+}_{l\nu}) \tag{2.59a}$$

as a consequence [according to Eq. (B6) below],

$$\gamma(\chi^{+}_{l\nu}) = (-1)^\nu\gamma(\chi'^{+}_{l\nu}) \text{ for } c_-\neq 0. \tag{2.59b}$$

On the other hand, if $c_-=0$ and $\varphi_1(x)=\varphi_2(x)=\varphi(x)$, then $\gamma(\chi_l)=0$ for odd l, and if $\gamma(\chi_l)\neq 0$ for even l Eq. (2.59b) cannot hold for odd ν. Similarly, the limit of the Clebsch-Gordan kernel V for $c_-\to 0$ does not commute with the one for $\chi\to\chi^{+}_{l\nu}$. The easiest way to treat the case $c_-=0$ is to reintroduce $c_-(\neq 0)$ as a regularization parameter, assuming that the poles for $g(\chi)$ at integer points (as the one for $\chi=[2,h]$) are shifted by an amount of the order of c_- (say, $\chi\to\chi_{c_-}=[2,h+O(c_-)]$), and letting c_- go to zero only in the final result (2.58).

4. Wightman positivity for the 4 point function

The representation (2.58) is particularly convenient in analyzing the positivity properties of the 4-point Wightman function (cf. Ref. 13). It is, of course, the full 4-point (Wightman or Schwinger) function that exhibits positivity, and not just the proper vertex function Γ. So, our first task will be to write down the counterpart of (2.58) for the Schwinger function

$$s(x_1,x_2,x_3,x_4) = \langle\phi_1(x_1)\phi_2(x_2)\phi_1(x_3)\phi_2(x_4)\rangle_0,$$

where ϕ_1 and ϕ_2 are spinless Euclidean fields with dimension parameters c_1 and c_2 (we can have $\phi_1=\phi_2$ as a special case).

Let us assume that there is a scalar (Euclidean) field $\phi_3(x)$ with $c_3<0$ such that the 3-point function $\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle_0$ does not vanish. (For the φ^3 model of Sec. I A we would have $\phi_1=\phi_2=\phi_3$, $c_1=c_2=c_3$.) Then, the nonamputated 1PI function $G_{1PI}(x_1,x_2;x_3,x_4)$ has a "shadow pole"[15] for $c=-c_3$, which is canceled by a singularity of the 1-particle reducible Green's function.[10] On the other hand, according to (1.45), (1.57), and (1.61) the conformal partial wave $s_0(\chi)$ of the disconnected Schwinger function $s_{11}(x_1-x_3)s_{22}(x_2-x_4)$ is 1. That gives

$$s(x_1,x_2,x_3,x_4) = s_{12}(x_1-x_2)s_{12}(x_3-x_4)$$
$$+2\pi\sum_{\chi_l}\frac{\rho_l(c_l)}{\sin\pi(l+c_l)}\operatorname*{Res}_{\chi=\chi_l}\left([1+g(\chi)]\int(dp)Q_{+}^{\tilde{\chi}}(x_1,x_2;-p)w_\chi(p)Q_{+}^{\tilde{\chi}}(x_3,x_4;p)\right), \tag{2.60}$$

where $g(\chi)$ is the conformal partial wave of $G_{1PI}(x_1,x_2;x_3,x_4)$ and the sum is over all poles of

$\{N_l^2(c_+, c_-; c)[1+g(\chi)]\}/[\Gamma(\delta_\chi + l + c_-)^2\Gamma(\delta_\chi + l - c_-)^2]$ in the right half plane c except the "shadow singularity" for $\chi = \tilde{\chi}_0 = [0, -c_3]$. We note that for a generalized free field [for which $g(\chi) = 0$] the disconnected Green's function $s_{11}(x_1 - x_3)s_{22}(x_2 - x_4) + s_{12}(x_1 - x_4)s_{21}(x_2 - x_3)$ is reproduced by the poles of $\Gamma(h - \delta_\chi + c_+)$ coming from the normalization factors in the Q's. It turns out that for interacting fields these poles are canceled by zeros of $1 + g(\chi; c_i) = 1/[1 - b(\chi, c_i)]$, i.e., by poles of $b(\chi, c_i)$. This is suggested by analysis of ultraviolet divergences in the skeleton expansion of the right-hand side of the equation:

$$b(\chi; c_i)V(x_3, c_3; x_4, c_4; x, \chi) = \frac{1}{2}\int dx_1 \int dx_2 V(x_1, -c_1; x_2, -c_2; x, \chi) B(x_1, \ldots, x_4).$$

[cf. (1.57) and (1.61)].

Now we are in a position to analyze the implications of the following (special case of) Osterwalder-Schrader positivity condition for the 4-point function.[13] Consider the space $\mathcal{S}_+ = \mathcal{S}_+(\mathrm{R}^{2h} \times \mathrm{R}^{2h})$ of test functions $f(x_1, x_2)$ of the Schwartz space $\mathcal{S}(\mathrm{R}^{4h})$ which vanish with all their derivatives unless $\sigma_1 > 0$, $\sigma_2 > 0$ $[\sigma_i = (x_i)_{2h}]$ and $x_1 \neq x_2$. Then, for any $f \in \mathcal{S}_+$, we have[33]

$$\int \cdots \int dx_1 \cdots dx_4 \bar{f}(\theta x_2, \theta x_1) s(x_1, \ldots, x_4) f(x_3, x_4) \geq 0 \quad [\theta(\vec{x}, \sigma) = (\vec{x}, -\sigma)]. \tag{2.61}$$

Inserting here $s(x_1, \ldots, x_4)$ by its expansion (2.60), we see that this positivity condition is satisfied, provided that the inequality (2.31) takes place and

$$\frac{\rho_l(c_l)}{\sin\pi(l + c_l)} \operatorname*{Res}_{\chi = \chi_l} g(\chi) > 0 \tag{2.62}$$

for all dynamical poles χ_l of $g(\chi)$, and finally that $[1 + g(\chi)]\operatorname{Res}\Gamma(h + c_+ - \delta_\chi) \geq 0$ at the poles of $\Gamma(h + c_+ - \delta_\chi)$.

C. The problem of crossing symmetry. Concluding remarks

1. Crossing symmetry and duality

The vacuum expansion (2.58) or (2.60) of the product $\varphi(x_1)\varphi(x_2)$ which satisfies the dynamical equations [in the (1, 2) channel] is not symmetric with respect to a permutation of the arguments x_1 and x_2 with any of the arguments $x_3, \ldots, x_n$. We are stuck here with the analog of a familiar problem of ordinary partial-wave analysis: It simplifies the unitarity equations but complicates the crossing symmetry condition. Yet, it should be stressed that the conformal expansion (as pointed out in Ref. 10) solves an infinite set of *coupled nonlinear* (integral) equations, while the problem of crossing symmetry can be reduced to a set of *noncoupled* (a finite number for each n) *linear* (integral) equations for the conformal partial waves.

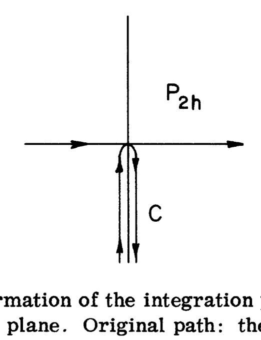

FIG. 7. Deformation of the integration path in the complex energy plane. Original path: the real p_{2h} axis; deformed path: the contour C.

In order to exhibit these symmetry equations we shall introduce another bit of graphical notation.

We shall represent the Clebsch-Gordan kernel V by a shadowed vertex [see Fig. 8(a)] and will write Eqs. (1.55) and (2.32) in the form given in Fig. 8(b). Then the crossing symmetry condition for the special case of the 4-point vertex function is expressed in Fig. 8(c). In the case of the $\varphi\varphi^* \to \varphi\varphi^*$ vertex function for the model considered in Sec. IA4 we would have $\gamma_{12}(\chi) = \gamma_{14}(\chi)$. In the case of the φ^3 model all γ_{ik} should be the same:

$$\gamma_{12}(\chi) = \gamma_{13}(\chi) = \gamma_{14}(\chi) = \gamma(\chi) \text{ for } \mathcal{L}_I = -\frac{1}{3!}g{:}\varphi^3{:}. \tag{2.63}$$

In order to find the crossing-symmetry equation

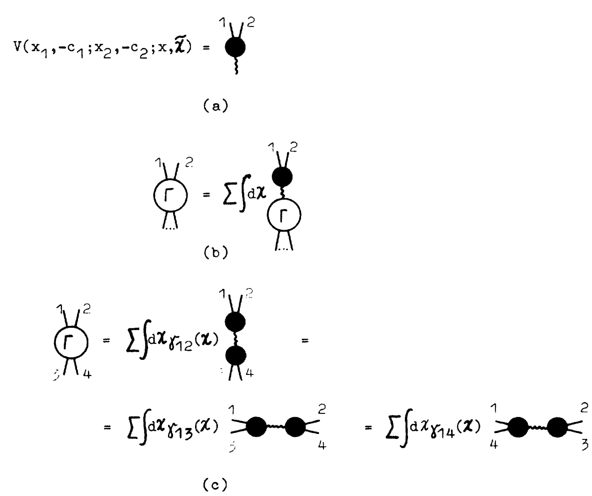

FIG. 8. The Clebsch-Gordan kernel and the partial-wave expansion of the vertex functions.

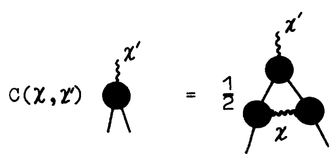

FIG. 9. Equation for the crossing kernel.

for $\gamma(\chi)$ we expand the kernel $F_\chi(x_1, x_3; x_2, x_4)$ [(2.3)] in conformal partial waves:

$$F_\chi(x_1, x_3; x_2, x_4) = \sum \int d\chi' C(\chi, \chi') F_{\chi'}(x_1, x_2; x_3, x_4). \tag{2.64}$$

To determine $C(\chi, \chi')$ we multiply both sides of (2.64) by $V(x_1, c_1; x_2, c_2; x, \chi'')$ and integrate over x_1 and x_2 using (1.59). After replacing χ'' by χ' we obtain the equation displayed in Fig. 9. From the involutive property of the crossing operation we deduce that

$$\sum \int d\chi'' C(\chi, \chi'') C(\chi'', \chi') = \delta(\chi, \chi'). \tag{2.65}$$

Inserting (2.64) into the equation in Fig. 8(c) we obtain in the symmetric case (2.63) the following linear integral equation for $\gamma(\chi)$:

$$\gamma(\chi) = \sum \int d\chi' C(\chi', \chi) \gamma(\chi'). \tag{2.66}$$

We could have alternatively formulated the crossing-symmetry condition as a duality property for the discrete vacuum expansion (2.58) [or (2.60)]. To do that we first need to continue both sides to Minkowski-space arguments with spacelike separations [since the inequalities (2.40) for different channels contradict each other]. That form of crossing symmetry makes obvious its relation to the local commutativity of the underlying fields.

The difficulty in treating the duality relation of the type displayed in Fig. 8(c) comes from the fact that an approximation of Γ involving only a finite number of poles in a given channel would not do. The reason is that the poles of the conformal partial wave in the crossed channel are reflected in the divergence of the infinite sum over residues in the direct channel.

2. A crossing-symmetric representation for the 4-point function

If we forget for a moment about the dynamical equations, it is not difficult to write down a crossing-symmetric representation for the conformal partial waves. It can be based on the known Mellin-transform representation of conformal-invariant Green's functions, proposed by Symanzik[4] and Mansouri.[19] For instance, the general conformal-invariant 4-point function, with dimensions restricted solely by Eq. (1.63), can be written in the form

$$\Gamma(x_1, \ldots, x_4) = \left(\frac{4}{x_{13}^2 x_{24}^2}\right)^{\delta} \left(\frac{16}{x_{12}^2 x_{23}^2 x_{14}^2 x_{34}^2}\right)^{(h-\delta)/2} \left(\frac{2}{x_{12}^2}\right)^{-c_{12}} \left(\frac{2}{x_{34}^2}\right)^{-c_{34}} \left(\frac{2}{x_{13}^2}\right)^{c_{24}} \left(\frac{2}{x_{24}^2}\right)^{c_{13}}$$
$$\times \left(\frac{x_{14}^2}{2}\right)^{c_{14}} \left(\frac{x_{23}^2}{2}\right)^{c_{23}} \int_{-\infty}^{\infty} \frac{d\sigma}{2\pi} \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} K(\sigma, \tau) \left(\frac{x_{13}^2 x_{24}^2}{x_{12}^2 x_{34}^2}\right)^{i\sigma} \left(\frac{x_{13}^2 x_{24}^2}{x_{14}^2 x_{23}^2}\right)^{i\tau}, \tag{2.67}$$

where $c_{ik} = \frac{1}{2}(c_i + c_k)$ as in (1.63). The right-hand side of (2.67) is independent of the real parameter δ provided that $K(z, w)$ is analytic in a strip domain along the real axes which includes the points $z = \sigma - i\delta$, $w = \tau - i(\delta/2)$ and that no poles arising from the x-dependent factors prevent us from shifting the integration path. In the φ^3 model under consideration we have

$$c_{ik} = c_\varphi. \tag{2.68}$$

The representation (2.12) can be made manifestly crossing symmetric by setting $\delta = \frac{1}{3}h - \frac{4}{3}c_\varphi$ and

$$K(\sigma, \tau) = f(\sigma, \tau, -\sigma - \tau), \tag{2.69a}$$

where

$$f(\alpha, \beta, \gamma) = f(\beta, \alpha, \gamma) = f(\alpha, \gamma, \beta). \tag{2.69b}$$

Inserting then (2.67) in Eqs. (1.57) and (1.61) and defining $\gamma(\chi)$, we obtain a conformal partial wave [depending on an arbitrary symmetric function $f(\alpha, \beta, \gamma)$] which satisfies automatically the crossing-symmetry equation (2.66). Using the integration formula of Ref. 4 we obtain

$$\gamma(\chi)=\frac{N_l(c_{12},c_-^{12};c)(2\pi)^{2h}}{2N_l(-c_{34},-c_-^{34};c)}\int_{-i\infty}^{i\infty}\frac{d\alpha}{2\pi}\int_{-i\infty}^{i\infty}\frac{d\beta}{2\pi}\int_{-i\infty}^{i\infty}\frac{d\sigma}{2\pi}\int_{-i\infty}^{i\infty}\frac{d\tau}{2\pi}\sum_{m=0}^{l}(-1)^m\binom{l}{m}$$
$$\times\frac{\Gamma(-\alpha-\beta)\Gamma(\alpha)\Gamma(-\alpha-\beta-\sigma-\tau-\frac{1}{2}(h-c+l)+\delta+c_{14})\Gamma(\alpha+\sigma+\frac{1}{2}(h-\delta)+c_-^{12})\Gamma(\beta+\frac{1}{2}(h-c+l)-c_-^{12})}{\Gamma(\frac{1}{2}(h+c+l)+c_-^{12})\Gamma(h-\frac{1}{2}(c+\delta-l)+\sigma)\Gamma(\delta+c_{24}-\sigma-\tau)\Gamma(\frac{1}{2}(h-\delta)-c_{14}+\tau)}$$
$$\times\frac{\Gamma(\beta+\tau+\frac{1}{2}(h-\delta)-c_{14})\Gamma(\frac{1}{2}(\delta+l-c)-\sigma-\alpha)\Gamma(\frac{1}{2}(c+\delta-l)+c_{13}-\tau-\beta+l-m)\Gamma(\frac{1}{2}(h-\delta)-c_{14}+\tau+\beta+m)}{\Gamma(h-\frac{1}{2}(c+\delta-l)-c_{13}+\tau+\beta)\Gamma(h-\frac{1}{2}(c+\delta-l)+\sigma+\alpha-m)\Gamma(\frac{1}{2}(c-l+\delta)+m-\alpha-\sigma)\Gamma(\delta+c_{13}-\alpha-\beta-\sigma-\tau)}$$
$$\times\Gamma(\tfrac{1}{2}(c+l+\delta)-\sigma-\alpha)K(-i\sigma,-i\tau). \tag{1.63$'$}$$

The difficulty now is to construct an f consistent with the pole structure of $\gamma(\chi)$ implied by the dynamical equations. This problem is not yet solved.

3. *Summary and discussion*

Our aim has been to construct a conformal-invariant quantum field theory satisfying (a) the dynamical equations in Figs. 3 and 4 (in a given channel), (b) Wightman (or Osterwalder-Schrader) positivity, and (c) crossing symmetry.

We were able to solve (a) and to incorporate some consequences of (b) by using the vacuum operator-product expansion, which can be written in the form

$$\varphi_2(x_2)\varphi_1(x_1)|0\rangle=s(x_1-x_2)|0\rangle+\sum_{\chi_l}C(\chi_l)\int dx\,\tilde{Q}_+^{\tilde{\chi}_l}(x_1,x_2;x)O_{\chi_l}(x)|0\rangle, \tag{2.70}$$

where we have set (for Minkowski-space coordinates with spacelike x_{12})

$$-[\Gamma(h-\delta_\chi-c_+)\Gamma(h-\delta_{\tilde\chi}-c_+)\Gamma(h-\delta_\chi-c_-)\Gamma(h-\delta_\chi+c_-)\Gamma(h-\delta_{\tilde\chi}-c_-)\Gamma(h-\delta_{\tilde\chi}+c_-)]^{1/2}\tilde{Q}_+^{\tilde\chi}(x_1,x_2;x)$$
$$=\left(\frac{2}{x_{12}^2}\right)^{h-\delta_{\tilde\chi}+c_+}D_l(\delta_{\tilde\chi}+c_-,z\cdot\nabla_1;\delta_{\tilde\chi}-c_-,z\cdot\nabla_2)$$
$$\times\int(dp)\theta_+(p)\left(\frac{-x_{12}^2}{p^2}\right)^{(l+c)/2}\int_0^1du[u(1-u)]^{h/2-1}\left(\frac{u}{1-u}\right)^{c_-}e^{ip\cdot(ux_{12}+x_2-x)}$$
$$\times J_{l+c}([-u(1-u)\,x_{12}^2p^2)]^{1/2}) \tag{2.71}$$

and

$$C(\chi_l)=\left(\frac{2\pi\rho_l(c_l)}{\sin\pi(l+c_l)}\operatorname*{Res}_{\chi=\chi_l}[\Gamma(h-\delta_\chi+c_+)\Gamma(h-\delta_{\tilde\chi}+c_+)(1+g(\chi))]\right)^{1/2},$$
$$\chi=[l,c],\quad \delta_\chi=\tfrac{1}{2}(h+c-l),\quad \chi_l=[l,c_l],\quad c_\pm=\tfrac{1}{2}(c_1\pm c_2)\ \ [\text{for } \varphi_1=\varphi_2=\varphi,\ c_-=0,\ c_+=c_\varphi]; \tag{2.72}$$

$O_{\chi_l}(x)$ are local (Hermitian) tensor fields, whose 2-point functions are given by the Fourier transform of (2.44):

$$\langle O_{\chi_l}(x)\otimes O_{\chi_{l'}}(x')\rangle_0=w_{\chi_l}(x-x')\delta_{ll'}\delta_{c_lc_{l'}}. \tag{2.73}$$

In the simplest φ^3 model the sum in (2.70) is over even values of l only and the first dynamical pole of $g(\chi)$ for $l=2$ comes from the stress-energy tensor. The positivity condition implies the inequality $c_l \geq h+l-2$ (2.31) and the reality of the coefficients $C(\chi_l)$ (2.72). The sum in (2.70) is over all poles of the expression in square brackets in the right-hand side of (2.72) with positive c_l, except for the scalar shadow pole ($c_0=-c_\varphi$ in the φ^3 model) which is omitted. We notice that the above construction automatically involves the positivity of the energy spectrum (cf. Ref. 12).

The crossing-symmetry condition implies a set of uncoupled linear integral equations for the conformal partial waves. The simplest of these equations—for the partial wave $\gamma(\chi)$ of the 1PI 4-point function $\Gamma(x_1,\ldots,x_4)$—is given by (2.66). It is satisfied by the general crossing-symmetric conformal-invariant 4-point functions (2.67)–(2.69). However, the problem of displaying simultaneously the pole structure, implied by the dynamical equations, and the permutation symmetry, reflecting the local commutativity of the underlying fields, is not solved. We conjecture that in carrying out a construction which takes into account all three requirements (a), (b) and (c) one should be able to discard the φ^3 model as inconsistent. The difficulty of this constructive problem should justify further study of simple soluble models[17,34] from the point of view of global operator-product expansions presented here.

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APPENDIX A: PARTIAL FOURIER TRANSFORM OF $V(x_1, x_2, x_3)$ AND RELATED FORMULAS

1. A differential formula for V. Fourier transform in x_3

To evaluate the Fourier transform of V in the third argument, it is convenient to replace the factor $(\lambda z)^l$ by a differential operator in x_1 and x_2 (cf. Refs. 10 and 11). To do that we use the identity

$$(a)_l (b)_l \left(\frac{2}{x_{13}^2}\right)^a \left(\frac{2}{x_{23}^2}\right)^b (\lambda z)^l = D_l(a, z\cdot\nabla_1; b, z\cdot\nabla_2)\left(\frac{2}{x_{13}^2}\right)^a \left(\frac{2}{x_{23}^2}\right)^b , \tag{A1}$$

where

$$D_l(a,\alpha;\ b,\beta) = \sum_{k=0}^{l} \binom{l}{k} (a+k)_{l-k}\,(b+l-k)_k\,(-\alpha)^k \beta^{l-k} = (a)_l(\alpha+\beta)^l F\left(a+b+l-1, -l; a; \frac{\alpha}{\alpha+\beta}\right); \tag{A2}$$

here F $(={}_2F_1)$ is the hypergeometric function and we are again using the Bateman shorthand notation $(a)_l$ (1.34). Equation (A1) is a simple consequence of the differentiation formula

$$(-z\cdot\nabla_1)^k \left(\frac{2}{x_{13}^2}\right)^a = (a)_k \left(\frac{2}{x_{13}^2}\right)^{a+k} (z\cdot x_{13})^k$$

and of the identity $(a)_k (a+k)_{l-k} = (a)_l$.

After that the calculation of the Fourier transform of V is reduced to the application of the following known relations:

$$\frac{\Gamma(a)\Gamma(b)}{(2\pi)^h}\int \left(\frac{2}{x_{13}^2}\right)^a \left(\frac{2}{x_{23}^2}\right)^b e^{-ipx_3} dx_3 = (2\pi)^{-h}\int_0^\infty \frac{d\alpha}{\alpha}\int_0^\infty \frac{d\beta}{\beta}\alpha^a\beta^b \int dx_3 \exp\left(-\alpha\frac{x_{13}^2}{2} - \beta\frac{x_{23}^2}{2} - ip\cdot x_3\right) \tag{A3a}$$

$$= \int_0^1 du\, u^{a-1}(1-u)^{b-1} e^{-ip\cdot[ux_1+(1-u)x_2]}\int_0^\infty d\lambda\, \lambda^{a+b-h-1}\exp\left\{-\tfrac{1}{2}\left[\lambda u(1-u)x_{12}^2 + \frac{p^2}{\lambda}\right]\right\},$$

$$\int_0^\infty d\lambda\, \lambda^{c-1}\exp\left[-\tfrac{1}{2}\left(\lambda\alpha^2 + \frac{\beta^2}{\lambda}\right)\right] = 2\left(\frac{\beta}{\alpha}\right)^c K_c(\alpha\beta) \tag{A3b}$$

(see Eq. 3.471.9 of Ref. 31). The result is

$$V_+^\chi(x_1,x_2;p,z) = \int V(x_1,c_1;x_2,c_2;x_3,\chi,z)e^{-ipx_3}dx_3$$
$$= 2A_l\left(\frac{2}{x_{12}^2}\right)^{h-\delta_x+c_+} D_l(\delta_x+c_-, z\cdot\nabla_1;\ \delta_x - c_-, z\cdot\nabla_2)\left(\frac{x_{12}^2}{p^2}\right)^{(l-c)/2}$$
$$\times\int_0^1 du[u(1-u)]^{h/2-1}\left(\frac{u}{1-u}\right)^{c_-} e^{-ip\cdot(ux_{12}+x_2)} K_{l-c}([u(1-u)x_{12}^2p^2]^{1/2}), \tag{A4}$$

where we have used the equation $h-2\delta_\chi = l-c$ and have set

$$A_l = A_l(c_+,c_-;c) = \frac{N_l(c_+,c_-;c)}{\Gamma(\delta_x+l+c_-)\Gamma(\delta_x+l-c_-)} = A_l(c_+,c_-;-c). \tag{A5}$$

For $x_{12}\to 0$ we can evaluate the main term in (A3) exactly. The result depends on the sign of $\mathrm{Re}\,c-l$. If V^χ corresponds to a physical 3-point function then the Wightman positivity condition (2.32) implies that for $h\geq 2$, $c-l$ is non-negative. In this case $2K_{l-c}(z)\sim\Gamma(c-l)(2/z)^{c-l}$ for $z\to 0$ and the small-distance behavior of V^χ is given by

$$V_+^\chi(x_1,x_2;p,z) \underset{x_{12}\to 0}{\approx} A_l\left(\frac{2}{x_{12}^2}\right)^{h-\delta_\chi+c_+} D_l(\delta_x+c_-,z\cdot\nabla_1;\delta_x-c_-,z\cdot\nabla_2)B(h-\delta_x+c_-,h-\delta_x-c_-)e^{-p\cdot x_2}\left(\frac{x_{12}^2}{2}\right)^{l-c}$$
$$\approx\frac{\Gamma(c)N_l(c_+,c_-;c)}{\Gamma(c-l)\Gamma(h-c+l)}(h+c-1)_l e^{-p\cdot x_2}\left(\frac{2}{x_{12}^2}\right)^{h-\delta_\chi+c_+}(z\cdot x_{12})^l. \quad \text{(A6)}$$

2. Derivation of Eq. (1.63) for the conformal partial wave

We shall first derive Eq. (1.63) for $\gamma(\chi)$ for

$$\mathrm{Re}\, c\geq l \quad \text{(A7)}$$

and then proceed by analytic continuation.

We start with Eq. (1.57) for $n=4$, and after insertion of (1.61) integrate both sides with respect to x. The result is given by (A6) with $p=0$ and $c_+\to -c_{34}$, $c_-\to -c_-$ on the left-hand side. Noting that [because of (1.62)]

$$\frac{N_l(c_{12},c_-;c)}{N_l(-c_{34},-c_-;c)}=\left(\frac{\Gamma(h-\delta_\chi+c_{12})\Gamma(h-\delta_{\tilde\chi}+c_{12})\Gamma(h-\delta_\chi+c_{34})\Gamma(h-\delta_{\tilde\chi}+c_{34})}{\Gamma(h-\delta_\chi-c_{12})\Gamma(h-\delta_{\tilde\chi}-c_{12})\Gamma(h-\delta_\chi-c_{34})\Gamma(h-\delta_{\tilde\chi}-c_{34})}\right)^{1/2}$$
$$=b_l(c_{12},c_{34};c)\quad\left[c_{ik}=\tfrac{1}{2}(c_i+c_k)\right], \quad \text{(A8)}$$

we obtain

$$\gamma(\chi)(x_{34}\cdot z)^l=\tfrac{1}{2}b_l(c_{12},c_{34};c)\left(\frac{x_{34}^2}{2}\right)^{h-\delta_{\tilde\chi}-c_{34}}\int dx_1\int dx_2\left(\frac{2}{x_{12}^2}\right)^{h-\delta_{\tilde\chi}+c_{12}}(z\cdot x_{12})^l\Gamma(x_1,x_2,x_3,x_4). \quad \text{(A9)}$$

Finally, we apply to both sides of (A9) the operator

$$\frac{1}{l!(h-1)_l}\left(\frac{x_{34}\cdot D}{x_{34}^2}\right)^l,$$

where D is the interior differentiation (1.30) on the complex light cone, and use

$$\frac{1}{l!(h-1)_l}(\eta\cdot D)^l(\zeta\cdot z)^l=H_l(\eta,\zeta)\equiv(\eta\cdot\zeta)^l F\left(-\frac{l}{2},\frac{1-l}{2};2-h-l;\frac{\eta^2\zeta^2}{(\eta\cdot\zeta)^2}\right)$$
$$=\frac{l!}{(h-1)_l}\left(\frac{1}{4}\eta^2\zeta^2\right)^{l/2}C_l^{h-1}\left(\frac{\eta\cdot\zeta}{\sqrt{\eta^2\zeta^2}}\right). \quad \text{(A10)}$$

Noting the normalization condition

$$C_l^{h-1}(1)=\frac{(2h-2)_l}{l!} \quad \text{(A11)}$$

for the Gegenbauer polynomials we end up with Eq. (1.63).

3. Splitting of $V^\chi(x_1,x_2;p)$ into two Q functions

Using the known relation (see e.g., Eq. 8.485 of Ref. 31)

$$2K_\nu(z)=-\frac{\pi}{\sin\pi\nu}[I_\nu(z)-I_{-\nu}(z)]$$

between modified Bessel functions, we obtain a splitting of V^χ into two Q functions, which have the properties (i)–(iii) of Sec. II B I. In order to prove eqs. (2.34) and (2.35), we will establish the following relation:

$$\left(\frac{2}{x_{12}^2}\right)^{(l+c)/2}\left[\frac{1}{l!(h-1)_l}G_\chi(p;z,D')D_l(\delta_{\tilde\chi}+c_-,z'\cdot\nabla_1;\delta_{\tilde\chi}-c_-,z'\cdot\nabla_2)\right]\left(\frac{x_{12}^2}{p^2}\right)^{(l+c)/2}$$
$$\times\int_{-1}^{1}dt\,\rho_{c_-}(t)e^{-ip\cdot x_t}I_{l+c}(\tfrac{1}{2}[(1-t^2)x_{12}^2p^2]^{1/2})$$
$$=\left(\frac{2}{x_{12}^2}\right)^{(l-c)/2}D_l(\delta_\chi+c_-,z\cdot\nabla_1;\delta_\chi-c_-,z\cdot\nabla_2)\left(\frac{p^2}{x_{12}^2}\right)^{(c-l)/2}\int_{-1}^{1}dt\,\rho_{c_-}(t)e^{-ip\cdot x_t}I_{c-l}(\tfrac{1}{2}[(1-t^2)x_{12}^2p^2]^{1/2}),\tag{A12}$$

where t is related to the integration variable u in (A4) and (2.35) by $t=2u-1$, and

$$\rho_{c_-}(t)=\frac{1}{2}\left(\frac{1+t}{2}\right)^{h/2-1+c_-}\left(\frac{1-t}{2}\right)^{h/2-1-c_-},\tag{A13}$$

$$x_t=\frac{1+t}{2}x_1+\frac{1-t}{2}x_2=\tfrac{1}{2}(x_1+x_2)-\tfrac{1}{2}tx_{12}.\tag{A14}$$

If we assume that (A12) is true, then multiplying both sides by $A_l(c_+,c_-;c)(2/x_{12}^2)^{c_++h/2}$ and using (A5) we obtain the counterpart of (2.34) for $V_-^{\tilde\chi}(x_1,x_2;-p)$ replaced by $V_+^{\chi}(x_1,x_2;p)$.

The proof of (A12) is rather tricky, since the equality does not hold for the $(t-)$ integrand. We shall only verify it for the leading terms in both sides for $x_{12}\to 0$. The validity of (A12) for arbitrary x_1 and x_2 would then follow from the covariance of Q_+^χ under the semigroup S defined in Ref. 12 [S consists of those transformations of $O^\dagger(2h+1,1)$ which leave the sign of the Euclidean time component x_{2h} invariant].

To find the small x_{12} behavior of each side of (A12) we use the power series expansion

$$I_\nu(z)=\left(\frac{z}{2}\right)^\nu\sum_{k=0}^{\infty}\frac{1}{k!\Gamma(\nu+k+1)}\left(\frac{z^2}{4}\right)^k$$

of the Bessel function and the relations

$$\frac{1}{2}\int_{-1}^{1}dt\left(\frac{1+t}{2}\right)^{a-1}\left(\frac{1-t}{2}\right)^{b-1}=B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},\tag{A15}$$

$$\frac{1}{\Gamma(c+l+1)}D_l(\delta_{\tilde\chi}+c_-,z\cdot\nabla_1;\delta_{\tilde\chi}-c_-,z\cdot\nabla_2)\left(\frac{x_{12}^2}{2}\right)^{l+c}=(-1)^l\frac{(h-c-1)_l}{\Gamma(c+1)}\left(\frac{x_{12}^2}{2}\right)^c(z\cdot x_{12})^l,\tag{A16}$$

$$\frac{1}{(h-1)_l l!}G_\chi(p;z,D')(x_{12}\cdot z')^l=G_\chi(p;z,x_{12})=\frac{(-1)^l l!}{(h-c-1)_l}\left(\frac{p^2}{2}\right)^c\left(\frac{2p\cdot zp\cdot x_{12}}{p^2}\right)^l P_l^{(c-l,h-2)}\left(1-\frac{p^2z\cdot x_{12}}{p\cdot zp\cdot x_{12}}\right);\tag{A17}$$

$$D_l(\delta_\chi+c_-,z\cdot\nabla_1;\delta_\chi-c_-,z\cdot\nabla_2)\left[\left(\frac{x_{12}^2}{2}\right)^k e^{-ip\cdot x_t}\right]=l!(h+c-1)_k(ip\cdot z)^{l-k}(-x_{12}\cdot z)^k e^{-ip\cdot x_t}P_{l-k}^{(\alpha_k^-,\alpha_k^+)}(t)+O(x_{12}^2(x_{12}\cdot z)^{k-1}),\tag{A18}$$

$$\alpha_k^\pm=\delta_\chi+k-1\pm c_-,\quad k=0,1,\ldots,l.$$

Here $P^{(\alpha,\beta)}(t)$ is the Jacobi polynomial; Eq. (A17) is a consequence of (1.33) (cf. Ref. 11(b)); in deriving (A18) we used the following relation between the Jacobi polynomials and the hypergeometric function

$$P_n^{(\alpha,\beta)}(t)=(-1)^n\frac{(\beta+1)_n}{n!}F\left(n+\alpha+\beta+1,-n;\beta+1;\frac{1+t}{2}\right)\tag{A19}$$

(see Eq. 8.962.1 of Ref. 31). Using further the integration formula (we caution the reader that the corresponding equation 7.391.3 of Ref. 31 contains an error)

$$\frac{1}{2}\int_{-1}^{1}dt\left(\frac{1-t}{2}\right)^\alpha\left(\frac{1+t}{2}\right)^{\beta+\nu}P_n^{(\alpha,\beta)}(t)=B(\alpha+n+1,\beta+n+1)\quad\text{for }\nu=n\tag{A20a}$$

$$=0\quad\text{for }\nu=0,\ldots,n-1,\tag{A20b}$$

we obtain the following small-x_{12} expression for the right-hand side of (A12):

$$\underset{x_{12}\to 0}{\approx} B(\delta_\chi+l+c_-,\delta_\chi+l-c_-)\exp\left(-ip\cdot\frac{x_1+x_2}{2}\right)\left(\frac{2}{x_{12}^2}\right)^{(l-c)/2}\left(\frac{p^2}{2}\right)^c\sum_{k=0}^{l}\binom{l}{k}\frac{(h+c-1)_k}{\Gamma(c+k-l+1)}\left(\frac{2p\cdot zp\cdot x_{12}}{p^2}\right)^{l-k}(-x_{12}\cdot z)^k. \tag{A21}$$

It follows from (A15)–(A17) and from the identity

$$l!P_l^{(c-l,h-2)}(\omega)=(c+1-l)_lF\left(h+c-1,-l;c+1-l;\frac{1-\omega}{2}\right)\quad\left(\omega=1-\frac{p^2z\cdot x_{12}}{p\cdot zp\cdot x_{12}}\right) \tag{A22}$$

(see Eq. 8.962.2 of Ref. 31) that the left-hand side of (A12) is also given by (A21) in the small x_{12} limit. This completes our proof of the representation (2.34).

APPENDIX B: IDENTITIES BETWEEN Q AND V FUNCTIONS FOR PARTIALLY EQUIVALENT REPRESENTATIONS

Equation (2.52) is equivalent to the relation

$$(z\cdot\nabla_3)^\nu V(x_1,c_1;x_2,c_2;x_3,\chi_{l\nu}^-,z)=\operatorname{sgn}\left[\left(\frac{1-\nu}{2}+c_-\right)_\nu\right]V(x_1,c_1;x_2,c_2;x_3,\chi_{l\nu}'^-,z) \tag{B1}$$

established in Ref. 11(c). They both follow from the remark that $\delta_\chi=-\frac{1}{2}(\nu-1)-l$ for both $\chi=\chi_{l\nu}^-$ and $\chi=\chi_{l\nu}'^-$ and from the identity

$$\begin{aligned}D_{l+\nu}\left(\frac{1-\nu}{2}-l+c_-,\alpha;\frac{1-\nu}{2}-l-c_-,\beta\right)&=\left(\frac{1-\nu}{2}+c_-\right)_\nu(\alpha+\beta)^\nu D_l\left(\frac{1-\nu}{2}-l+c_-,\alpha;\frac{1-\nu}{2}-l-c_-,\beta\right)\\&=\left(\frac{1-\nu}{2}-l+c_-\right)_{l+\nu}(\alpha+\beta)^{l+\nu}F\left(-l,-l-\nu;\frac{1-\nu}{2}-l+c_-;\frac{\alpha}{\alpha+\beta}\right)\end{aligned} \tag{B2}$$

for the polynomials (A2). To see that, we rewrite Eq. (1.42) for the Clebsch-Gordan kernel in terms of the differential operator D_l:

$$\begin{aligned}(2\pi)^hV(x_1,c_1;x_2,c_2;x_3,\chi,z)\\=\frac{N_l(c_+,c_-;c)}{(\delta_\chi+c_-)_l(\delta_\chi-c_-)_l}\left(\frac{2}{x_{12}^2}\right)^{h-\delta_\chi+c_+}D_l(\delta_\chi+c_-,z\cdot\nabla_1;\delta_\chi-c_-,z\cdot\nabla_2)\left[\left(\frac{2}{x_{13}^2}\right)^{\delta_\chi+c_-}\left(\frac{2}{x_{23}^2}\right)^{\delta_\chi-c_-}\right]\end{aligned} \tag{B3}$$

and use the identities $\delta_\chi+l=(1-\nu)/2$ (for $\chi=\chi_{l\nu}^{(\prime)-}$)

$$\frac{(\delta_\chi+c_-)_{l+\nu}(\delta_\chi-c_-)_{l+\nu}}{(\delta_\chi+c_-)_l(\delta_\chi-c_-)_l}=(-1)^\nu\left[\left(\frac{1-\nu}{2}+c_-\right)_\nu\right]^2, \tag{B4}$$

$$\frac{N_l(c_+,c_-;1-h-l-\nu)}{N_{l+\nu}(c_+,c_-;1-h-l)}\left(\frac{1-\nu}{2}+c_-\right)_\nu^2=\left[\frac{(\delta_\chi+l-c_+)_\nu(\frac{1}{2}(1-\nu)+c_-)_\nu}{(\delta_\chi+l+c_+)_\nu(\frac{1}{2}(1-\nu)+c_-)_\nu}\right]^{1/2}\left|\left(\frac{1-\nu}{2}+c_-\right)_\nu\right|=\left|\left(\frac{1-\nu}{2}+c_-\right)_\nu\right| \tag{B5}$$

for the normalization factors. Noting further that the translation invariance of $V(x_1,x_2,x_3)$ allows us to replace $z\cdot\nabla_3$ by $-z\cdot(\nabla_1+\nabla_2)$ and using

$$\left(\frac{1-\nu}{2}-c_-\right)_\nu=(-1)^\nu\left(\frac{1-\nu}{2}+c_-\right)_\nu \tag{B6}$$

[cf. (B4)] we complete the proof of (B1). Equation (2.52) then follows by replacing ∇_3 by ip and $c_\pm$ by $-c_\pm$.

Applying the differential operator $(z\cdot\nabla)^\nu$ to both sides of the equation

$$\gamma(\chi_{l\nu}^-)V(x_3,-c_3;x_4,-c_4;x,\chi_{l\nu}^-,z)=\frac{1}{2}\int dx_1\int dx_2V(x_1,c_1;x_2,c_2;x,\chi_{l\nu}^-,z)\Gamma(x_1,x_2,x_3,x_4) \tag{B7}$$

[which is obtained by analytic continuation in c of (1.57) and (1.61)], and using (B1), (1.62), and (B6) we obtain

$$\left(\frac{1-\nu}{2}+c_-\right)_\nu \gamma(\chi_{l\nu}^-)=\left(\frac{1-\nu}{2}-c_-\right)_\nu \gamma(\chi_{l\nu}'^-). \tag{B8}$$

Using again (1.61) and the relation (2.39) between $\Gamma_\chi(p;x_3,x_4)$ and $Q_\Gamma^\chi(p;x_3,x_4)=\gamma(\chi)Q_-^\chi(x_3,x_4;p)$ [where the last equality is a consequence of (1.61) and (1.64)], we find

$$(-ipz)^\nu Q_\Gamma^{\chi_{l\nu}^-}(p,z;x_3,x_4)=\operatorname{sgn}\left[\left(\frac{1-\nu}{2}-c_-\right)_\nu\right] Q_\Gamma^{\chi_{l\nu}'^-}(p,z;x_3,x_4), \tag{B9}$$

which coincides with (2.53) for $n=4$. For arbitrary n Eq. (2.53) can be justified in the framework of the skeleton perturbation theory.

We note that for $c_-=0$ the above argument [as well as Eqs. (B1) and (B9)] requires a modification since

$$\left(\frac{1-\nu}{2}\right)_\nu = 0 \text{ for odd } \nu \tag{B10}$$

and the sign function in the above formulas is not defined (cf. the remark at the end of Sec. II B 3).

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$$\varphi(x) = Z_3^{-1/2} : \varphi_u(x) : ,$$

$$: \varphi_u(x) : = -i \frac{\delta}{\delta J(x)} \{ \langle T \exp[i \int \varphi_u(x) J(x) dx] \rangle_0^{-1} \times T \exp[i \int \varphi_u(x) J(x) dx] \} |_{J=0} .$$

It should be noted that Eqs. (1.13)–(1.15) do not contain any parameter (like coupling constant), but just relate Green's functions among themselves.

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[28]Other normalization conventions, consistent with the property

$$n(\chi) n(\tilde{\chi}) = (c)_{h-1} (-c)_{h-1} [(h+1-1)^2 - c^2]$$

[and hence with Eq. (1.38)] were also considered in the last three papers of Ref. 11 where the notation $n_0(x)$ is used for the choice (1.32); see, in this connection, also K. Koller, Commun. Math. Phys. 40, 15 (1975).

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