Integral equations for extended solutions in field theory: Monopoles and dyons

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In this paper we give a simple method to obtain extended solutions of nonlinear classical field theories. The technique applies to problems where the boundary values at either end of the domain are specified, and consists of converting the set of field equations to a system of coupled nonlinear integral equations that can be solved numerically by simple iteration. As an illustration, the 't Hooft monopole and the Julia-Zee dually charged monopole system are studied in detail. The physical structure of these extended solutions and the possible effects of quantum fluctuations are briefly discussed.

I. INTRODUCTION

The 't Hooft nonsingular magnetic monopole[1,2] in a spontaneously broken non-Abelian gauge theory has aroused a great deal of interest as an example of an extended solution in a moderately realistic four-dimensional field theory. Recent related work includes the electrically as well as magnetically charged "dyons" studied by Julia and Zee,[3] and the extended nonlinearly coupled scalar theories discussed by Lee.[4] In each case, a set of coupled nonlinear ordinary differential equations is obtained from the Euler-Lagrange equations by imposing the requirements that the solution be time-independent and spherically symmetric. These equations are thenceforth treated classically. The additional requirements that the solution have a locally integrable energy density, and that it also have a finite total energy, then impose boundary conditions at the origin and at infinity.

It has not proved possible to find general analytic solutions to these problems, as has been done for a variety of analogous equations in two-dimensional field theories,[5] although Prasad and Sommerfield[6] have succeeded in solving the 't Hooft-Julia-Zee system in terms of simple functions in an unphysical limit of the theory. Thus, it appears that further analysis of such problems will require at least some use of numerical techniques.

The fact that one has to deal with a boundary-value problem implies that one cannot trivially integrate the field equations by computer. The main purpose of this paper is to show that the numerical solution of such problems is nevertheless rather straightforward, if one converts the set of nonlinear coupled differential equations to a system of nonlinear integral equations that can be solved iteratively. We apply this technique here to the 't Hooft and Julia-Zee equations, but its range of application appears to be a good deal wider.

The organization of this paper is as follows: In Sec. II, we summarize the 't Hooft-Julia-Zee system. In the next section, we give a general prescription for converting such a system of nonlinear differential equations to a system of coupled integral equations. In order that such integral equations have a convergent iterative solution, it is necessary that the trial functions used in the first iteration be a fair approximation to the actual solution. In Sec. IV, we find such an approximate solution by a simple variational calculation, and then in Sec. V we solve the 't Hooft-Julia-Zee system iteratively. We conclude with a discussion of our results and some comments on related problems. The Appendix contains a discussion of the general convergence properties of our approach, illustrated by an especially simple example—the "kink" solution of the $(\phi^4)_2$ theory—which is known analytically.[7]

II. THE 't HOOFT-JULIA-ZEE MONOPOLE

In this section, we will summarize the features of the monopole system that are relevant for our calculations.

't Hooft[1] considered an SO(3) gauge theory with a Yang-Mills triplet W_μ^a ($a=1,2,3$ isospace indices, $\mu=0,\ldots,3$ spacetime indices) coupled to a triplet of Higgs scalars ϕ^a and with the electromagnetic charge operator identified with the third component of isospin. If one breaks the symmetry so that the scalar field gets a nonzero vacuum expectation value, two components of the vector field acquire a mass M_W. The third component of the vector field stays massless and is identified with the ordinary photon field.

't Hooft considered static solutions to this system of the form

$$\phi^a = \frac{r_a}{r}\tilde{\phi}(r) \tag{2.1}$$

and

$$W_\mu^a = \epsilon_{0\mu ab}\frac{r_b}{r}\tilde{W}(r)\,, \tag{2.2}$$

where $\epsilon_{\mu\nu\alpha\beta}$ is the completely antisymmetric Levi-Civita symbol with $\epsilon_{0123}=+1$. Note that $\tilde{W}(r)$ and $\tilde{\phi}(r)$ are functions of $|\vec{r}|$ only, and that we are working in the Lorentz frame in which the monopole is at rest. Julia and Zee added a nonvanishing time component (electric field) to the vector field,

$$W_0^a=\frac{r_a}{r}\tilde{V}(r). \tag{2.3}$$

One now can write the 't Hooft system in terms of one independent variable $x=M_W r$ and a dimensionless parameter $\beta=M_\Phi^2/M_W^2$. In the Julia-Zee case, there is another dimensionless parameter λ, defined below, which is related to the electric charge of the dyon. If we now rescale the fields in the following way

$$\phi(x)=\frac{\tilde{\phi}(r)}{M_W},\quad W(x)=\frac{e\tilde{W}(r)}{M_W},\quad V(x)=\frac{\tilde{V}(r)}{M_W} \tag{2.4}$$

and substitute them in the Euler-Lagrange equations, we obtain the field equations

$$x^2W''+2xW'-2W-x(1+xW)(\phi^2-V^2)-3xW^2-x^2W^3=0\,, \tag{2.5a}$$

$$x^2\phi''+2x\phi'-2\phi+\frac{\beta x^2}{2}\phi-4xW\phi-2x^2W^2\phi-\tfrac{1}{2}\beta x^2\phi^3=0\,, \tag{2.5b}$$

$$x^2V''+2xV'-2V-4xVW-2x^2W^2V=0\,. \tag{2.5c}$$

The mass of the monopole or dyon is found to be

$$E=\frac{M_W}{\alpha}C(\beta) \tag{2.6}$$

with $\alpha=e^2/4\pi$ the fine-structure constant and the mass integral $C(\beta)$ given by

$$C(\beta)=\int_0^\infty dx\left[(xW'+W)^2+2W^2\left(1+\frac{xW}{2}\right)^2+\tfrac{1}{2}x^2\phi'^2+\tfrac{1}{2}x^2V'^2+(1+xW)^2(\phi^2+V^2)+\frac{\beta x^2}{8}(\phi^2-1)^2\right]. \tag{2.7}$$

The observable fields can be constructed using 't Hooft's gauge-invariant electromagnetic field tensor $F_{\mu\nu}$.[1] Since $F_{\mu\nu}$ satisfies the free-space Maxwell equations everywhere except at the origin, the electric and magnetic fields are known everywhere once we determine their behavior for $x\to\infty$. One finds that the magnetic field is radial and corresponds to a magnetic charge of $4\pi/e$ at the origin. There is also a radial electric field if $V(x)\neq 0$,[3] corresponding to a total electric charge

$$Q=-\frac{e}{\alpha}\theta \tag{2.8}$$

with the charge integral

$$\theta=2\int_0^\infty dx\,V(1+xW)^2. \tag{2.9}$$

The system (2.5) has the trivial singular solution

$$\phi=\delta\,, \tag{2.10a}$$

$$W=-1/x\,, \tag{2.10b}$$

$$V=\eta\,, \tag{2.10c}$$

with δ and η constant. Note that if $\beta\neq 0$, then $\delta=1$ in order that (2.5b) is satisfied asymptotically. We are now interested in another, *nonsingular* solution which behaves for large x like the singular one but remains finite at the origin in order to have a finite mass. Because the equations are nonlinear, there may be more than one solution with the same asymptotic behavior.

For the case $\beta=0$, the following analytic solution was found by Prasad and Sommerfield[6]:

$$\phi=\cosh\gamma(\lambda\coth\lambda x-1/x)\,, \tag{2.11a}$$

$$W=\lambda/\sinh\lambda x-1/x\,, \tag{2.11b}$$

$$V=\sinh\gamma(\lambda\coth\lambda x-1/x)\,, \tag{2.11c}$$

where γ is a free parameter and λ is a scale parameter. (Note that this remains a solution if we replace the hyperbolic functions by circular functions, but this solution is singular and thus of no physical interest.) Unfortunately, the theory becomes rather unphysical in the limit $\beta=0$ because this corresponds to either an infinite vector mass (resulting in an infinite mass for the monopole or dyon) or a massless scalar field (never observed). The solution (2.11) provides us, however, with a nice test case for our technique.

In order that the energy density be integrable at the origin, it is necessary that ϕ, W, and V are finite as $x\to 0$. The requirement of finite total field energy implies that the behavior of the fields as $x\to\infty$ is as in Eqs. (2.10). Since the solution is to be regular at the origin, we substitute power series expansions into Eqs. (2.5) and find that the even terms in each function must vanish. Thus

$$\begin{aligned}\phi(x\to 0)&=ax+O(x^3)\,,\\ W(x\to 0)&=bx+O(x^3)\,,\\ V(x\to 0)&=cx+O(x^3)\,,\end{aligned} \tag{2.12}$$

and the fields vanish at the origin.

III. GENERAL DESCRIPTION OF ITERATIVE PROCEDURE

As we explained in the Introduction, we are interested in the numerical solution of a set of coupled ordinary second-order nonlinear differential equations for the various wave functions; for example, the set (2.5). The boundary values are given, say, for $x=a$ and $x=b$; in our example $a=0$ and $b=\infty$, and the boundary values there are (2.10) and (2.12). To convert the set of n differential equations to a system of integral equations, we first separate the linear and nonlinear terms in the equations, and write

$$L_i y_i = -f_i(x, y, y') \tag{3.1}$$

(with no summation over repeated indices here or in the other equations of this section). The y_i are the n different fields ($i=1,\ldots,n$), f_i consists of the nonlinear terms of the ith equation, and the L_i are the linear differential operators.

We now solve the linear homogeneous equations

$$L_i y_i = 0 \tag{3.2}$$

for homogeneous boundary conditions. From those we construct the Green's functions G_i for the linear problem, obeying

$$L_i G_i(x, x') = -\delta(x-x') . \tag{3.3}$$

If it were true that

$$y_i(a) = y_i(b) = 0 \tag{3.4}$$

for $i=1,\ldots,n$, then the system (3.1) would be equivalent to

$$y_i(x) = \int_a^b G_i(x,x') f_i(x', y(x'), y'(x'))\, dx' , \quad i=1,\ldots,n. \tag{3.5}$$

If, however, we have nonhomogeneous boundary conditions, as we do in the example considered here, we proceed as follows. We write

$$y_i = a_i + z_i , \tag{3.6}$$

where a_i is a function which satisfies the nonhomogeneous boundary conditions, and which is otherwise arbitrary and can be chosen to be as simple as one wants. The z_i's are unknown but they now satisfy the homogeneous boundary conditions. We now rewrite (3.2) in the following form:

$$L_i z_i = -f_i - P_i , \tag{3.7}$$

where

$$P_i = L_i a_i \tag{3.8}$$

is a function we can calculate once a_i is given.

From (3.5) and (3.8) we now find

$$y_i(x) = a_i(x) + \int_a^b G_i(x,x')[f_i(x', y(x'), y'(x')) + P_i(x')]dx' , \tag{3.9}$$

which is our set of coupled nonlinear integral equations.

There are different ways one can solve these integral equations[8] depending on the particular problem, knowledge about the solutions, and the available computer facilities. In the calculations that will be described in the next sections, we simply iterated the system (3.9) using a suitably chosen a_i as the first iteration.

It is hard to give a quantitative description of the convergence of the method because it depends strongly on the number and properties of the equations. For a detailed discussion of the convergence criteria, the interested reader is referred to the Appendix. We mention here a few general features. The convergence of the simple iteration scheme is first order, which means that after one is close enough to the actual solution the relation between successive iterations becomes

$$y_i - y_i^{(r)} = a(y_i - y_i^{(r-1)}) , \tag{3.10}$$

where $y_i^{(r)}$ is the rth iteration of y_i and a is a characteristic value for the system and has to satisfy $|a| < 1$ for convergence. Taking the functions a_i in Eq. (3.9) to be close to the final solution, the contribution of the integral will always be small—which increases the accuracy and decreases the total number of iterations. However, the value of a in Eq. (3.10), and thus the speed of convergence, is unaffected by the choice of a_i. Finally, as one may expect, the convergence worsens with an increasing number of coupled equations.

IV. VARIATIONAL CALCULATION

It is desirable that the initial approximation $y_i^{(0)} = a_i$ to the solution of the integral equations (3.9) be reasonably near the exact solution, as explained in the last section. A simple way to find such an initial approximation is to choose a set of simple trial functions which satisfy the boundary conditions and depend upon a small number of adjustable parameters, and substitute them in the mass integral (2.7). The values of the adjustable parameters that minimize the mass integral then determine the initial approximation for the integral equations.

We take as our trial functions for the 't Hooft-Julia-Zee problem the following set:

$$\phi = \frac{\delta x}{(x^2+a)^{1/2}}, \tag{4.1a}$$

$$W = \frac{-x}{x^2+b}, \tag{4.1b}$$

$$V = \frac{\eta x}{(x^2+c)^{1/2}}. \tag{4.1c}$$

The mass integral $C(\beta)$ can now be calculated analytically:

$$C(\beta, \delta, \eta) = \frac{\pi}{64}\left[\frac{21}{\sqrt{b}} + 2\beta a\sqrt{a} + \delta^2 f(a,b) + \eta^2 f(c,b)\right], \tag{4.2}$$

where

$$f(a,b) = \left\{2\sqrt{a}\left[(a-b)^2 - 16b^2\right] + \frac{16b^2(b+a)}{\sqrt{b}}\right\}\frac{1}{(a-b)^2}.$$

It is also simple to express the charge integral (2.9) in terms of the parameters a, b, and c. We find

$$\theta = \eta b^2\left\{\frac{\sqrt{c}}{b(c-b)} + \frac{1}{2(c-b)^{3/2}}\ln\left[\frac{\sqrt{c}-(c-b)^{1/2}}{\sqrt{c}+(c-b)^{1/2}}\right]\right\}, \quad c>b$$

(4.3)

$$\theta = \eta b^2\left\{\frac{\sqrt{c}}{b(c-b)} + \frac{1}{(b-c)^{3/2}}\left[\frac{\pi}{2} - \arctan\left(\frac{\sqrt{c}}{(b-c)^{1/2}}\right)\right]\right\}, \quad c<b$$

Minimizing (4.2) with respect to a, b, and c for fixed β and η, we obtain the results presented in Table I (for $\eta=0$) and in Fig. 4 (for a range of values of η). Of course, the values of the energy integral obtained in this way give only upper limits to the true monopole or dyon mass. One can improve on this upper limit by increasing the number of parameters, and indeed 't Hooft[1] obtained considerably lower values, given also in Table I, using a total of six parameters for ϕ and W.[9]

V. APPLICATION TO 't HOOFT-JULIA-ZEE MONOPOLE

We now apply our procedure to the 't Hooft-Julia-Zee monopole, and solve the system with the boundary conditions given in (2.10) and (2.12). For computation, it is convenient to define the functions

$$M = 1 - \frac{\phi}{\delta}, \qquad N = 1 - \frac{V}{\eta}, \tag{5.1}$$

and solve in terms of these functions which go to zero rather rapidly for large x.[10]

Let us first consider the homogeneous equations that correspond to (3.2). For W we have the equation

$$x^2W'' + 2xW' - (2+\lambda^2x^2)W = 0 \tag{5.2}$$

with $\lambda^2 = \delta^2 - \eta^2$. Note that we must have $\eta^2 < \delta^2$ in order that W vanish rather than oscillate as $x \to \infty$.[3] (Oscillatory behavior leads to an infinite field-energy integral.) The solutions to (5.2) are the modified spherical Bessel functions of order $n=1$, the general equation for such functions being

$$x^2f''(x) + 2xf'(x) - [n(n+1) + x^2]f(x) = 0. \tag{5.3}$$

The two independent Bessel functions of order $n=1$ are defined as follows:

$$i_1(x) = \frac{\cosh x}{x} - \frac{\sinh x}{x^2}, \qquad k_1(x) = \frac{\pi}{2x}e^{-x}\left(1+\frac{1}{x}\right). \tag{5.4}$$

Thus i_1 vanishes at $x=0$ and diverges for $x\to\infty$, and k_1 has the opposite behavior. We can write the Green's function that satisfies homogeneous boundary conditions for $x=0$ and $x=\infty$, and has the correct discontinuity for $x=x'$, as

$$G_W(x,x') = \begin{cases} \frac{2\lambda}{\pi} i_1(\lambda x)k_1(\lambda x'), & x'>x \\ \frac{2\lambda}{\pi} i_1(\lambda x')k_1(\lambda x), & x'<x. \end{cases} \tag{5.5}$$

For M, we find a similar linear equation

$$x^2M'' + 2xM' - (2+\epsilon^2x^2)M = 0 \tag{5.6}$$

with

$$\epsilon^2 = \frac{\beta}{2}(3\delta^2 - 1), \tag{5.7}$$

and the Green's function becomes

$$G_M(x,x') = \begin{cases} \frac{2\epsilon}{\pi} i_1(\epsilon x)k_1(\epsilon x'), & x'>x \\ \frac{2\epsilon}{\pi} i_1(\epsilon x')k_1(\epsilon x), & x'<x. \end{cases} \tag{5.8}$$

Finally, for N we have

$$x^2N'' + 2xN' - 2N = 0 \tag{5.9}$$

and the Green's function is simply

$$G_N(x,x') = \begin{cases} \frac{x'}{3x^2}, & x'<x \\ \frac{x}{3x'^2}, & x'>x. \end{cases} \tag{5.10}$$

Note that in the case $\beta=0$, the Green's function for the M field reduces to that for the N field; and if $\lambda=0$, the Green's function for the W field also reduces to that for the N field.

Let us now write our final equations:

$$W(x)=A_W(x)+\int_0^\infty G_W(x,x')[F_W(x')+P_W(x')]\,dx' ,$$

$$M(x)=A_M(x)+\int_0^\infty G_M(x,x')[F_M(x')+P_M(x')]\,dx' , \tag{5.11}$$

$$N(x)=A_N(x)+\int_0^\infty G_N(x,x')[F_N(x')+P_N(x')]\,dx' ,$$

where the nonlinear sources are given by

$$F_W(x)=-x(1+xW)[\delta^2(M^2-2M)-\eta^2(N^2-2N)] - 3xW^2-x^2W^3-x^2(\delta^2-\eta^2) ,$$

$$F_M(x)=2+2xW(2+xW)(1-M) +\frac{\beta x^2}{2}\,[\delta^2M^2(3-M)+\delta^2-1] , \tag{5.12}$$

$$F_N(x)=2+2xW(2+xW)(1-N) .$$

As we explained in Sec. III, the functions $A(x)$ must satisfy the boundary conditions (2.10) and (2.12), but are otherwise arbitrary. We choose them to be of the form used in our variational calculation:

$$A_W(x)=-\frac{x}{x^2+b} ,$$

$$A_M(x)=1-\frac{x}{(x^2+a)^{1/2}} , \tag{5.13}$$

$$A_N(x)=1-\frac{x}{(x^2+c)^{1/2}} .$$

The quantities P then take the following form:

$$P_W(x)=\frac{2xb(3x^2-b)}{(x^2+b)^3}-(\lambda^2x^2+2)A_W(x) ,$$

$$P_M(x)=\frac{ax(x^2-2a)}{(x^2+a)^{5/2}}-(\epsilon^2x^2+2)A_M(x) , \tag{5.14}$$

$$P_N(x)=\frac{cx(x^2-2c)}{(x^2+c)^{5/2}}-2A_N(x) .$$

VI. RESULTS AND DISCUSSION

Before we start a detailed discussion of the results, we plot in Fig. 1 a sample calculation for $\beta=0$ and $\eta=0$. To illustrate the convergence of our method, we purposely started in this example with a very poorly chosen set of initial functions, namely $W=0$ and $M=0$. The $n=30$ line coincides with the exact analytic solution (2.11). The figure caption gives the energy associated with each curve; as one would expect, the energy decreases as the number of iterations increases. The solution is convergent within roundoff errors after 30 iterations, and the energy agrees with the exact value 1 to within 0.1%.

We found iterative solutions of the system (5.11) for various values of the parameters β, η, and δ, subject to the following constraints. For $\beta=0$, we chose $\delta=1$; correspondingly, in the exact solution (2.11), $\lambda=1/\cosh\gamma$. If $\beta\neq 0$, then necessarily $\delta=1$, as explained after Eq. (2.10). In either case, $\eta\leq\delta$, as noted after Eq. (5.2).

Let us first consider the case $\eta=0$, corresponding to the 't Hooft monopole. The results for the mass integral are given in Table I. They are about 5% lower than the ones obtained with our simple variational calculation of Sec. IV, but they agree very well with the numbers obtained in

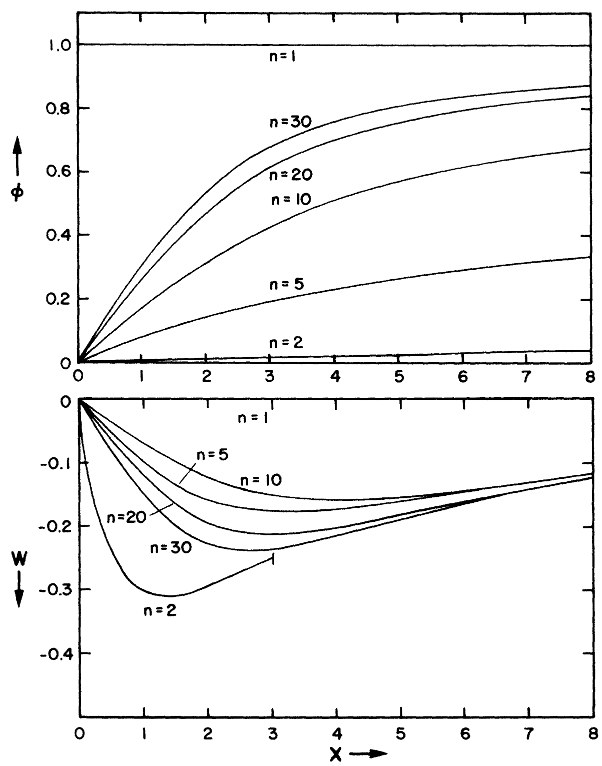

FIG. 1. Sample calculation for $\beta=0$, $\eta=0$. We start with initial functions $\phi^{(1)}=1$, $W^{(1)}=0$, purposely chosen to be rather poor approximations to the actual solution in order to illustrate convergence. For clarity, the curve for $W^{(2)}$ has not been drawn in beyond $x=3$, since it crosses the $W^{(30)}$ and $W^{(20)}$ curves; some points on this curve are $W^{(2)}(6)=-0.156$, $W^{(2)}(8)=-0.121$. The energy $C^{(n)}$ in the nth iteration (in units of M_W/α) is calculated from Eq. (2.7); the results are $C^{(1)}=\infty$, $C^{(2)}=10.09$, $C^{(5)}=3.014$, $C^{(10)}=1.196$, $C^{(20)}=1.007$, $C^{(30)}=1.000$.

TABLE I. Monopole mass in units of M_W/α as a function of $\beta = M_\phi^2/M_W^2$.

β	Variational 2 parameters [a]	Variational 6 parameters [b]	Integral equation	Analytic solution
0	1.06	...	1.000	1
0.1	1.15	1.1	1.106	...
1	1.30	...	1.238	...
10	1.49	1.44	1.433	...

[a] Method of Sec. IV.
[b] 't Hooft (see Ref. 1).

't Hooft's more elaborate variational calculation for $\beta = 0.1$ and $\beta = 10$. The calculated wave functions ϕ and W for $\eta = 0$ and various values of β are plotted in Fig. 2.

For $\eta \neq 0$, the monopole acquires an electric as well as a magnetic charge, and thus becomes a "dyon." The electric charge varies continuously with η: For various values of β, we chose values of $\eta \leq \delta = 1$, found solutions to Eqs. (2.5) using our integral equation method, and determined the corresponding electric charge from Eqs. (2.8) and (2.9). In Fig. 3, we have plotted the resulting wave functions ϕ, V, and W for $\beta = 0$ and 1 with $\theta = 0.586$.

Values of the mass and charge integrals for various values of β and η are given in Table II. For $\beta = 0$, we can compare our numerical results with those corresponding to the analytic solution (2.11); the agreement is excellent. In Fig. 4, we have plotted the mass integral C versus the charge integral θ for several different values of β. In the same figure, we have also plotted for comparison the two points obtained by 't Hooft and also three points that were calculated numerically for $\beta = 0.5$ by Julia and Zee[3] using a different technique; the agreement is excellent with the 't Hooft points (as already mentioned) and with all but the last of the Julia-Zee points.[11] Finally, the results of our

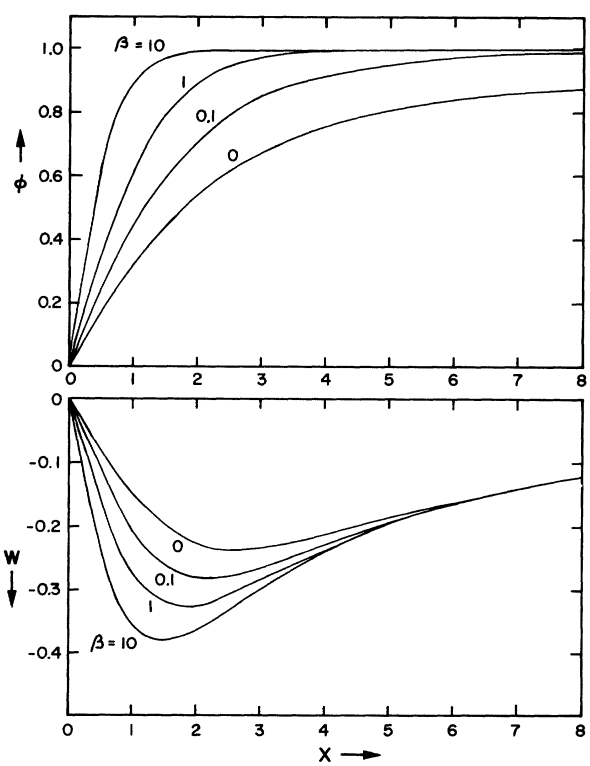

FIG. 2. Monopole solutions ϕ, W as a function of $x = rM_W$ for various values of $\beta = M_\phi^2/M_W^2$.

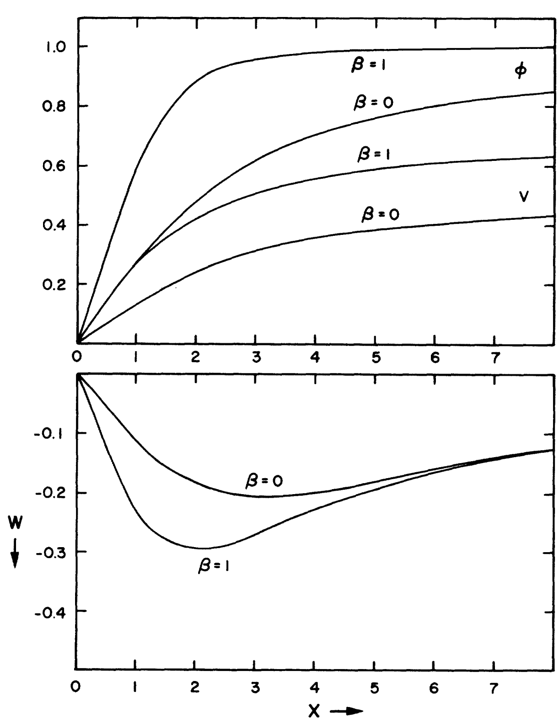

FIG. 3. Solutions $\phi(x)$, $V(x)$, and $W(x)$ for dyons with $\beta = 0$ and 1, for charge integral $\theta = 0.586$. (This value of θ corresponds to $\eta^2 = 0.5$ for $\beta = 1$, and $\eta^2 = 0.256$ for $\beta = 0$.)

TABLE II. Energy and charge integrals for monopole/dyon.

	β	$\eta^2=0.1$		$\eta^2=0.5$		$\eta^2=0.9$	
		C	θ	C	θ	C	θ
Analytic solution [a]	0	1.054	0.334	1.414	1.000	3.162	3.000
Integral equations	0	1.053	0.333	1.413	1.000	3.154	3.000
	0.1	1.150	0.266	1.382	0.718	1.765	1.176
	1	1.275	0.235	1.456	0.586	1.713	0.892
	10	1.466	0.204	1.617	0.496	1.829	0.749

[a] In Eq. (2.11) we have set $\lambda = 1/\cosh\gamma$. (In the integral equation solution this corresponds to $\delta = 1$ for $\beta = 0$ as well as $\beta > 0$.)

simple variational calculations, which are also plotted in Fig. 4, are seen to be rather poor for increasing charge. This happens because the trial functions (4.1) increasingly differ in shape from the actual solutions as η (and θ) increase; that this is so is apparent from the fact that proper asymptotic behavior for $W(x)$ requires $\eta<1$, as noted above, while the variational problem can be solved for arbitrary η.

Indeed, since the charge and mass of the dyon increase smoothly with η for a given value of β, but η is constrained by $\eta<\delta=1$ for $\beta\neq 0$, we find that the curves of $C(\beta,\theta)$ end at some maximum value $C(\beta,\theta_{max}(\beta))$. This, however, gives a rather misleading picture of the physical situation, because it does not explicitly reflect a "mass renormalization" of the vector field due to the presence of the nonvanishing V field [recall that $V(x)\to\eta$ away from the dyon]. The quantity M_W is only that part of the mass of the charged vector field which arises from the nonvanishing of the scalar field ϕ [$\phi(x)\to\delta$ away from the dyon]. Taking both ϕ and V into account, the mass of the Yang-Mills field is $M_W^*=\lambda M_W$, where $\lambda^2=\delta^2-\eta^2$. This is illustrated by the asymptotic behavior of the vector field far away from the dyon; from Eq. (2.5a) we see[10] that $W(x)$ approaches its asymptotic behavior $-1/x$ exponentially in λx, not x:

$$W(x)\sim\frac{e^{-\lambda x}-1}{x}. \tag{6.1}$$

That the true vector mass is M_W^* far from the dyon

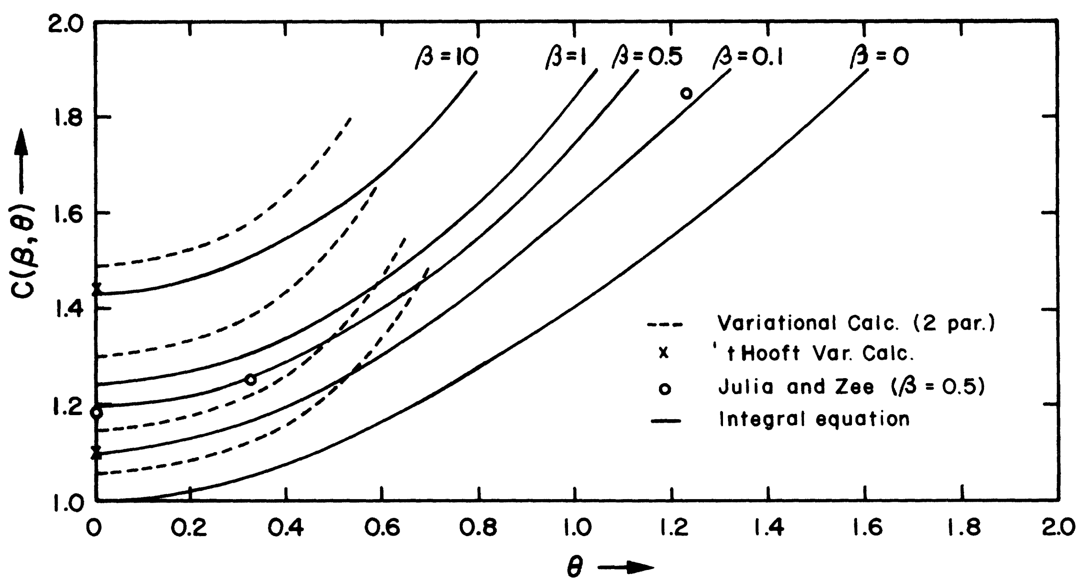

FIG. 4. Energy $C(\beta,\theta)$ of the monopole or dyon, measured in units of M_W/α, as a function of $\beta = M_\phi{}^2/M_W{}^2$ and of the charge integral $\theta=(-e/\alpha)Q$. The dashed lines are the energies obtained from the simple variational calculation of Sec. IV, with $\beta=0$ for the lowest-lying dashed line and $\beta=0.1$, 1, and 10, respectively, for the higher dashed lines. The solid lines are the results of the integral equation calculation. The solid line for $\beta=0$ coincides with the results of Prasad and Sommerfield. For comparison, we also display the energies calculated by 't Hooft for $\beta=0.1$ and 10 and $\theta=0$, and by Julia and Zee for $\beta=0.5$.

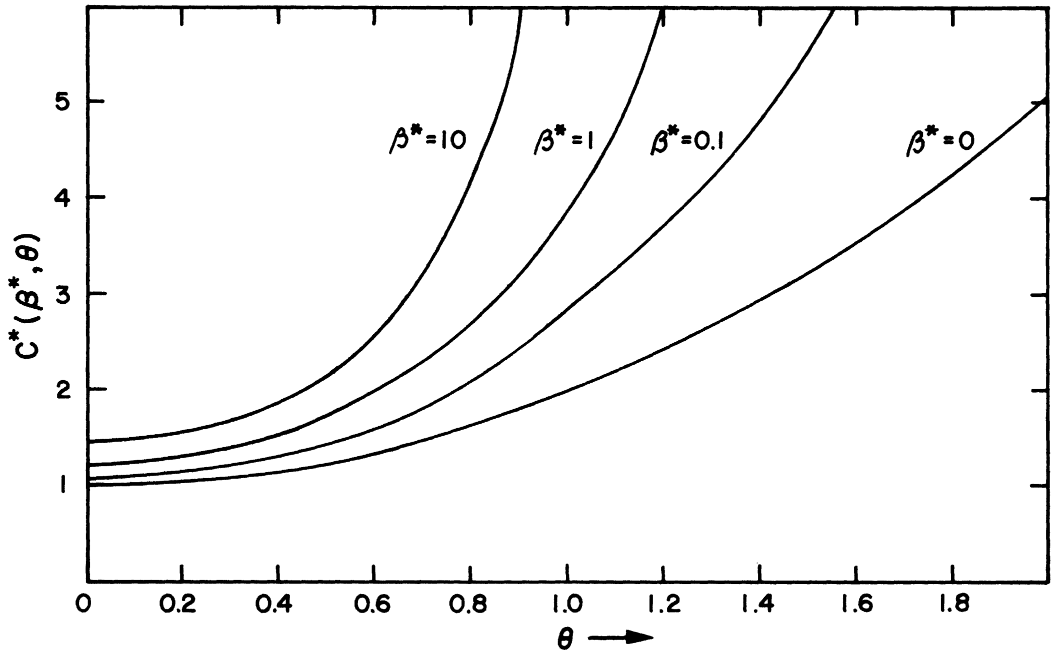

FIG. 5. Energy $C^*(\beta^*,0)$ of the monopole or dyon, measured in units of M_W^*/α, as a function of the charge integral θ for several values of $\beta^*=M_\phi^2/M_W^{*2}$. ($M_W^*=\lambda M_W$ is the mass of the charged vector field far from the dyon.)

can also be seen directly from the Lagrangian.

Physically, it seems most reasonable to express the dyon mass E in terms of the actual vector mass M_W^*, and as a function of $\beta^*=M_\phi^2/M_W^{*2}=\beta/\lambda^2$; i.e.,

$$E=\frac{M_W}{\alpha}C(\beta,\theta)=\frac{M_W^*}{\alpha}C^*(\beta^*,\theta)\,. \qquad (6.2)$$

In Fig. 5, we plot $C^*(\beta^*,\theta)$ for several values of β^*, and now we see that the dyon mass is unbounded as its charge increases if M_W^* is held fixed.

It is interesting at this point to examine the stability of the dyon. One question which arises[3] is whether the dyon with electric charge $|Q|=qe$ is stable against decay into a dyon of smaller charge plus some number of free charged Yang-Mills bosons. Since the mass of a Yang-Mills boson far away from the dyon is M_W^*, this stability condition can be written

$$E_{\text{dyon}}(q)\le E_{\text{dyon}}(q-n)+nM_W^*,\quad 1\le n\le q\,. \qquad (6.3)$$

In order that this equation be satisfied, it is sufficient to require that

$$\frac{d}{dx}E_{\text{dyon}}(x)\le M_W^*,\quad 0\le x\le q\,. \qquad (6.4)$$

If we keep M_ϕ and M_W^* fixed while varying $q=\theta/\alpha$, we can rewrite (6.4) as

$$\frac{d}{d\theta}C^*(\beta^*,\theta)\le 1\,. \qquad (6.5)$$

The largest values of θ for which this condition is satisfied can be determined graphically from Fig. 5; for example, for $\beta=1$, the largest such θ is roughly 0.3. This corresponds to an electric charge $Q=-e\theta/\alpha\simeq-40e$. Equation (6.5) can be solved precisely for the analytic solution of Prasad and Sommerfield,[6] for which $C^*(\beta=0,\theta)=1+\theta^2$; the result is $\theta_{\text{max stable}}=\frac{1}{2}$, which corresponds to $Q=-68.5e$.

A treatment of the quantum-field-theoretic corrections to the extended solutions we have considered here is beyond the scope of this paper. We will, however, give a brief discussion of the physical structure of these solutions, and mention possible quantum effects.

At distances from the monopole which are large compared to the Compton wavelength of the charged vector field, almost all of the field energy arises simply from the $\frac{1}{2}(\vec{E}^2+\vec{B}^2)$ energy of the ordinary electromagnetic field.[12] The magnetic field maintains its inverse-square-law form all the way to $r=0$, but at distances less than ~$1/M_W$ the charged components of the Yang-Mills field effectively contribute a negative energy density which cancels what would otherwise be a divergence. The resulting magnetic field energy is ~g^2M_W, where $g=e/\alpha$ is the magnetic charge of the monopole. If the time component of the Yang-Mills field has been chosen to be nonzero as in Eq. (2.3), then the resulting "dyon" possesses an extended electric charge distribution which peaks at $\sim 1/M_W$ and vanishes for $r \to 0$ and $r \to \infty$. Thus the electric field energy density is nonsingular, and the total electric field energy is $\sim Q^2 M_W$. We can therefore understand physically the roughly quadratic dependence of the dyon energy on the charge integral $\theta = (-\alpha/e)Q$ displayed in Figs. 4 and 5, and also why the electric charge needs to be fairly large to increase the energy significantly.

Since the fields are quite large in the vicinity of the monopole or dyon, it may be that the quantum fluctuations and vacuum polarization effects will be large and will produce substantial corrections to the classical energy. This is especially so since the removal of the divergence in the magnetic field energy is a somewhat delicate matter. Thus the significance of the classical solutions in the presence of quantum effects is unclear and requires further study.

A particularly interesting question is the influence of quantum effects on the time component of the vector field, W_0^a, whose direction in isotopic space is assumed in the dyon solution to be parallel to that of the scalar field ϕ^a. Far away from the dyon, the magnitude of W_0^a is assumed to go to a constant value proportional to that of ϕ^a (the proportionality constant being $\eta < 1$). Of course, the asymptotic value of ϕ^a is dictated by the parameters of the potential $V(\phi)$ which appears in the Lagrangian, and this value is stable against small fluctuations. This is not true for W_0^a, however, since there can be no analogous potential $V(W_0^a)$. The only term in the Lagrangian which is relevant is the term $+\frac{1}{2}(e\epsilon^{abc} W_0^b W_i^c)^2$, whose sign, because of the Minkowski metric, is such as to give a negative mass to the charged spatial components of the vector field, as we already mentioned. Since W_i^a vanishes asymptotically in our classical extended solution, however, this term contributes nothing to the energy of the time component far from the dyon.

We thus conclude that, classically, the stability of the nonvanishing W_0^a field is neutral, like that of a marble which can rest anywhere on a flat surface. The situation is evidently rather like that considered by Coleman and Weinberg;[13] an effective potential, calculated perhaps in perturbation theory, may determine the actual vacuum expectation value of W_0^a. The most exciting possibility is that the same term mentioned above, $+\frac{1}{2}(e\epsilon^{abc}W_0^b W_i^c)^2$, will now tend to give W_0^a a nonvanishing vacuum expectation value in the presence of vacuum fluctuations in W_i^a. If the e^4 term in the effective potential is of the opposite sign, then that would give the time component a vacuum expectation value proportional to $1/e$.

ACKNOWLEDGMENTS

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APPENDIX

In this Appendix we will discuss in some detail the convergence properties of the simple iteration method for solving the system of integral equations

$$y_i(x) = a_i(x) + \int G_i(x,x')[F_i(x', y_j(x')) + P_i(x')]\,dx' . \tag{A1}$$

The iterative procedure is determined by the relation

$$y_i^{(r)}(x) = a_i(x) + \int G_i(x,x')[F_i(x', y_j^{(r-1)}(x')) + P_i(x')]\,dx' , \tag{A2}$$

where $y_i^{(r)}$ is the rth iteration of y_i. Subtracting (A2) from (A1) we obtain after we expand $F_i(x', y_j^{(r-1)}(x'))$ to first order about $y_j(x')$ the following expression:

$$y_i - y_i^{(r)} = \int G_i(x,x') \sum_j \frac{\partial F_i}{\partial y_j}[y_j(x') - y_j^{(r-1)}(x')]\,dx' . \tag{A3}$$

Here we have assumed that the F_i are once differentiable in the y_j and that the $y_j^{(r)}$ are sufficiently near the actual solution that the second-order terms are negligible. If we introduce a function

$$\Delta_i^{(r)} = y_i - y_i^{(r)} \tag{A4}$$

we can write (A3) in vector notation as

$$\Delta^{(r)} = K\Delta^{(r-1)} , \tag{A5}$$

where K stands for the linear integral operator

$$(Kh)_i = \sum_j \int G_i(x,x') \frac{\partial F_i}{\partial y_j} h_j\,dx' . \tag{A6}$$

Suppose that the operator K defines a complete set of eigenfunctions Φ_s with corresponding eigenvalues λ_s:

$$K\Phi_s = \lambda_s \Phi_s . \qquad \text{(A7)}$$

Expanding our initial iteration in the eigenfunctions

$$\Delta^{(0)} = \sum_{s=1}^{\infty} a_s \Phi_s \qquad \text{(A8)}$$

and inserting in Eq. (A5) we find

$$\Delta^{(r)} = K^r \sum_{s=1}^{\infty} a_s \Phi_s = \sum_{s=1}^{\infty} a_s \lambda_s{}^r \Phi_s . \qquad \text{(A9)}$$

From this relation we learn that the convergence of our simple iteration procedure is essentially governed by the largest eigenvalue λ^* of the operator K. The procedure is convergent if $\lambda^* < 1$ and divergent if $\lambda^* > 1$. For large enough r we also can write

$$\frac{\Delta^{(r)}}{\Delta^{(r-1)}} \simeq \lambda^* , \qquad \text{(A10)}$$

which means that the value of a in Eq. (3.10) is equal to λ^*. Another consequence of Eq. (A9) is that the functions a_i of (A1) do not directly affect the convergence properties of the method, but of course they do influence the total number of iterations needed to obtain an accurate result.

Let us briefly discuss a simple example. We consider a pure scalar theory in one space and one time dimension with quartic self-interaction, which leads to the field equation

$$2y'' + y(1 - y^2) = 0 , \qquad \text{(A11)}$$

where we have set the parameters equal 1. An analytic solution is the "kink",[11]

$$y = \tanh\left(\frac{x}{2}\right) , \qquad \text{(A12)}$$

which we will use as a test case for the method of Sec. III. For computational purposes, it is convenient to consider the function

$$g = 1 - y \qquad \text{(A13)}$$

for which Eq. (A11) becomes

$$g'' - g = \tfrac{1}{2} g^2 (g - 3). \qquad \text{(A14)}$$

The Green's function for the linear problem is

$$G(x, x') = \begin{cases} \tfrac{1}{2} e^{x - x'}, & x' > x \\ \tfrac{1}{2} e^{x' - x}, & x' < x \end{cases} \qquad \text{(A15)}$$

and the nonlinear source is

$$F(x) = \tfrac{1}{2} g(x)^2 [3 - g(x)], \qquad \text{(A16)}$$

leading to the iteration scheme

$$g - g^{(r)} = \int_{-\infty}^{\infty} G(x, x') \tfrac{3}{2} g (2 - g)(g - g^{(r-1)}) dx' . \qquad \text{(A17)}$$

Use of the analytic solution (A12) in the kernel gives us an explicit form for the integral operator K, and the eigenvalue problem (A7) for K becomes

$$\int_{-\infty}^{\infty} G(x, x') \frac{3}{2 \cosh^2 (x'/2)} \Phi_s (x') dx' = \lambda_s \Phi_s (x). \qquad \text{(A18)}$$

To find solutions, it is easiest to write the above equation as the equivalent differential equation

$$\Phi_s'' - \Phi_s = \frac{-3}{2\lambda_s \cosh^2 (x/2)} \Phi_s , \qquad \text{(A19)}$$

where we impose the boundary conditions that the eigenfunctions vanish at $x = -\infty$ and $x = +\infty$. There are two sets of solutions. The first set Φ_s^e are even polynomials in $\cosh^2(x/2)$ of order s $(s = 1, 2, \ldots)$:

$$\Phi_s^e = \sum_{n=1}^{s} \alpha_n^{(s)} \cosh^{-2n}\left(\frac{x}{2}\right), \qquad \text{(A20)}$$

where the coefficients are determined by the recursion relation

$$\alpha_{n+1}^{(s)} = \frac{s(2s+1) - n(2n+1)}{2n(n+2)} \alpha_n^{(s)} . \qquad \text{(A21)}$$

The corresponding eigenvalues are found to be

$$\lambda_s^e = \frac{3}{s(2s+1)} . \qquad \text{(A22)}$$

For this set, the largest eigenvalue is $\lambda_1^e = 1$. The iteration scheme would not converge, but we have to remember that we look for an antisymmetric solution of the original field equation (A11) and our first guess will therefore also be antisymmetric. This implies that the difference $g - g^{(0)}$ has no even components.

Equation (A19) has also a set of odd eigenfunctions Φ_s^o:

$$\Phi_s^o = \tanh\left(\frac{x}{2}\right) \sum_{n=1}^{s} \beta_n^{(s)} \cosh^{-2n}\left(\frac{x}{2}\right), \qquad \text{(A23)}$$

with

$$\beta_{n+1}^{(s)} = \frac{(s+1)(2s+1) - (n+1)(2n+1)}{2n(n+2)} \beta_n^{(s)} . \qquad \text{(A24)}$$

The eigenvalues become

$$\lambda_s^o = \frac{3}{(s+1)(2s+1)} \qquad \text{(A25)}$$

and the largest eigenvalue, which determines the speed of convergence of Eq. (A17), is $\lambda_1^o = \frac{1}{2}$. This is indeed the value we found from explicit numerical and analytical calculations. For example, if we take

$$y^{(0)} = \tanh^3\left(\frac{x}{2}\right),$$

then a little calculation gives

$$y^{(r)}=\left(1-\frac{1}{2^r}\right)\tanh\left(\frac{x}{2}\right)+\frac{1}{2^r}\tanh^3\left(\frac{x}{2}\right)$$

so that indeed

$$(y-y^{(r)})=\tfrac{1}{2}(y-y^{(r-1)}).$$

A similar analysis can be made for more complicated systems like the 't Hooft-Julia-Zee monopole, but the analysis is neither simpler nor more illuminating than the numerical computation itself. From our numerical results, it is possible to estimate the largest eigenvalue λ^*. For the monopole with $\beta=0$ and $\eta=0$, for example, we find $\lambda^*\approx 0.85$.

Let us make a final remark about existence and uniqueness of the solutions. To prove uniqueness and existence, one proceeds along a path similar to the one followed above. If the kernel of Eq. (A6) in the specified domain of the variable and for a certain range of values for the fields y_i is bounded such that its eigenvalues are smaller than 1, it follows that there exists a unique solution within that range and that domain.[8] We will not pursue such questions any further in this paper.

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[10]The asymptotic behaviors of M and N follow from Eqs. (5.6) and (5.9): $M\sim e^{-\epsilon x}/x$, $N\sim 1/x^2$. The asymptotic behavior of W is given in Eq. (6.1). We have verified that the wave functions obtained from our integral equations have these asymptotic behaviors.

[11]The three Julia-Zee points plotted in Fig. 3 are their values for what we call $C(\beta,\theta)$ as given in their paper, Ref. 3; these are to be compared with the solid curve $\beta=0.5$. Prasad and Sommerfield, Ref. 6, quoted the values of $C(\beta,\theta)$ from Julia and Zee's report, which are different. They make a comparison of those values with their values of what we call $C^*(\beta=0,\theta)$, which is not entirely appropriate.

[12]The contribution of the electromagnetic field energy is actually one half the total mass of the monopole or dyon; half the energy resides in the scalar field. This "equipartition theorem" can be verified exactly for the analytic solution (2.11), but it is not true in general.

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