Regular solution of 't Hooft's magnetic monopole model in curved space*

P. van Nieuwenhuizen and David Wilkinson
Institute for Theoretical Physics at State University of New York, Stony Brook, New York 11794

M. J. Perry
DAMTP, Cambridge, England
(Received 5 November 1975)

A regular solution of 't Hooft's magnetic monopole model in curved space is presented. As in flat space, an analytical solution seems impossible, but by using the Einstein equations to eliminate the gravitational degrees of freedom, a positive-definite energy functional is constructed, which at its minimum gives both the required solution and the energy of the system. To first order in the gravitational constant G, the energy can be expressed in terms of the flat-space solutions only, whereas for general G a positive lower bound is derived.

I. INTRODUCTION

The interest in nonperturbative solutions of classical nonlinear field theories (solitons) is due to the expectation that they might provide a better starting point for quantum corrections than the usual plane-wave solutions. In this article we consider a simple model of general relativity with dynamical sources: 't Hooft's magnetic monopole model in curved space.[1] This model has such a high degree of symmetry, even in curved space, that one may expect that the formalism becomes simple and will yield explicit results. The aim of this article is to find a nonperturbative, regular, and localized static solution in curved space which is the extension of 't Hooft's flat-space solution.

In flat space the field equations of the model are already so nonlinear that only in a limiting case (the case of vanishing self-coupling of the Higgs scalars) is an analytical solution known.[2] The strategy adopted by 't Hooft to prove the existence of a solution in flat space was to use the negative-definite character of the Lagrangian density for this static problem. The function which maximizes the Lagrangian is a solution of the Euler-Lagrange equations and at the same time the energy is obtained because, for this static system, the Lagrangian is the negative of the energy. In curved space the Lagrangian of our static system is not negative-definite. In this article we construct a positive-definite energy functional which yields at its minimum both the required solution and its energy, precisely as in flat space.

The model that we consider has been discussed previously by Bais and Russell,[3] by Cho and Freund,[4] and by Cordero and Teitelboim.[5] Bais and Russell observed that the asymptotic solution of the field equations actually satisfies the field equations everywhere, as in the case of flat space. The result is the Reissner-Nordström metric which is singular at the origin. We are here interested in a solution which is everywhere regular. Cho and Freund noted that one can transform by a singular gauge transformation the Wu-Yang gauge field[6] asymptotically to the usual Dirac magnetic monopole vector potential.[7] They concluded that any solution in curved space must yield far away a Reissner-Nordström geometry. Our solution satisfies this criterion. Cordero and Teitelboim have used this model to apply Dirac's techniques for constrained Hamiltonian systems to a simpler and more tractable subclass of motions. Whereas in their approach the emphasis is on a canonical formulation rather than on particular solutions, we will restrict ourselves to the ground-state solution only, in the hope that this narrowing of subject will be profitable for the explicit calculations we will have to perform.

Our conventions are as follows. The Yang-Mills coupling constant will be denoted by e. The Einstein equations read $G_{\mu\nu} = -8\pi G T_{\mu\nu}$, where $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$, G = Newton's constant, and $T_{\mu\nu} = -2(-g)^{-1/2}(\delta I/\delta g^{\mu\nu})$ is the energy tensor with T_{00} non-negative. We set $\hbar = c = 1$, and our flat-space metric is $(-1, +1, +1, +1)$.

II. THE MODEL

We consider 't Hooft's magnetic monopole model[1] in curved spacetime. It describes interacting Yang-Mills bosons W_μ^a, an SO_3 triplet of Higgs scalars Q^a, and gravitons $g_{\mu\nu}$. Since we seek a static solution, all properties of the system may be described by a Lagrangian which is the sum of the usual Einstein Lagrangian L^E and the covariant matter Lagrangian L^M,

$$L^E = -(16\pi G)^{-1}\int d^3x(-g)^{1/2}R\,, \tag{1}$$

$$L^M = -(4\pi)^{-1}\int d^3x(-g)^{1/2}[\tfrac{1}{4}G^a_{\mu\nu}G^{\mu\nu a} + \tfrac{1}{2}(D_\mu Q^a)(D^\mu Q^a) + V(Q)]\,. \tag{2}$$

Lorentz indices are raised by $g^{\mu\nu}$, $G^a_{\mu\nu} \equiv \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + e\epsilon^{abc} W^b_\mu W^c_\nu$ is the Yang-Mills tensor, and $D_\mu Q^a \equiv \partial_\mu Q^a + e\epsilon^{abc} W^b_\mu Q^c$ is the Yang-Mills covariant derivative. Both $G^a_{\mu\nu}$ and $D_\mu Q^a$ are also gravitationally covariant. The potential $V(Q)$ leads to spontaneous symmetry breaking and contains the dimensional parameter μ needed in order to obtain a classical solution with nonzero energy,

$$V(Q) = \tfrac{1}{2}\mu^2 Q^a Q^a + \tfrac{1}{8}k(Q^a Q^a)^2 + \tfrac{1}{2}k\mu^4 = \tfrac{1}{8}k(Q^a Q^a - F^2)^2 , \tag{3}$$

where $\mu^2 < 0 < k$ and $F^2 \equiv -2\mu^2/k$. We have added a constant term to $V(Q)$ so that we may use Einstein's equations without cosmological constant, with energy tensors which vanish asymptotically. We look for time-independent solutions of the Wu-Yang-'t Hooft[1,6] form

$$Q^a(r) = r^{-1} r^a p(r) , \quad W^a_0 = 0, \quad W^a_i = \epsilon_{iab} r^b r^{-2}[-e^{-1} + v(r)] , \tag{4}$$

where the Cartesian index i runs from 1 to 3. Since the energy tensor of our system is rotationally symmetric in flat space (see Appendix), we assume that in curved space the metric is also rotationally symmetric and given by

$$g_{\mu\nu} = (-e^\nu, e^\lambda, r^2, r^2 \sin^2\theta), \quad \mu = \nu = t, r, \theta, \phi . \tag{5}$$

Evaluating $G^a_{\mu\nu}$ and $D_\mu Q^a$ in polar coordinates and substituting these expressions into L^M in (2) yields (see Appendix)

$$L^M = -\int_0^\infty dr\, e^{(\nu+\lambda)/2} r^2 (e^{-\lambda} U_1 + U_2) , \tag{6}$$

where U_1 and U_2 do not depend explicitly on the metric,

$$U_1 = r^{-2} v'^2 + \tfrac{1}{2} p'^2 , \tag{7}$$

$$U_2 = \tfrac{1}{2} e^{-2} r^{-4} (e^2 v^2 - 1)^2 + r^{-2} e^2 v^2 p^2 + \tfrac{1}{8} k (p^2 - F^2)^2 . \tag{8}$$

The primes in these equations denote differentiation with respect to r. The structure of the matter Lagrangian L^M in Eq. (6) is clear; the derivatives are "covariantized" by means of $g^{rr} = e^{-\lambda}$, and a factor $\sqrt{-g}$ multiplies the integrand in order to make a scalar density.

The gravitational part of the Lagrangian is given in polar coordinates by[8]

$$L^E = -(4G)^{-1} \int_0^\infty dr [r^2 (\nu' e^{(\nu-\lambda)/2})' + 4r (e^{(\nu-\lambda)/2})' + 2(e^{(\nu-\lambda)/2} - e^{(\nu+\lambda)/2})] . \tag{9}$$

For purposes to be explained below we prefer the alternative gravitational Lagrangian

$$L^{E\prime} = -(4G)^{-1} \int_0^\infty dr\, r(\nu' + \lambda')(e^{(\nu+\lambda)/2} - e^{(\nu-\lambda)/2}) . \tag{10}$$

Since the integrands of L^E and $L^{E\prime}$ differ by a total derivative

$$L^E - L^{E\prime} = -(4G)^{-1} \int_0^\infty dr [(2r + r^2\nu') e^{(\nu-\lambda)/2} - 2r e^{(\nu+\lambda)/2}]' , \tag{11}$$

they give the same field equations. Introducing the convenient gravitational variables $x = \frac{1}{2}(\nu - \lambda)$ and $y = \frac{1}{2}(\nu + \lambda)$ one obtains the total Lagrangian

$$L^{E\prime} + L^M = -\int_0^\infty dr [(2G)^{-1} r y' (e^y - e^x) + r^2 (e^x U_1 + e^y U_2)] . \tag{12}$$

Variation of the Lagrangian with respect to x, y, v, and p yields the field equations

$$y' = 2G r U_1 , \tag{13}$$

$$[r(e^y - e^x)]' = 2G r^2 e^y (U_1 + U_2) , \tag{14}$$

$$(v' e^x)' = e^y [r^{-2} v (e^2 v^2 - 1) + e^2 v p^2] , \tag{15}$$

$$(r^2 p' e^x)' = e^y [2e^2 v^2 p + \tfrac{1}{2} k r^2 p (p^2 - F^2)] . \tag{16}$$

These equations are invariant under $v \to -v$; this discrete symmetry corresponds in the original Lagrangian to a gauge rotation about the radial direction over an angle π. A second discrete symmetry, under $p \to -p$, cannot be related to a gauge transformation because, for example, the gauge-invariant $Q^a G^a_{\mu\nu}$ would change sign.

The boundary conditions at infinity follow from the requirement that our solutions be localized. The metric is therefore asymptotically flat and the Lagrangian converges for large r if the flat-space boundary conditions hold:

$$p \to \pm F, \quad v \to 0, \quad y \to 0, \quad x \to 0 \text{ as } r \to \infty . \tag{17}$$

Actually, the last condition $x \to 0$ follows from the three other conditions, as the reader may verify from Eq. (14). The sign p at infinity distinguishes between magnetic monopoles and antimonopoles; from now on we choose the plus sign. We specify the boundary conditions at the origin by the requirement that the Lagrangian $L^{E\prime} + L^M$ be stationary with respect to arbitrary variations of v, p, x, and y at $r = 0$. The Lagrangian determines in this way its own "natural" boundary conditions at the origin, arising from the partial integration used in deriving the Euler-Lagrange equations

$$r(e^y - e^x) = 0 \text{ at } r = 0 , \tag{18}$$

$$v' e^x = 0, \quad r^2 p' e^x = 0 \text{ at } r = 0 . \tag{19}$$

As found previously by Bais and Russell[3] and Cho and Freund,[4] an exact solution of the four field equations (13–16) is given by the Reissner-Nordström solution for a magnetic point charge of strength $1/e$ and arbitrary mass M,

$$v=0,\quad p=F,\qquad y=0,\quad e^{x}=1-2GMr^{-1}+Ge^{-2}r^{-2}. \tag{20}$$

This solution satisfies our boundary conditions at infinity, (17), but at $r=0$ it is singular and violates the boundary condition (18). We are rather interested in a regular solution for which the total energy is calculable.

III. POSITIVE-DEFINITE ENERGY FUNCTIONAL

Unlike the case in flat space, the total curved-space Lagrangian $L^{E'}+L^{M}$ in (12) is not a negative-definite functional. If we can find for our curved-space system a negative-definite Lagrangian which attains its supremum, then there exists a solution because at the extremum the Euler-Lagrange equations are satisfied. In order to construct such a Lagrangian we will eliminate the gravitational degrees of freedom x and y by using the two Einstein field equations (13) and (14). Then it will turn out that $L^{E'}+L^{M}$, as a function of the matter variables v and p only, is negative-definite.

We now proceed to eliminate the gravitational degrees of freedom. Integration of the two Einstein field equations in (13) and (14) with the boundary conditions in (17) and (18) yields

$$y=-2G\int_r^\infty d\rho\,\rho U_1(\rho)\,, \tag{21}$$

$$r(e^{y}-e^{x})=2G\int_0^r d\rho\,\rho^2 e^{y(\rho)}[U_1(\rho)+U_2(\rho)]\,. \tag{22}$$

Inserting the first of these equations (21) into the Lagrangian $L^{E'}+L^{M}$ in (12) we obtain the curiously simple result

$$L^{E'}+L^{M}=-E(v,p)\equiv-\int_0^\infty dr[r^2(U_1+U_2)e^{y}]\,, \tag{23}$$

where y, U_1, and U_2 are given explicitly in terms of v and p by Eqs. (7), (8), and (21). Note that the dependence of $L^{E'}+L^{M}$ on the variable x has dropped out completely, without using Eq. (22) to express x in terms of v and p. Since $e^{y}=\sqrt{-g}$, the effect of adding $L^{E'}$ to L^{M} has just been to "de-covariantize" the derivative terms $(v')^2$ and $(p')^2$ in L^{M}.

The functional $E(v,p)$ is manifestly positive for any functions v, p and so has a greatest lower bound E. We assume that there exist functions v, p for which this minimum value is attained. In Sec. IV we briefly discuss this assumption and show that $E>0$ for any value of the gravitational constant G. We now show that E is the total energy of the system, as suggested by the notation. From the integrated Einstein equation (22) we have

$$\lim_{r\to\infty}(2G)^{-1}r(e^{y}-e^{x})=\int_0^\infty dr\,r^2e^{y}(U_1+U_2)=E\,, \tag{24}$$

since the integral is just the energy functional $E(v,p)$. From the asymptotic boundary conditions (17) and the field equations (13)–(16) we easily find that the asymptotic form of the metric is the Reissner-Nordström solution (20). Inserting these asymptotic values of x and y into Eq. (24), we see that E is indeed the total energy.

In the literature[8] there exist two expressions for the total energy of a system in terms of the energy tensor $T_{\mu\nu}$

$$E=-4\pi\int_0^\infty dr\,r^2T^0_{\ 0}\,, \tag{25}$$

$$E=-\int d^3x\sqrt{-g}\,(T^0_{\ 0}-T^1_{\ 1}-T^2_{\ 2}-T^3_{\ 3})\,, \tag{26}$$

which hold for rotationally symmetric or static systems, respectively. For systems such as ours, which are both rotationally symmetric and static, these formulas may be verified by using the Einstein equations $G_{\mu\nu}=-8\pi GT_{\mu\nu}$ in polar coordinates to express the integrands in terms of the gravitational variables λ and ν. In each case the result is a total derivative which may be related to the energy using the asymptotic property $\lambda=-\nu=2GEr^{-1}$, provided the solution is sufficiently regular at the origin. Our formula $E=-(L^{E'}+L^{M})$ follows only from the particular dependence of the matter Lagrangian on the gravitational variables λ and ν [see Eq. (6)]. Of course, formulas (25) and (26) also hold in flat space; in this case their equivalence may be proved directly, using energy-momentum conservation in flat-space polar coordinates

$$(r^3T^r_{\ r})'=r^2(T^r_{\ r}+T^\theta_{\ \theta}+T^\phi_{\ \phi})\,. \tag{27}$$

Contrary to the situation in point-particle mechanics, in our system the stress components $T^j_{\ i}$ do not vanish in flat space. However, the total stress $\int r^2T^i_{\ i}$ vanishes, as expected, for an isolated system. For our magnetic monopole system it is shown in the Appendix that $T^0_{\ 0}=-(e^{-\lambda}U_1+U_2)/4\pi$, so that in flat space both our formula $E=-(L^{E'}+L^{M})$ and Eq. (25) give the same result $E=\int_0^\infty dr\,r^2(U_1+U_2)$.

Finally we investigate whether the solution we have found, i.e., the one determined by the minimum of the energy functional $E(v,p)$, is regular. In the next section we will demonstrate that this solution does satisfy the field equations (13)–(16) and the boundary conditions (17)–(19). The most general solution of the set of field equations is a six-parameter function. The three boundary conditions at the origin (18)–(19) leave a regular solution which still depends on three parameters y_0, v_2, and p_1,

$$e^y = y_0(1+y_2r^2+\cdots),$$

$$e^x = y_0(1+x_2r^2+\cdots),$$

$$v = e^{-1}+v_2r^2+\cdots,$$

$$p = p_1r+\cdots,$$

where the other parameters (y_2, x_2, etc.) can be expressed in terms of v_2 and p_1. [That the functions v, rp, x, and y contain only even powers of r follows from (13)–(16).] The three boundary conditions at infinity (17) will then determine our regular solution. Note that the scale of e^y and e^x is undetermined by the boundary conditions at the origin and the field equations.

IV. PROPERTIES OF THE ENERGY FUNCTIONAL

In the preceding section we showed that a solution of the field equations (13)–(16) is given by the functions v and p, which realize the infimum of the energy functional $E(v,p)$

$$E(v,p)=\int_0^\infty dr\left[r^2(U_1+U_2)\exp\left(-2G\int_r^\infty \rho U_1 d\rho\right)\right]. \tag{28}$$

The functions U_1 and U_2 are positive-definite and were given in (7) and (8); U_1 is quadratic in v' and p' while U_2 contains only v and p. In this section we discuss the following aspects of this functional:

(1) We construct a nonzero lower bound for $E(v,p)$, so that the energy E of the ground state is positive for any value of G. We also show that the minimizing functions v and p are bounded by $|ev|\leq|1$ and $|p|\leq F$ everywhere.

(2) We derive an exact expression for the energy E to first order in G in terms of the flat-space solutions v_0 and p_0 only.

(3) We verify explicitly that v and p minimize $E(v,p)$ then they do solve the original field equations and boundary conditions.

The form of the functional $E(v,p)$ in (28) might lead one to believe that its value could come arbitrarily near zero when v and p are trial functions which, though bounded, oscillate so rapidly that they make U_1 arbitrarily large. In this case the greatest lower bound on $E(v,p)$ would be zero, and no minimizing functions v and p could exist. We show that this cannot happen by constructing a nonzero lower bound for $E(v,p)$. For any $b>a$ we define a quantity

$$E_2(a,b)=\inf_{v,p}\int_a^b dr[r^2U_2(v,p)], \tag{29}$$

which is positive and nonzero since U_2 is a sum of three positive terms.[9] Let $\delta>0$ be such that

$$\delta = 2GE_2(\delta,\infty). \tag{30}$$

Such a δ certainly exists, since $E_2(\delta,\infty)$ is a decreasing function of δ which tends to zero as δ goes to infinity. We now show that $E_2(\delta,\infty)$ is a lower bound for the functional $E(v,p)$. Observing that the first term in $E(v,p)$, which involves only U_1, is almost a total derivative, we obtain

$$E^{(1)}(v,p,R)\equiv\int_0^R dr\left[r^2U_1\exp\left(-2G\int_r^\infty \rho U_1 d\rho\right)\right]$$
$$=(2G)^{-1}\exp\left(-2G\int_R^\infty \rho U_1 d\rho\right)\int_0^R dr\left[1-\exp\left(-2G\int_r^R \rho U_1 d\rho\right)\right]. \tag{31}$$

Since the term in square brackets is a positive decreasing function of r, a lower bound is obtained by replacing $\int_0^R$ by $\int_0^\delta$ and then taking the value of the integrand at $r=\delta$,

$$E^{(1)}(v,p,R)\geq(2G)^{-1}\exp\left(-2G\int_R^\infty \rho U_1 d\rho\right)\delta\left(1-\exp\left(-2G\int_\delta^R \rho U_1 d\rho\right)\right). \tag{32}$$

A similar manipulation on the second term in $E(v,p)$, which contains both U_1 and U_2, leads to

$$E^{(2)}(v,p,R)\equiv\int_0^R dr\left[r^2U_2\exp\left(-2G\int_r^\infty \rho U_1 d\rho\right)\right]$$
$$\geq\int_\delta^R dr\left[r^2U_2\exp\left(-2G\int_r^\infty \rho U_1 d\rho\right)\right]\geq\exp\left(-2G\int_\delta^\infty \rho U_1 d\rho\right)E_2(\delta,R). \tag{33}$$

Adding $E^{(1)}(v,p,R)$ and $E^{(2)}(v,p,R)$ we obtain

$$E(v,p,R)\geq(2G)^{-1}\delta\exp\left(-2G\int_R^\infty \rho U_1 d\rho\right)+[E_2(\delta,R)-(2G)^{-1}\delta]\times\exp\left(-2G\int_\delta^\infty \rho U_1 d\rho\right). \tag{34}$$

Taking the limit $R\to\infty$ and recalling the definition of δ in Eq. (3) gives the required result

$$E(v,p)\geq E_2(\delta,\infty). \tag{35}$$

We now show that the minimizing functions v and p are bounded by $|ev|\leq 1$ and $|p|\leq F$. When $|ev|>1$ we may define a new function $\hat{v}$ by reflecting v about the axes $ev=\pm 1$,

$$e\hat{v}=2-ev \text{ if } ev>1,\qquad e\hat{v}=-2-ev \text{ if } ev<-1. \tag{36}$$

After repeated reflections one obtains $|e\hat{v}|\leq 1$. Since $\hat{v}'=\pm v'$ except at the reflection points we have $U_1(\hat{v},p)=U_1(v,p)$, but $U_2(\hat{v},p)\leq U_2(v,p)$, as one easily verifies. Hence replacing the function v by $\hat{v}$ yields a lower value of the energy functional. By reflecting the function p about the lines $p=\pm F$ in the same manner, we may restrict the space of trial functions by $|ev|\leq 1$ and $|p|\leq F$.

Next we calculate the energy of our system in the weak-field limit $G\to 0$. We find that the first-order terms in G may be computed exactly in terms of the flat-space solution, without performing an explicit first-order calculation. Expanding the exponential in the energy functional in (28), we obtain

$$E=\int_0^\infty dr[r^2(U_1^{(1)}+U_2^{(1)})]-2G\int_0^\infty dr\left[r^2(U_1+U_2)\int_r^\infty \rho U_1 d\rho\right], \tag{37}$$

where the superscripts in the first term indicate that we must consider U_1 and U_2 to be expressed in terms of the curved-space solution to first order in G, whereas in the second term the zero-order flat-space functions v_0, p_0, are sufficient. Now the first term, regarded as a functional of v and p, is just the functional which we would have to minimize in flat space, and so it is stationary with respect to arbitrary variations of v and p about the flat-space solution. Thus the value of the first term is the flat-space energy E_0, plus corrections of order G^2. Using spherical symmetry to rewrite the second term we then have, correct to first order in G,

$$E=E_0-\frac{G}{8\pi^2}\int_{\rho>r}\int d^3r\,d^3\rho\,\frac{[U_1(r)+U_2(r)]U_1(\rho)}{|\vec{r}-\vec{\rho}|} \tag{38}$$

in which U_1 and U_2 are expressed in terms of the flat-space solutions v_0 and p_0.

Finally we verify that if v and p minimize the energy functional $E(v,p)$ then they do yield a solution of the original field equations (13)–(16) with the boundary conditions (17)–(19). For the gravitational variables this is evident: Equations (21) and (22) define functions y and x which satisfy the Einstein field equations (13) and (14) and the boundary conditions (17) and (18). The minimizing functions v and p satisfy the Euler-Lagrange equations and natural boundary conditions at $r=0$ obtained by varying the energy functional $E(v,p)$. For brevity writing only the results for v, we obtain the integro-differential equation

$$\left\{v'\left[e^y-2Gr^{-1}\int_0^r d\rho\,\rho^2 e^{y(\rho)}(U_1(\rho)+U_2(\rho))\right]\right\}'=e^y[r^{-2}v(e^2v^2-1)+e^2vp^2], \tag{39}$$

and the boundary condition at the origin

$$v'\left[e^y-2Gr^{-1}\int_0^r d\rho\,\rho^2 e^{y(\rho)}(U_1(\rho)+U_2(\rho))\right]=0 \text{ at } r=0. \tag{40}$$

In these equations y is expressed in terms of v and p by Eq. (21). Inserting the second integrated Einstein equation (22) into this result, we see that v satisfies the original matter field equation (15) and its boundary condition at the origin in (19); the boundary condition $v\to 0$ at infinity is clearly required in order that the functional $E(v,p)$ be convergent. Note that the boundary condition (18) at the origin, which leads to the integrated Einstein equation (22), is essential for verifying the original field equations, even though only the first Einstein equation (21) was used to eliminate the gravitational variables from the Lagrangian.

V. CONCLUSIONS

We eliminated in closed form the gravitational variables from the total Lagrangian $L^{E'}+L^M$, where L^M describes 't Hooft's magnetic monopole in curved space and $L^{E'}$, which differs from the usual Einstein Lagrangian L^E by a total divergence, contains only first-order derivatives. The resulting functional is particularly simple; it is the flat-space Lagrangian multiplied by $-(-g)^{1/2}$,

$$E(v,p)=4\pi\int_0^\infty dr\, r^2 T_{00}(v,p,\lambda=\nu=0)\times\exp\left(-2G\int_r^\infty d\rho\,\rho^{-1}[(v')^2+\tfrac{1}{2}\rho^2(p')^2]\right).$$

It is positive-definite, and the functions v and p which minimize it yield a regular and localized solution. The minimal value of $E(v,p)$ is the energy of the ground state, and we derived a positive lower bound for it. Physically this means that although bringing matter closely together yields a large negative Newtonian energy, this is compensated for by the positive energy which resides in the curving of space. In particular the vacuum is stable for any value of G, in agreement with general theorems.[10] By inserting the flat-space solutions as trial functions into $E(v,p)$, we see that the over-all effect of gravitation is to bind the ground state better. The solution is stable according to the usual topological arguments.[16] We found an explicit expression for the order-G corrections to the ground-state energy. It is not of the usual form $\int\rho(r)\rho(r')|\vec{r}-\vec{r}'|^{-1}d^3r\,d^3r'$, nor of the form $T_{\mu\nu}\times$(graviton propagator)$\times T_{\mu\nu}$ because the flat-space solution also has, apart from a given energy density, internal structure, as witnessed by the nonvanishing of the stress components T_{ij}.

In the text we restricted ourselves to the two Einstein equations which determine the radial and time components λ and ν of the metric, although there exists a third Einstein equation for static solutions. In source-free gravitation this third equation (involving $G_2{}^2=G_3{}^3$) follows from the other two by means of the Bianchi identities. In our problem, this third equation follows from the other two Einstein equations and energy-momentum conservation.[11]

The boundary conditions at infinity followed by requiring that the solution be localized and the Lagrangian convergent. The boundary conditions at the origin, both in the original problem and in the reduced problem with functional $E(v,p)$, were obtained by letting the functional determine its own "natural" boundary conditions.

The role of spontaneous symmetry breaking is to set a scale for a nonzero energy solution. If all fields would tend to zero asymptotically, the minimum value of $E(v,p)$ would be zero. The fixed nonzero asymptotic value of the Higgs field serves as a peg to prevent this trivial solution.

Our results do not hold for an electrically charged magnetic monopole (a dyon) with W_0^a of the Julia-Zee[12] form $W_0^a=e^{-1}r^a r^{-2}\phi(r)$. The reason is that in this case even in flat space the Lagrangian is not positive-definite and is different from minus the energy.

APPENDIX

Below we give a few steps in the derivation of the Lagrangian and energy tensors in polar coordinates.

The Yang-Mills and Higgs fields in polar coordinates are obtained by transforming the Lorentz indices but not the isospin indices. Evaluating $W^a_{\mu'}(r')=(\partial x^\mu/\partial x'^{\mu'})W^a_\mu$, one has

$$\begin{aligned}&\vec{W}_t=\vec{W}_r=0,\\&\vec{W}_\theta=(-\sin\phi,\cos\phi,0)(-e^{-1}+v),\\&\vec{W}_\phi=(-\cos\phi\cos\theta,-\sin\phi\cos\theta,\sin\theta)\sin\theta(-e^{-1}+v),\\&\vec{Q}=(\cos\phi\sin\theta,\sin\phi\sin\theta,\cos\theta)p.\end{aligned}\tag{A1}$$

The tensors $D_\mu Q^a$ are given in polar coordinates by

$$\begin{aligned}&D_t\vec{Q}=0,\\&D_r\vec{Q}=(\cos\phi\sin\theta,\sin\phi\sin\theta,\cos\theta)p',\\&D_\theta\vec{Q}=(\cos\phi\cos\theta,\sin\phi\cos\theta,-\sin\theta)evp,\\&D_\phi\vec{Q}=(-\sin\phi\sin\theta,\cos\phi\sin\theta,0)evp.\end{aligned}\tag{A2}$$

The nonvanishing components of the Yang-Mills tensor in polar coordinates are equal to

$$\begin{aligned}&\vec{G}_{r\theta}=(-\sin\phi,\cos\phi,0)v',\\&\vec{G}_{r\phi}=(-\cos\phi\cos\theta,-\sin\phi\cos\theta,\sin\theta)\sin\theta\, v',\\&\vec{G}_{\theta\phi}=(\cos\phi\sin\theta,\sin\phi\sin\theta,\cos\theta)\sin\theta(-e^{-1}+ev^2).\end{aligned}\tag{A3}$$

Note the orthogonality relations between the isovectors in (A2) and (A3). From Eqs. (A2) and (A3) one obtains the Lagrangian L^M given in (6).

The energy tensor $T_{\mu\nu}=-2(-g)^{-1/2}(\delta L^M/\delta g^{\mu\nu})$ is rotationally invariant, hence it is diagonal in polar coordinates and satisfies $T^\theta_\theta=T^\phi_\phi$. For the Yang-Mills energy tensor one finds

$$\begin{aligned}&T_\mu{}^\nu(\mathrm{YM})=-(4\pi)^{-1}(F+C,-F+C,-C,-C),\\&\mu=\nu=t,r,\theta,\phi\\&F=r^{-2}\exp(-\lambda)(v')^2,\quad C=\tfrac{1}{2}e^{-2}r^{-4}(e^2v^2-1)^2.\end{aligned}\tag{A4}$$

This tensor is of course traceless. The terms proportional to C with v set equal to zero constitute the Maxwell energy tensor for an electric or magnetic[13-15] point charge e^{-1}. [We introduced in Eq. (2) the factor $(4\pi)^{-1}$ in order that the energy density for a point charge e^{-1} be equal to $(8\pi)^{-1}(er^2)^{-2}$.] Hence the one massless Yang-Mills mode behaves asymptotically as a (magnetic) point charge e^{-1}. The fields v fall off exponentially for large r, as in flat space; this is due to the mass of the two remaining Yang-Mills modes.

For the energy tensor of the Higgs fields one finds in polar coordinates

$$T_\mu{}^\nu\,(\text{Higgs}) = -(4\pi)^{-1}(P+I+V, -P+I+V, P+V, P+V), \tag{A5}$$

$$P=\tfrac{1}{2}\exp(-\lambda)(p')^2,\quad I=e^2r^{-2}v^2p^2,\quad V=\tfrac{1}{8}k(p^2-F^2)^2.$$

All components vanish exponentially fast for large r, in agreement with the massive character of the Higgs fields. Consequently, the term V in Eq. (2) is also convergent. (The $1/r$ term in p, found in Ref. 2, is due to the limiting case $k=\mu^2=0$ considered there.) This is due to our handling of the constant term in V; had we not included it in V it would have given a cosmological constant. (Note that V contributes to $T_\mu{}^\nu$ as $-V\delta_\mu^\nu$.)

It is interesting to note the implication of the equality of the two energy formulas in (25) and (26). Consider flat space. The first equation reduces to the volume integral of the energy density, as expected. The second formula, however, yields twice the Yang-Mills terms and only the self-interaction term of the Higgs scalars

$$\begin{aligned} -4\pi T_0^0 &= (F+C)+(P+I+V),\\ -4\pi(T_0^0-T_1^1-T_2^2-T_3^3) &= 2(F+C)-2V. \end{aligned} \tag{A6}$$

This is thus equipartition of energy between the Yang-Mills and Higgs modes, which becomes exact in the limit of vanishing scalar self-coupling.

---

*Work supported in part by NSF Grant No. MPS-74-13208 A01.

[1]G. 't Hooft, Nucl. Phys. B79, 276 (1974).

[2]M. K. Prasad and C. M. Sommerfield, Phys. Rev. Lett. 35, 760 (1975).

[3]F. A. Bais and R. J. Russell, Phys. Rev. D 11, 2692 (1975).

[4]Y. M. Cho and P. G. O. Freund, Phys. Rev. D 12, 1588 (1975).

[5]P. Cordero and C. Teitelboim (unpublished).

[6]T. T. Wu and C. N. Yang, in *Properties of Matter under Unusual Conditions*, edited by H. Mark and S. Fernbach (Interscience, New York, 1969).

[7]P. A. M. Dirac, Proc. R. Soc. London A133, 60 (1931).

[8]L. D. Landau and E. M. Lifshitz, *Classical Theory of Fields* (Addison-Wesley, Reading, Mass., 1962).

[9]One can solve this variational problem exactly and finds that the minimizing r^2U_2 is given by $\frac{1}{8}kr^2F^4$ for $0<Fr<(\frac{1}{4}e^2k)^{-1/4}$ and by $(2e^2r^2)^{-1}$ for $Fr>(\frac{1}{4}e^2k)^{-1/4}$.

[10]D. R. Brill and S. Deser, Ann. Phys. (N.Y.) 50, 548 (1968).

[11]When the matter field equations are satisfied, $T^\nu_{\mu;\nu}=0$ and one may copy the source-free proof by simply replacing $G_{\mu\nu}$ by $\tilde{G}_{\mu\nu}\equiv G_{\mu\nu}+8\pi G\,T_{\mu\nu}$ and using the two Einstein equations $\tilde{G}_0^0=\tilde{G}_r^r=0$.

[12]B. Julia and A. Zee, Phys. Rev. D 11, 2227 (1975).

[13]The electromagnetic $T_{\mu\nu}$ is invariant under infinitesimal duality rotations (see Ref. 14) $F_{\mu\nu}\rightarrow F_{\mu\nu}+\delta\lambda\,{}^*F_{\mu\nu}$ which interchange electric and magnetic fields. The transformation of A_μ follows (see Ref. 15) from this by choosing the gauge

$$A_\mu(x)=\int f^\nu(y)F_{\mu\nu}(x-y)d^4y$$

with $\partial_\nu f^\nu(y)=\delta^4(y)$.

[14]S. Deser, M. T. Grisaru, P. van Nieuwenhuizen, and C. C. Wu (unpublished).

[15]J. Schwinger (unpublished).

[16]J. Arafune, P. G. O. Freund, and C. J. Goebel, J. Math. Phys. 16, 433 (1975).