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Hyperfine splitting of ground-state charmonium*

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The mass splitting of $\psi(3100)$ and its pseudoscalar partner η_c is examined within the context of nonrelativistic linear-potential models for bound charmed quarks. If the Coulomb-type short-range gluon interaction is the sole contribution to this hyperfine splitting, then a mass splitting of 15–25 MeV is obtained. However, if the long-range forces that confine quarks also produce shorter-range spin-spin interactions, then the ψ-η_c mass splitting is predicted to be in the range 40–80 MeV. If one abandons the assignment of $\psi'(3700)$ as the first radial excitation of $\psi(3100)$, then the η_c is predicted to lie 150–300 MeV below $\psi(3100)$. This possibility has the consequence that the rate predicted for the yet-unobserved decay $\psi \to \eta_c + \gamma$ is uncomfortably large.

I. INTRODUCTION

One of the most attractive interpretations[1] of the newly discovered ψ, ψ', and ψ'' resonances[2-4] is that they are bound states of quark-antiquark ($c\bar{c}$) pairs, with the quarks carrying new quantum numbers generically called charm.[5-6] The $\psi(3100)$ is assumed to be the lowest $^3S_1$ bound state of the $c\bar{c}$ pair,[1] with $\psi'(3700)$ and $\psi''(4200)$ the radial excitations of the ground state. Following this proposal, a number of potential models were considered,[7] with the linear potential receiving the greatest attention. A common feature of all these models is the result that the low-lying levels of the ψ system are nonrelativistic, with the charmed quark mass heavy, i.e., in the 1.5–2.0 GeV range. An immediate by-product of these models is an extensive spectroscopy of $c\bar{c}$ states which as yet has not been observed, aside from the ψ, ψ', and predicted ψ''. If in fact $\psi(3100)$ is the lowest $c\bar{c}$ bound state with $J^{PC} = 1^{--}$ (also called ϕ_c), then it is essential[1,8] that there exist a $c\bar{c}$ bound state with $J^{PC} = 0^{-+}$ (η_c or paracharmonium). It is the purpose of this paper to present an estimate of the mass of the η_c within the context of the nonrelativistic linear potential models of quark confinement.

The alternatives to a dynamical calculation of the η_c mass are SU(4) mixing schemes[8,9] based on a Gell-Mann–Okubo type formula and the identification of η_c as a pure $c\bar{c}$ state. Choosing η, X^0, and η_c as orthogonal states, Lee and Quigg obtained[9] $M_{\eta_c} = 3122$ MeV, while if η, E, and η_c are the orthogonal states of the mixing scheme, $M_{\eta_c} = 3066$ MeV is obtained.[9] (It is extremely difficult to understand $M_{\eta_c} > M_{\phi_c}$ within the framework of potential models.) However, De Rújula, Georgi, and Glashow have emphasized[10] that the transitions $\bar{q}_i q_i \to$ gluons $\to q_j q_j$ make a contributions to the mass matrix of the isoscalar mesons which varies rapidly with energy if asymptotic freedom is relevant to this mechanism. As a result the eigenvectors of the $I = 0$ meson mass matrix *need not be orthogonal*.[11] Therefore, η_c is essentially pure $c\bar{c}$, but η and η' need not be orthogonal states. As a consequence, no direct prediction of M_{η_c} results, and one must appeal to more detailed dynamics to understand the mass of η_c.

By postulating that bound $c\bar{c}$ heavy quarks have a Coulomb-type (or positronium-like) spectroscopy, Appelquist and Politzer[1] predicted the mass difference of "orthocharmonium" and "paracharmonium" to be roughly $\alpha_s^4 m_c \sim 10$–30 MeV (based on the estimate $\alpha_s \sim 0.2$–0.3 from asymptotic freedom). However, the Coulomb-type mechanism underestimates the $\psi(3100)$–$\psi'(3700)$ mass splitting by a factor of 10, so that the prediction of the charmonium hyperfine splitting may also be an underestimate. Indeed, Appelquist *et al.*[1] make use of the fact that in the Coulomb model the hyperfine splitting is proportional to $|\phi(0)|^2$, the wave function at the origin of the $c\bar{c}$ system, *but* use empirical data to obtain $|\phi(0)|^2$ and predict

$$M_{\phi_c} - M_{\eta_c} = \frac{1}{6}\left[\frac{9}{\alpha^2}\frac{\Gamma(\psi \to e^+e^-)}{m_\psi}\right]^{4/3} m_\psi \simeq 80\text{–}90 \text{ MeV}, \tag{1.1}$$

where

$$\Gamma(\psi \to e^+e^-) = 16\pi\alpha^2 \frac{|\phi(0)|^2}{m_\psi^2} e_Q^2 \tag{1.2}$$

in the three-color quark model [$e_Q^2 = \frac{4}{9}$ in the simplest SU(4) model[6]]. From the assumption that the ϕ_c-η_c hyperfine interaction comes *entirely* from the short-distance $1/r$ part of the $q\bar{q}$ interaction, but by using this assumption in a different way

from earlier calculations, De Rújula *et al.*[10] relate the ϕ_c-η_c hyperfine splitting to the ρ-π mass splitting and obtain

$$M_{\phi_c} - M_{\eta_c} = (M_\rho - M_\pi)\left(\frac{m_p}{m_c}\right)^2 \sim 27 \text{ MeV}, \tag{1.3}$$

where m_p and m_c are the masses of nucleon type and charmed quark, respectively.

The assumption that the charmonium hyperfine interaction comes *only* from the short-range (Coulomb-type) gluon interactions is controversial. The basis of this particular point of view is abstracted from lattice gauge models,[12] where the spin-spin interaction is exponentially damped relative to the leading spin-independent force which confines quarks.[13] Certainly there are other possibilities even within the framework of linear potential models of quark confinement. For example, it has been conjectured[14] that since the $c\bar{c}$ system is nonrelativistic, the skeleton expansion of the Bethe-Salpeter kernel might be truncated at the single *dressed* ladder. This is suggested by QED, where the single (undressed) ladder gives the Schrödinger equation along with the lowest-order hyperfine and other relativistic corrections. In this case the same gluon exchange mechanism which gives rise to quark confinement also leads to the spin-spin and other relativistic corrections to the nonrelativistic $c\bar{c}$ system. Here the spin-spin interaction is shorter range than the leading spin-independent potential, but it is by no means exponentially damped.

In this paper we will study the lowest-order relativistic corrections to the $c\bar{c}$ system based on a single dressed ladder approximation to the Bethe-Salpeter kernel. In Sec. II we discuss the effective Hamiltonian for the computation of the lowest-order relativistic corrections to the $c\bar{c}$ system. In Sec. III we discuss the perturbation expansion of the charmonium energy levels in terms of the natural expansion parameters of the system, which is essentially (v/c), the average velocity of the quark in the ground state. Using a variational wave function, analytic expressions for the ground-state relativistic corrections are presented. Numerical estimates of the ϕ_c-η_c mass splitting are presented in Sec. IV based on the results of Sec. III. We argue in the Appendix that the analytical expressions of Sec. III reproduce a computor calculation of the matrix elements to an accuracy of better than 2%. Our conclusions are presented in Sec. V.

II. AN EFFECTIVE HAMILTONIAN

Let us present an effective Hamiltonian suitable for the computation of the lowest-order relativistic corrections to the nonrelativistic charm-anticharm bound states. In contrast to quantum electrodynamics (QED), a systematic derivation of such a Hamiltonian has not as yet been given from first principles. Therefore, we must make some *ad hoc* hypotheses to proceed. We assume the following:

(1) After summing over the color indices of the quarks (to obtain color singlets), the Bethe-Salpeter kernel for the $c\bar{c}$ system can be well approximated by a single dressed ladder[14] with an *effective* Abelian gluon propagator[15]

$$D_{\mu\nu}(q) = g_{\mu\nu} d(q^2) \tag{2.1}$$

in Feynman gauge.

(2) The nonrelativistic limit of the system is obtained by treating the kernel in the instantaneous (single-time) approximation.

(3) The effective quark-gluon vertex, to lowest order in the natural expansion parameters of the system (to be elaborated further in Sec. III), is given by

$$\Gamma_\mu = \text{const} \times \gamma_\mu . \tag{2.2}$$

These assumptions are motivated by an analogous treatment of positronium in QED, but their justification is still a matter for the future. In particular, in QED the radiative corrections to the electron-photon vertex make contributions to positronium energy levels which are higher order in α than the lowest-order fine-structure and spin-spin interactions. This well-known result is the basis for assumption (3).

It is a straightforward matter to obtain the effective Hamiltonian to leading order in $(v/c)^2$ from our hypotheses (1)–(3). (We found the procedure outlined by Schwinger[16] particularly useful.) The static Fourier transform of $d(q^2)$ is

$$\int d^3r\, e^{-i\vec{q}\cdot\vec{r}} d(\vec{q}^{\,2}) = \text{constant} \times V(r), \tag{2.3}$$

where the constants in (2.2) and (2.3) are chosen so that $V(r)$ is the static potential of the nonrelativistic Schrödinger equation. The result in the center of mass of the $q\bar{q}$ system is[17]

$$H=m_1+m_2+\frac{\vec{p}^2}{2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)-\frac{1}{8}\left(\frac{1}{m_1^3}+\frac{1}{m_2^3}\right)(\vec{p}^2)^2+V(r)$$
$$+\left\{\frac{1}{m_1m_2}\vec{p}\cdot\vec{\Lambda}\cdot\vec{p}+\frac{1}{8}\left(\frac{1}{m_1^2}+\frac{1}{m_2^2}\right)\nabla_1^2V(r)+\frac{1}{4}\left(\frac{1}{m_1^2}+\frac{2}{m_1m_2}\right)\frac{1}{r}\frac{dV(r)}{dr}\vec{\sigma}_1\cdot\vec{L}+\frac{1}{4}\left(\frac{1}{m_2^2}+\frac{2}{m_1m_2}\right)\frac{1}{r}\frac{dV(r)}{dr}\vec{\sigma}_2\cdot\vec{L}\right.$$
$$\left.-\frac{1}{4m_1m_2}\left(\vec{\sigma}_1\cdot\hat{r}\,\vec{\sigma}_2\cdot\hat{r}-\frac{1}{3}\vec{\sigma}_2\cdot\vec{\sigma}_2\right)\left[\frac{d^2V(r)}{dr^2}-\frac{1}{r}\frac{dV(r)}{dr}\right]+\frac{1}{6m_1m_2}\vec{\sigma}_1\cdot\vec{\sigma}_2\nabla^2V(r)\right\}. \tag{2.4}$$

Here m_1 ($\vec{r}_1$) and m_2 ($\vec{r}_2$) are the masses (coordinates) of the quark and antiquark, respectively, with $r=|\vec{r}_1-\vec{r}_2|$. The dyadic operator

$$\vec{\Lambda}=\left[\vec{I}-\frac{1}{\nabla_1^2}\vec{\nabla}_1\vec{\nabla}_1\right]V(r), \tag{2.5}$$

with $\vec{I}$ the unit dyadic. It is understood that $\vec{p}=-i\vec{\nabla}_1$, where $\vec{\nabla}_1$ is the gradient with respect to $\vec{r}_1$.

A reasonable assumption for the potential, which can be abstracted from speculative analyses of non-Abelian gauge theories, is[7]

$$V(r)=ar+C-\alpha_s/r. \tag{2.6}$$

The linear part of the potential represents the long-range part of the $c\bar{c}$ forces (acting between color singlets), and the Coulomb-type piece characterizes the short-range gluon exchange, as expected in gauge theories which are asymptotically free.[18] The constant C subsumes those *spin-independent* interactions not included explicitly in the r-dependent part of the potential.[19] In practical applications to ψ spectroscopy the (α_s/r) term can be considered a perturbation, since it only contributes[20] about 60 MeV to the $\psi'(3700)-\psi(3100)$ energy difference for $\alpha_s\simeq 0.2$, a typical value suggested by asymptotic freedom.

Let us concentrate on the relativistic corrections to the lowest $^3S_1$ and $^1S_0$ states of the $c\bar{c}$ system, orthocharmonium and paracharmonium. The effective Hamiltonian appropriate to the diagonal matrix elements of the $n=1$, $^3S_1$ states of the $c\bar{c}$ system simplifies to

$$(H)_{^3S_1}=2m+\frac{\vec{p}^2}{2m}-\frac{1}{4}\frac{(\vec{p}^2)^2}{m^3}+V(r)$$
$$+\left[\frac{1}{m^2}\vec{p}\cdot\vec{\Lambda}\cdot\vec{p}+\frac{5}{12m^2}\nabla^2V(r)\right], \tag{2.7}$$

with $m_1=m_2=m$ the mass of the charmed quark. The ortho-paracharmonium mass splitting, to lowest order in $(v/c)^2$, is given by

$$\nabla E=\frac{1}{6m^2}[\langle\vec{\sigma}_1\cdot\vec{\sigma}_2\rangle_{^3S_1}-\langle\vec{\sigma}_1\cdot\vec{\sigma}_2\rangle_{^1S_0}]\langle\langle\nabla^2V(r)\rangle\rangle$$
$$=\frac{2}{3m^2}\langle\langle\nabla^2V(r)\rangle\rangle, \tag{2.8}$$

where the expectation value is to be taken with the $n=1,S$ state eigenfunction of the nonrelativistic Hamiltonian, $2m+\vec{p}^2/2m+V(r)$. In principle there is also a contribution to ΔE from $c\bar{c}\to$ gluons $\to c\bar{c}$ annihilation; however, with $\alpha_s\sim 0.2$–0.3 this is negligible compared to the energy shift given by Eq. (2.8). We therefore neglect the gluon annihilation mechanism in what follows.

III. EXPANSION PARAMETERS

Positronium served as a suggestive analog to Appelquist and Politzer,[1] who predicted the existence of charmonium bound states of $c\bar{c}$ pairs. The positronium atom is a nonrelativistic system bound by Coulomb forces whose energy levels can be expressed as a perturbation series with successive terms of order

$$\alpha^2m_e,\ \alpha^4m_e,\ \alpha^5m_e,\ldots \tag{3.1}$$

corresponding to the Bohr levels, fine-structure corrections, and Lamb shift, respectively, where m_e is the mass of the electron. (The Schwinger correction to the electron magnetic moment makes its first contribution to energy levels in order α^5m_e.) Equation (3.1) can also be identified with the series expansion whose terms are ordered

$$(v/c)^2m_e,\ (v/c)^4m_e,\ (v/c)^5m_e,\ldots, \tag{3.2}$$

where v is the average electron velocity in the positronium atom.

It is not unreasonable to expect an expansion similar to (3.2) to hold for the nonrelativistic bound states of charm quarks and antiquarks. One cannot use the analog of Eq. (3.1) directly, since the Coulomb model of Appelquist and Politzer[1] is inadequate in that it underestimates level spacings of $\psi'(3700)$ and $\psi(3100)$ by a factor of 10. If the linear potential given by Eq. (2.6) gives an adequate phonemenological description of the low-lying levels of the $c\bar{c}$ system, with the Coulomb-type term, α_s/r, a perturbation, then we argue

that an expansion similar to (3.1) holds for the nonrelativistic $c\bar{c}$ system but with *two* expansion parameters, α_s and $(a/m^2)^{1/3}$. Semiclassical arguments, virial theorems,[21] or dimensional analysis allow us to identify

$$(v/c)^2 \sim \left(\frac{a}{m^2}\right)^{2/3} \tag{3.3}$$

for the average velocity of a charmed quark in the nonrelativistic states of charmonium bound by the potential

$$V_0 = ar + C. \tag{3.4}$$

Indeed the S-wave radial excitations of this system, computed by the WKB method,[21] are

$$E = 2m + \left[\frac{3\pi}{\sqrt{2}}(n' + \tfrac{3}{4})\right]^{2/3}\left(\frac{a}{m^2}\right)^{2/3} m + C, \tag{3.5}$$

where n' is the number of radial nodes of the wave function. The energy levels of the leading Regge trajectory, obtained from the loaded string model,[22] are a series in $(T_0/m^2)^{2/3}$ in the nonrelativistic limit, where T_0 is the rest tension of the string, so that the dependence of the energy levels on the parameter $(a/m^2)^{2/3}$ is the same as the loaded string as expected from intuitive arguments based on (3.1)–(3.4). Let us define the dimensionless parameter

$$\beta = \left(\frac{a}{m^2}\right)^{1/3}. \tag{3.6}$$

If the Coulomb-type term α_s/r appearing in (2.6) is treated as a perturbation in the system bound by the potential (3.4), then the Hamiltonian given by (2.7) leads to the expansion for low-lying levels

$$E \propto \beta^2,\ \beta\alpha_s,\ \beta^4,\ \beta^3\alpha_s, \ldots, \tag{3.7}$$

which is the perturbation expansion in the parameters β and α_s, accurate to $O((v/c)^4, \alpha_s)$, and neglecting terms of order $(v/c)^5$ or α_s^2. In analogy with positronium, we conjecture that nontrivial corrections to the effective quark-gluon vertex (2.2) make their first contribution to charmonium energy levels in order $(v/c)^5$, and hence can be neglected.[23] [This is to be expected if form-factor or anomalous-magnetic-moment corrections to Eq. (2.2) vanish with the momentum transfer to the quarks which will make them higher order in (v/c) than the terms we keep.]

We can go beyond these intuitive arguments to establish the connection between (3.7) and the Hamiltonian (2.7). Although the Schrödinger equation for S-wave bound states, with the potential (3.4), *can* be solved analytically in terms of Airy functions, the matrix elements of the relativistic correction terms in (3.7) *cannot* be evaluated in terms of special functions, but require numerical computation. However, we can finesse these difficulties and exhibit an *analytic* estimate of the relativistic corrections, with the correct dependence on the expansion parameters of the problem, by using a complete set of harmonic-oscillator functions as a basis for the computation of matrix elements. Of course, in this basis the "unperturbed" Hamiltonian

$$H_0 = 2m + \frac{\vec{p}^2}{2m} + ar + C \tag{3.8}$$

is *not* diagonal, so that a *given* eigenfunction of (3.8) requires an infinite series of oscillator functions. In practice, we can express the *ground-state* wave function of (3.8), to an excellent approximation, in terms of a *single* oscillator wave function by using a variational principle to select the optimal oscillator interaction strength. This procedure leads to *at most* a 5% error in the numerical evaluation of the energy shifts given by (2.7) and (2.8), and gives the analytical behavior on the expansion parameters expected. (See the Appendix.)

Consider the three-dimensional harmonic oscillator, restricted to S wave,

$$\left(-\frac{1}{m}\frac{d^2}{dr^2} + \tfrac{1}{2}Kr^2\right) u_n(r) = (2n - \tfrac{1}{2})\omega u_n(r), \tag{3.9}$$

where $\hbar = 1$, $n = 1, 2, 3, \ldots$, and

$$\psi_n(r) = \frac{u_n(r)}{r} \tag{3.10}$$

is the S-wave oscillator wave function, with

$$\omega = \left(\frac{2K}{m}\right)^{1/2}. \tag{3.11}$$

The normalized solution to (3.9) is

$$u_n(r) = \frac{1}{\sqrt{4\pi}}\left[\frac{2\xi}{\pi^{1/2}2^{2n-1}(2n-1)!}\right]^{1/2} H_{2n-1}(\xi r)e^{-\frac{1}{2}\xi^2 r^2}, \tag{3.12}$$

where $\xi = (\frac{1}{2}m\omega)^{1/2} = (\frac{1}{2}mK)^{1/4}$, and $H_n(x)$ is the Hermite polynomial. The ground-state wave function for the system (3.8) is given by

$$\phi_0(r) = \sum_{n=0}^{\infty} c_n \psi_n(r), \tag{3.13}$$

where c_n are constants, and the $\psi_n(r)$ are given by

(3.10)–(3.12). Let us make the approximation that

$$\phi_0(r) \simeq \psi_0(r), \tag{3.14}$$

with the optimal choice of the interaction strength K given by the variational principle

$$\delta_K \langle \psi_0 | H_0 | \psi_0 \rangle = 0 \tag{3.15}$$

with normalized solutions $\langle \psi_0 | \psi_0 \rangle = 1$. Since

$$\begin{aligned}\langle \psi_0 | H_0 | \psi_0 \rangle &= \frac{3}{2}\left(\frac{2K}{m}\right)^{1/2} + \langle \psi_0 | (ar - \tfrac{1}{2}Kr^2) | \psi_0 \rangle \\ &= \frac{3}{4}\left(\frac{2K}{m}\right)^{1/2} + \frac{2a}{\sqrt{\pi}}\left(\frac{2}{mK}\right)^{1/4},\end{aligned} \tag{3.16}$$

we find from (3.15) that the oscillator strength for this approximation is

$$K = \left(\frac{128}{81\pi^2}\right)^{1/3}\left(\frac{a}{m^2}\right)^{4/3} m^3. \tag{3.17}$$

Thus, for the ground state

$$\begin{aligned} m\left(\frac{v}{c}\right)^2 \sim \omega &= \left(\frac{2K}{m}\right)^{1/2} \\ &= \left(\frac{32}{9\pi}\right)^{1/3}\left(\frac{a}{m^2}\right)^{2/3} m, \end{aligned} \tag{3.18}$$

as expected from the considerations which led to (3.3), (3.6), and (3.7). [A more detailed discussion of the approximation (3.14) and (3.15), together with an estimate of its accuracy, is to be found in the Appendix.]

The approximate ground-state wave function given by (3.12), (3.14), and (3.17) allows an explicit computation of the diagonal matrix elements of (2.7) and (2.8) in terms of the parameters of the model specified by the potential (2.6). The results are, using the definition (3.6),

$$\left\langle \phi_0 \left| \left(\frac{p^2}{2m} + ar\right) \right| \phi_0 \right\rangle \simeq 3\left(\frac{1}{2\pi}\right)^{1/3} \beta^2 m, \tag{3.19}$$

$$\left\langle \phi_0 \left| \frac{\alpha_s}{r} \right| \phi_0 \right\rangle \simeq \left(\frac{16}{3\pi^2}\right)^{1/3} \alpha_s \beta m, \tag{3.20}$$

$$\frac{1}{4m^3}\langle \phi_0 | (p^2)^2 | \phi_0 \rangle \simeq \frac{5}{8}\left(\frac{2}{3\pi^2}\right)^{1/3} \beta^4 m, \tag{3.21}$$

$$\frac{1}{m^2}\langle \phi_0 | \vec{p} \cdot \vec{\Lambda} \cdot \vec{p} | \phi_0 \rangle \simeq \left[-\frac{4}{3\pi}\alpha_s \beta^3 + 4\left(\frac{2}{3\pi^2}\right)^{1/3} \beta^4 \right] m, \tag{3.22}$$

$$\begin{aligned}\frac{1}{m^2}\langle \phi_0 | \nabla^2 V(r) | \phi_0 \rangle &= \frac{1}{m^2}\left\langle \phi_0 \left| \left(4\pi\alpha_s \delta^3(\vec{r}) + \frac{2a}{r}\right) \right| \phi_0 \right\rangle \\ &\simeq \frac{4\pi\alpha_s}{m^2} |\phi_0(0)|^2 + \frac{2a}{m^2}\left\langle \psi_0 \left| \frac{1}{r} \right| \psi_0 \right\rangle \\ &= \left[\alpha_s \beta^3 + 4\left(\frac{2}{3\pi^2}\right)^{1/3} \beta^4 \right] m, \end{aligned} \tag{3.23}$$

where we have used $|\phi_0(0)|^2 = ma/4\pi$ appropriate to the linear potential.

Inserting (3.19)–(3.23) into (2.7) we find for the energy of the $n = 1$, $^3S_1$ state

$$\begin{aligned}\langle {}^3S_1 | H | {}^3S_1 \rangle \simeq\ & 2m + C + 3\left(\frac{1}{2\pi}\right)^{1/3} \beta^2 m \\ & - \left(\frac{16}{3\pi^2}\right)^{1/3} \alpha_s \beta m + \left(\frac{2}{3\pi^2}\right)^{1/3} \left(\tfrac{121}{24}\right) \beta^4 m \\ & + \frac{1}{12\pi}(5\pi - 12)\alpha_s \beta^3 m. \end{aligned} \tag{3.24}$$

The estimate of the ortho-paracharmonium mass splitting obtained from (2.8) and (3.23) is

$$\begin{aligned}\Delta E &= M_{\phi_c} - M_{\eta_c} \\ &\simeq \frac{2}{3}\left[4\left(\frac{2}{3\pi^2}\right)^{1/3} \beta^4 + \alpha_s \beta^3 \right] m. \end{aligned} \tag{3.25}$$

Note that we have verified the form of the perturbation expansion, (3.7), expected from general principles. Numerical estimates of ΔE will be made in the next section.

There are of course additional relativistic corrections to be considered; retardation corrections in the ladder approximation, vertex corrections to (2.2), etc., but these are all expected to be at least of order (v/c) smaller, i.e., of over-all order $\beta^5, \beta^4\alpha_s, \beta^3\alpha_s^2, \ldots$.

IV. HYPERFINE SPLITTING OF CHARMONIUM

Equation (3.25) presents our prediction for the mass splitting of the ground states of orthocharmonium and paracharmonium. Since the dominant term in ΔE varies as $\beta^4 m = a^{4/3} m^{-5/3}$ its value is sensitive to the particular values of the parameters. We shall consider several typical models employing the linear potential[7] so as to be able to survey the range of values of ΔE that result. Using the energy of the first radial excitation of $\psi(3100)$ [usually taken as $\psi'(3700)$], and the leptonic decay rate $\Gamma(\psi \to l^+ l^-)$ as input, one may fix the parameters of a potential model of the $c\bar{c}$ system. Since $\Gamma(\psi \to l^+ l^-) \sim |\phi(0)|^2 \sim am$ in the linear-potential model, in fact $\Delta E \sim m^{-3}$, if the leptonic decay rate is considered fixed. If $\psi'(3700)$ is *not* the first

radial excitation of ψ,[24] it has been shown[25] that the charmed quark mass must be light, $m < 1$, GeV say, as compared to the usual range occurring in the charm models where $1.5 \lesssim m \lesssim 2$ GeV. Consequently, if $\psi'(3700)$ is not a radial excitation of ψ, the pseudoscalar partner of ψ will be considerably lower than otherwise expected. We now turn to a summary of the predictions of ΔE resulting from a number of models. A discussion of the accuracy of our calculation is to be found in the Appendix.

A. Model of Eichten *et al.*[7]

These authors obtained

$$\begin{aligned} a &= 0.194 \text{ GeV}^2, \\ \alpha_s &= 0.2, \\ m &= 1.6 \text{ GeV}, \\ \beta &= 0.42, \end{aligned} \tag{4.1}$$

for which we find

$$\begin{aligned} \Delta E &= (55.5 + 16.2) \text{ MeV} \\ &= 71.2 \text{ MeV}, \end{aligned} \tag{4.2}$$

where 55.5 MeV comes from the β^4 term, and 16.2 MeV is from the short-range part of the potential. (If $\alpha_s \simeq 0.3$, then the short-range contribution is increased to ~ 25 MeV.) An alternative solution is[7]

$$\begin{aligned} a &= 0.194 \text{ GeV}^2, \\ m &= 2.0 \text{ GeV}, \\ \alpha_s &= 0, \\ \beta &= 0.362, \end{aligned} \tag{4.3}$$

for which Eq. (3.25) predicts

$$\Delta E = 38 \text{ MeV}. \tag{4.4}$$

B. Model of Kang and Schnitzer[7]

Kang and Schnitzer studied a potential model, based on theoretical speculations originating from gauge theories, and found

$$\begin{aligned} a &= 0.30 \text{ GeV}^2, \\ m &= 2.0 \text{ GeV}, \\ \alpha_s &= 0, \\ \beta &= 0.42. \end{aligned} \tag{4.5}$$

These parameters, used in conjunction with Eq. (3.25), imply

$$\Delta E = 67.5 \text{ MeV} \tag{4.6}$$

for the orthocharmonium-paracharmonium mass splitting.

C. If $\psi'(3700)$ is not a radial excitation of ψ (Refs. 24 and 25)

To raise the energy of the radial excitation of ψ to 4.2 GeV or higher, one must increase the parameter a and decrease m so as to keep $\Gamma(\psi \to l^+ l^-)$ fixed. For example, if the energy of the first radial excitation of ψ is ~ 4.2 GeV, then

$$\begin{aligned} a &= 0.4 \text{ GeV}^2, \\ \alpha_s &= 0, \\ m &= 1 \text{ GeV} \end{aligned} \tag{4.7}$$

will be a typical set of parameters which makes this possible in the linear potential model. Although $\beta = 0.74$, nevertheless consider Eq. (3.25) to be applicable, obtaining

$$\Delta E = 330 \text{ MeV}, \tag{4.8}$$

which may be somewhat of an overestimate in view of the known ρ-π and the K^*-K mass splittings. Nonetheless the qualitative effect is evident.

V. CONCLUSIONS

We have presented a number of estimates of the mass splitting of $\psi(3100)$ and its pseudoscalar partner, η_c, based on nonrelativistic potential models of bound charmed quarks, together with lowest-order relativistic corrections. If the Coulomb-type short-range gluon interactions is the *sole* contribution to the charmonium hyperfine splitting, we find

$$\Delta E \sim 15\text{–}25 \text{ MeV}, \tag{5.1}$$

in agreement with other authors.[10,20] [Compare Eq. (1.3) with the second term of Eq. (4.2), using $\alpha_s = 0.2\text{–}0.3$.] On the other hand, if the long-range forces that confine quarks also produce shorter-range spin-spin interactions, as suggested by the Bethe-Salpeter equation, then our estimate of the ϕ_c-η_c mass splittings is

$$\Delta E \sim 40\text{–}80 \text{ MeV}. \tag{5.2}$$

[See Eqs. (4.2), (4.4), and (4.6).]

If $\psi'(3700)$ is not the first radial excitation of $\psi(3100)$,[24] then the mass splittings of the pseudoscalar η_c from $\psi(3100)$ must be considerably larger than that given by Eqs. (5.2)–(5.1). Considering Eq. (4.8) as an overestimate, we suggest that this would require

$$\Delta E \sim 150\text{–}300 \text{ MeV}, \tag{5.3}$$

say, together with *light* charmed quarks.[25] (By way of comparison, the ρ-π splitting is ~ 600 MeV, while the K^*-K splitting is ~ 300 MeV.) A very uncomfortable consequence of Eq. (5.3) is that one would predict[8] a large rate for the transition

$$\psi \to \eta_c + \gamma, \tag{5.4}$$

which in fact has not been observed. An absolute *lower* limit on the mass of η_c can be found by assuming[8] that the width of the mode (5.4) be no greater than the total width $\Gamma(\psi \to \text{all})$. Gaillard, Lee, and Rosner[8] require $M_{\eta_c} > 2.7$ GeV or $M_{\eta_c} > 2.95$ GeV, depending on how the rate for (5.4) is computed.

As yet the assumptions presented in Sec. II, which are the basis of our calculations, do not have secure theoretical support. We find a natural expansion parameter $(a/m^2)^{2/3}$ for the problem (cf. Sec. III) which suggests that it might be possible to find an expansion of the complete Bethe-Salpeter kernel in terms of this (or a similar) parameter for nonrelativistic quark systems. The difficulty with this idea is that $(a/m^2)^{1/3}$ does not represent a constant in the underlying Lagrangian, contrary to the role of the charge e in electrodynamics. Nonetheless the simplicity of the connection of $(a/m^2)^{1/3}$ with (v/c) indicates that such a search might not be futile.

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APPENDIX

Let us use Eqs. (3.10), (3.12), and (3.17) to represent the eigenfunctions of the Hamiltonian (3.8). The ground-state wave function is given by (3.13), with a similar expansion for the excited S states of the linear potential. The energy matrix for the S states

$$\langle \psi_m | H_0 | \psi_n \rangle = \begin{bmatrix} 3 & 0 & -\frac{1}{\sqrt{30}} & \cdots \\ 0 & \frac{16}{3} & -\frac{\sqrt{5}}{6} & \cdots \\ -\frac{1}{\sqrt{30}} & \frac{\sqrt{5}}{6} & \frac{89}{12} & \cdots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \end{bmatrix} \frac{3}{4}\left(\frac{2K}{m}\right)^{1/2}, \tag{A1}$$

with K given by (3.17). The diagonalization of (A1) will give the eigenvalues and eigenfunctions of the linear potential in terms of oscillator wave functions. From the small values of the off-diagonal matrix elements, we expect Eq. (3.14) to be a very good approximation.

Suppose we truncate the energy matrix (A1), and approximate it by the 3×3 submatrix whose matrix elements are explicitly shown. Diagonalizing this submatrix, we find

$$\phi_0(r) \simeq \psi_0(r) + 0.021\psi_1(r) + 0.049\psi_2(r), \tag{A2}$$

which shows a 5% admixture of $\psi_1(r)$ and $\psi_2(r)$ in the ground-state wave function $\phi_0(r)$. [A diagonal matrix element $\langle \phi_0 | \delta V | \phi_0 \rangle$, computed with (3.13), differs from $\langle \psi_0 | \delta V | \psi_0 \rangle$ by terms of order $2\sum_{n\neq 0} c_n \langle \psi_0 | \delta V | \psi_n \rangle$, neglecting terms of $O(c_n c_m)$.]

A further test is given by the wave function at the origin, where

$$|\phi_0(0)|^2 = \frac{ma}{4\pi} = 0.0795ma \tag{A3}$$

for the linear potential. Our approximation (3.14) together with (3.17) gives

$$|\phi_0(0)|^2 \simeq |\psi_0(0)|^2 = \frac{2ma}{3\pi^2} = 0.0675ma, \tag{A4}$$

while (A2) with (3.17) gives

$$|\phi_0(0)|^2 \simeq |\psi_0(0) + 0.021\psi_1(0) + 0.049\psi_2(0)|^2 = 0.0734ma, \tag{A5}$$

which suggest reasonable convergence to the exact answer (A3).

Finally we note that Borenstein[17] has evaluated Eq. (3.20) *numerically* by computer, using tabulated values of the Airy function for ϕ_0, which is the *exact* S-wave ground-state wave function for the Hamiltonian (3.8). His numerical value differs from our expression (3.20) by only 2%. Since the dependence on βm follows from dimensional analysis alone, we conclude that the over-all numerical constant in (3.20) represents the actual value to within 2%. Therefore, our analytic evaluation of (3.23), which follows from (3.21) and (A3), must be accurate to better than 2%. Since ΔE, given by (2.8), only requires (3.23) for its evaluation, we can be confident that the numerical values presented in Sec. IV correctly represent our model, and are not an artifact of the variational calculation.

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