# Enhancement of asymmetry in lepton annihilation due to resonances

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We show that the asymmetry in $e + \bar{e} \to 2\pi$, $3\pi$ due to interference between one-photon and two-photon terms is quite large near the $f$, $A_1$, and $A_2$ resonance positions, but goes through a zero. The analysis of this asymmetry can yield important information about the resonances and theories describing them.

## I. INTRODUCTION

The lepton annihilation process is a very rich and relatively clean source of information about weak, electromagnetic, and strong interactions. For example, the study of asymmetry in $e\bar{e}$ annihilation into hadron resonances can provide evidence for parity-violating interaction[1] due to neutral currents in gauge theories.[2] It can also provide interesting knowledge about the interference between the charge conjugation $C = -1$ hadron states produced by the one-photon intermediate state and the $C = +1$ hadron states produced by the two-photon intermediate states. The observation of this effect in terms of asymmetries in the states produced is important for several reasons:

(i) It would be a test of quantum electrodynamics.

(ii) It would allow us to determine the properties of $C = +1$ hadron resonances, including their quantum numbers, without having to worry about the side effects due to "nonaligned" hadrons.

(iii) At the resonance positions, the interference depends only on the absorptive part of the two-photon amplitude and hence can be estimated reliably, without having to depend unduly on the strong-interaction models.

(iv) Away from the resonances, it provides us with a test of the validity of the techniques for the off-shell continuation of photon amplitudes.

The asymmetries due to the interference between the one-photon and two-photon amplitudes are generally expected to be of the order of the fine-structure constant $\alpha$, i.e., about 1%. However, because of some infrared effects, these asymmetries may be significantly larger.[3] For example, the forward-backward asymmetry in $e + \bar{e} \to \pi^+ + \pi^-$, due to the interference between the one-photon amplitude and the nonresonant two-photon amplitude, and the interference between the electron and pion bremsstrahlung amplitudes, is[3] about 3.5%. It is interesting to note that the asymmetries may be considerably enhanced near the $C = +1$ resonance positions and may be as large as 20%, in which case they would be easily observable.

At the $C = +1$ resonance positions, the one-photon amplitude interferes primarily with the absorptive part of the resonant two-photon amplitude. Away from the resonance, however, the dispersive part of the two-photon resonant amplitude, which is quite large, will give the dominant contribution to the asymmetry. Unfortunately, the evaluation of the dispersive part of the two-photon amplitude is quite complicated. However, one can obtain the leading contributions coming from the soft-photon region, and the implications of these results are expected to be fairly reliable.

In this note, we discuss the results of an analysis of the asymmetry due to the interference between one-photon and two-photon amplitudes for

$$e(p_1) + \bar{e}(p_2) \to \pi^+(k_1) + \pi^-(k_2) \quad (1)$$

$$\to \pi^+(k_1) + \pi^-(k_2) + \pi^0(k_3) \quad (2)$$

in the region of $C = +1$ resonances with $J = 0, 1, 2$. The lepton mass is neglected in all our calculations.

One first notes that the spin-averaged interference between the $J = 0$ resonance term and the single photon term is proportional to the lepton mass and is neglected for the processes (1) and (2). The main results of the analysis are the following:

(i) The asymmetry in the forward and backward events for process (1), near the $f$-meson resonance,[4] is of the order of 15%, but goes to zero within about half a width from the resonance.

(ii) The asymmetry, as a function of the azimuthal angle defined later, for process (2) near the $A_2$-meson resonance is about 20% under certain restrictions, and goes to zero within a width away from the resonance.

(iii) The corresponding asymmetry for process (2) is nearly zero at the $A_1$-meson resonance, but is expected to be quite large, about 15%, away from the resonance.

(iv) The asymmetry for process (2) has another zero in between the positions of the $A_1$ and $A_2$ resonances, because of interference between the contributions from these resonances.

It would indeed be very useful to observe these effects, i.e., the large asymmetries and the zeros in them, experimentally.

## II. ASYMMETRY IN $e(p_1)+\bar{e}(p_2)\to\pi^+(k_1)+\pi^-(k_2)$

With the assumption of vector-meson dominance, the one-photon amplitude for this process is

$$T_1=e^2\bar{v}(p_2)(\not k_1-\not k_2)u(p_1)\frac{m_\rho^2}{s(m_\rho^2-s-i\Gamma_\rho)}, \tag{3}$$

where $p=p_1+p_2$, $s=p^2$, $m_\rho$ is the $\rho$-meson mass, and $\Gamma_\rho$ is related to its width. The region of major interest for the analysis of the asymmetry in this process is around the $f$-meson resonance with $I=0$ and $J=2$, which is produced through the two-photon intermediate state. Therefore one must calculate the two-photon amplitude including the final-state interaction in the $I=0$, $J=2$ $\pi\pi$ state. It may be noted that at the resonance, the asymmetry is dominantly due to the interference of the one-photon amplitude with the "absorptive" part of the two-photon $I=0$, $J=2$ resonant amplitude, while away from the resonance, it is due mainly to the interference of the one-photon amplitude with the "dispersive" part of the two-photon, resonant and nonresonant amplitudes. We evaluate the required absorptive and dispersive parts of the two-photon amplitude separately.

*Absorptive part.* The absorptive part of the required two-photon amplitude is obtained from Cutkosky rules[5]:

$$\mathrm{abs}T_2=\frac{e^2}{8\pi^2}\int d^4q_1d^4q_2\delta^4(p-q_1-q_2)\delta(q_1^2)\theta(q_{10})\delta(q_2^2)\theta(q_{20})\bar{v}(p_2)\gamma_\nu\frac{1}{\not p_1-\not q_1}\gamma_\mu u(p_1)B_{\mu\nu}, \tag{4}$$

$$B_{\mu\nu}=(q_1\cdot q_2g_{\mu\nu}-q_{2\mu}q_{1\nu})B_1+[(q_2\cdot Q)^2g_{\mu\nu}-Q_\mu Q_\nu q_1\cdot q_2+Q_\mu q_{1\nu}q_2\cdot Q-Q_\nu q_{2\mu}q_2\cdot Q]B_2, \tag{5}$$

where $Q=\frac{1}{2}(k_1-k_2)$ and $B_1,B_2$ are invariant amplitudes for the process

$$\gamma_\mu(q_1)+\gamma_\nu(q_2)\to\pi^+(k_1)+\pi^-(k_2) \tag{6}$$

described by $B_{\mu\nu}$. The helicity amplitudes[6] are related to these invariant amplitudes by

$$T_{1,1}=-\frac{s}{2}\left(B_1+\frac{s-4\mu^2}{8}B_2\right),\qquad T_{1,-1}=\frac{s(s-4\mu^2)}{16}(\sin^2\theta')B_2, \tag{7}$$

where the subscripts for $T$ describe the helicities of $\gamma_\mu(q_1)$ and $\gamma_\nu(q_2)$, $\mu$ is the pion mass, and $\theta'$ is the scattering angle in the center-of-mass (c.m.) frame. We have used the representations

$$\epsilon_{+1}(q_1)\equiv(0,-1/\sqrt{2},-i/\sqrt{2},0),\qquad \epsilon_{-1}(q_1)\equiv(0,1/\sqrt{2},-i/\sqrt{2},0),\qquad \epsilon_{\pm1}(q_2)=-\epsilon_{\mp1}(q_1) \tag{8}$$

for the photon polarization vectors with helicity $\pm1$.

The Born terms for the helicity amplitudes, due to pion-exchange diagrams and the seagull diagram, are

$$T^B_{1,1}=-e^2|\vec{k}_1|^2(\sin^2\theta')\left(\frac{1}{q_1\cdot k_1}+\frac{1}{q_2\cdot k_1}\right)+2e^2,\qquad T^B_{1,-1}=e^2|\vec{k}_1|^2(\sin^2\theta')\left(\frac{1}{q_1\cdot k_1}+\frac{1}{q_2\cdot k_1}\right), \tag{9}$$

where $|\vec{k}_1|=\frac{1}{2}(s-4\mu^2)^{1/2}$.

For obtaining the contribution of the $f$ meson to $T_{1,1}$ and $T_{1,-1}$ we first carry out the partial-wave projection of $T^B_{1,1}$ and $T^B_{1,-1}$ and retain only the $J=2$ term. This results in

$$T^B_{1,1}\approx-\tfrac{5}{8}(1-3\cos^2\theta')\int_{-1}^{1}d\cos\theta(1-3\cos^2\theta)(e^2|\vec{k}_1|^2\sin^2\theta)\left(\frac{1}{q_1\cdot k_1}+\frac{1}{q_2\cdot k_1}\right), \tag{10}$$

which tends to zero as $\mu \to 0$, and

$$T^B_{1,-1} \approx \tfrac{15}{16}\sin^2\theta' \int_{-1}^{1} d\cos\theta(\sin^2\theta)(e^2|\vec{k}_1|^2\sin^2\theta)\left(\frac{1}{q_1\cdot k_1}+\frac{1}{q_2\cdot k_1}\right). \tag{11}$$

These relations, according to (7), lead us to the corresponding invariant amplitudes

$$B_2^B \approx -\frac{30e^2}{s(s-4\mu^2)}\left[-\frac{10}{3}+\frac{s}{2|\vec{k}_1|^2}+\left(1-\frac{s}{4|\vec{k}_1|^2}\right)^2\frac{2|\vec{k}_1|}{s^{1/2}}\ln\frac{s^{1/2}-2|\vec{k}_1|}{s^{1/2}+2|\vec{k}_1|}\right] \tag{12}$$

and, for $\mu \to 0$,

$$B_1^B \approx -\frac{s}{8}B_2^B. \tag{13}$$

However, this $B_1^B$, which is independent of the scattering angle $\theta$, does not contribute to the absorptive part (4) of $T_2$.

The final-state interaction of the two pions is incorporated by writing dispersion relations[7] for $sB_2^{I=0}(s)D_{\pi\pi}(s)$, where $B_2^{I=0}(s)$ is the $I=0$ projection of $B_2$ and $D_{\pi\pi}(s)$ is the $D$ function for the $I=0$, $J=2$ partial-wave $\pi\pi$ scattering amplitude. This is justified by the analyticity properties of $B_2^B$ in (12). Approximating the left-hand discontinuity by the $I=0$ projection of $B_2^B$, we get

$$sB_2(s)D_{\pi\pi}(s)=\frac{2}{3\pi}\int_{-\infty}^{0}\frac{s'D_{\pi\pi}(s')\,\mathrm{Im}B_2^B(s')}{s'-s}ds', \tag{14}$$

where the factor $\frac{2}{3}$ is due to isospin projections. Now, because of the steep behavior of (12) near $s\approx 0$, we may put $D_{\pi\pi}(s')\approx D_{\pi\pi}(0)$ and obtain

$$B_2(s)\approx\frac{D_{\pi\pi}(0)}{D_{\pi\pi}(s)}\tfrac{2}{3}B_2^B(s). \tag{15}$$

We saturate the $\pi\pi$ partial-wave scattering amplitude with the $f$ meson so that we have, in the region of interest,

$$B_2(s)\approx\left(\frac{m_f^2}{m_f^2-s-i\Gamma_f}\right)\tfrac{2}{3}B_2^B(s), \tag{16}$$

where $m_f$ is the mass of the $f$ meson and $\Gamma_f$ is related to its width, and this allows us to calculate the absorptive part of the amplitude in (4). Finally, in the limit of $\mu\to 0$, one has

$$\mathrm{abs}T_2\approx e^2\bar{v}(p_2)(\not{k}_1-\not{k}_2)u(p_1)\frac{1}{s}\left(-i\frac{10\alpha m_f^2\cos\theta}{9(m_f^2-s-i\Gamma_f)}\right), \tag{17}$$

where $\alpha$ is the fine-structure constant, and $\theta$ is the scattering angle for process (1) in the c.m. frame.

*Dispersive part.* For evaluating the dispersive part of the amplitude, we use the soft-photon calculation of Brown and Mikaelian.[3] They show that, in the soft-photon approximation, the amplitude is essentially a point-pion amplitude multiplied by the single-photon form factor.

Consider the point-pion two-photon amplitude without the final-state interaction, which we call $T_2^0$:

$$T_2^0=-\frac{ie^4}{(2\pi)^4}\int\frac{d^4q_1}{q_1^2}\frac{d^4q_2}{q_2^2}\delta^4(q_1+q_2-p)\bar{v}(p_2)\gamma_\nu\frac{1}{\not{p}_1-\not{q}_1-m}\gamma_\mu u(p_1)$$
$$\times\left[\frac{(2k_1-q_1)_\mu(-2k_2+q_2)_\nu}{(k_1-q_1)^2-\mu^2}+\frac{(2k_2-q_1)_\mu(-2k_1+q_2)_\nu}{(k_2-q_1)^2-\mu^2}-2g_{\mu\nu}\right], \tag{18}$$

where $m$ is the lepton mass. This integral is quite complicated to evaluate. However, one can easily obtain the leading contributions which one expects to come from the soft-photon region, i.e., $q_1\approx 0$ or $q_2\approx 0$, or the soft-lepton region, i.e., $q_1 \approx p_1$, or the soft-pion region, i.e., $q_1 \approx k_1$ or $q_1 \approx k_2$. The leading contribution, including the soft-photon bremsstrahlung processes, on multiplication by the pion form factor will give[3]

$$T_2^0 \approx e^2 \bar{v}(p_2)(\not{k}_1 - \not{k}_2)u(p_1)\, \frac{1}{s}\frac{m_\rho^2}{m_\rho^2 - s - i\Gamma_\rho}\left[-\frac{e^2}{2\pi^2}\ln\left(\frac{s}{\lambda}\right)\ln(\tan\tfrac{1}{2}\theta)\right] \tag{19}$$

where $\lambda = 4(\Delta E)^2$, $\Delta E$ is the maximum energy of the soft photon emitted, and is taken to be 1% of $\frac{1}{2}s^{1/2}$. The leading contributions of the $\ln(s/m^2)$ or $\ln(s/\mu^2)$ type are not present in the soft-lepton and the soft-pion contributions to the amplitude.

The final-state interaction is introduced in the $I = 0$, $J = 2$ state by subtracting the $I = 0$, $J = 2$ projection from (19), and adding a resonant $I = 0$, $J = 2$ amplitude, analogous to the absorptive part. The resulting dispersive part of the two-photon amplitude is

$$\begin{aligned} \mathrm{disp}T_2 = {} & e^2 \bar{v}(p_2)(\not{k}_1 - \not{k}_2)u(p_1)\, \frac{1}{s}\frac{m_\rho^2}{m_\rho^2 - s - i\Gamma_\rho} \\ & \times\left[-\frac{e^2}{2\pi^2}\ln\left(\frac{s}{\lambda}\right)\right]\left[\ln(\tan\tfrac{1}{2}\theta) + \tfrac{5}{6}\cos\theta - \tfrac{5}{6}(\cos\theta)\frac{m_f^2}{m_f^2 - s - i\Gamma_f}\right] \end{aligned} \tag{20}$$

*Asymmetry.* From the expressions (3), (17), and (20), one has, for the total amplitude for process (1),

$$\begin{aligned} T \approx e^2 \bar{v}(p_2)(\not{k}_1 - \not{k}_2)u(p_1)\, \frac{1}{s}\Bigg\{ & \left(\frac{m_\rho^2}{m_\rho^2 - s - i\Gamma_\rho}\right)\left[1 - \frac{2\alpha}{\pi}\ln\left(\frac{s}{\lambda}\right)\left(\ln(\tan\tfrac{1}{2}\theta) + \tfrac{5}{6}\cos\theta - \tfrac{5}{6}\cos\theta\frac{m_f^2}{m_f^2 - s - i\Gamma_f}\right)\right] \\ & - i\,\frac{10\alpha\cos\theta\, m_f^2}{9(m_f^2 - s - i\Gamma_f)}\Bigg\}\,. \end{aligned} \tag{21}$$

The spin-averaged differential cross section from this amplitude, to order $\alpha^3$, is

$$\frac{d\sigma}{d\cos\theta} = \frac{d\sigma_0}{d\cos\theta}(1 + \delta_1 + \delta_2), \tag{22}$$

where

$$\frac{d\sigma_0}{d\cos\theta} = \left(\frac{\pi\alpha^2\sin^2\theta}{4s}\right)\frac{m_\rho^4}{(m_\rho^2 - s)^2 + \Gamma_\rho^2},$$

$$\delta_1 = -2\mathrm{Re}\left[\frac{2\alpha}{\pi}\ln\left(\frac{s}{\lambda}\right)[\ln(\tan\tfrac{1}{2}\theta) + \tfrac{5}{6}\cos\theta]\right],$$

$$\delta_2 = +2\mathrm{Re}\left\{\left[\frac{2\alpha}{\pi}\ln\left(\frac{s}{\lambda}\right)\tfrac{5}{6}\cos\theta + i\,\frac{10\alpha\cos\theta}{9m_\rho^2}(m_\rho^2 - s + i\Gamma_\rho)\right]\frac{m_f^2}{m_f^2 - s + i\Gamma_f}\right\}.$$

A more rigorous calculation should also include symmetric contributions due to vertex and propagator corrections which, as has been noted by Brown and Mikaelian,[3] may change the asymmetry significantly, say by 20–25%. The approximations introduced in the analysis of the final-state strong interaction also may introduce errors of this order.

At the $f$ meson resonance, the contribution to the asymmetry is primarily from the absorptive part, since $\delta_1$ is quite small, as also the first term in $\delta_2$. The value of $\delta_2$ for $\theta \approx 0$ is 0.25 at $s \approx m_f^2$ which is in good agreement with an earlier estimate[8] based on the universal coupling of the $f$ meson and another[9] based on the prediction of the quark model, but disagrees with the estimate[10] based on the Cabibbo-Radicati sum rule or finite-energy sum rules and duality. Our result appears to support the universal coupling of the $f$ meson.

Away from the resonance, the contribution of the dispersive term to $\delta_2$ begins to dominate, while $\delta_1$ remains small. Furthermore, $\delta_2$ has a zero near $s \approx m_f^2$ so that the asymmetry itself is found to have a zero at about $s \approx m_f^2 - \frac{1}{4}\Gamma_f$ . It is convenient to represent this behavior in terms of an asymmetry parameter

$$A = \frac{F-B}{F+B}, \quad (23)$$

where $F$ and $B$ are the forward and backward events, respectively. We find that $A \approx 9\%$ of the partial cross section of $3\times10^{-33}$ cm$^2$, at the $f$-meson position, with $m_f = 1270$ MeV and $\Gamma_f = 170 m_f$ (MeV)$^2$, but $A \approx 0$ at $s^{1/2} \approx 1230$ MeV. The expected behavior of the asymmetry parameter near the resonance is shown in Fig. 1, for two cases, one for $\theta < \pi/4$ or $\theta > 3\pi/4$, and another for all allowed values of $\theta$.

The detailed observation of the rather striking behavior of the asymmetry parameter which is quite large, about $\pm 15\%$, but goes through a zero near the position of the $f$-meson resonance, would provide important information about the amplitude for $f \to 2\gamma$ and a test for off-shell effects.

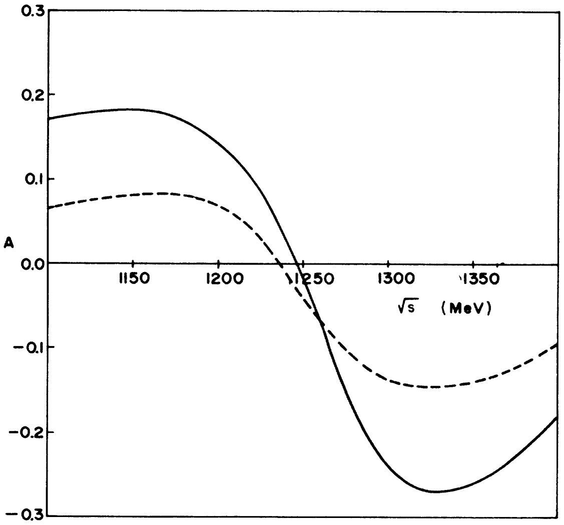


FIG. 1. Asymmetry $A$ in $e+\bar{e} \to \pi^+ + \pi^-$. The dashed line is for all allowed values of $\theta$, while the solid line is for $\theta < \pi/4$ or $\theta > 3\pi/4$.

## III. ASYMMETRY IN $e(p_1) + \bar{e}(p_2) \to \pi^+(k_1) + \pi^-(k_2) + \pi^0(k_3)$

With the assumption of vector-meson dominance, the one-photon amplitude for this process is

$$T_1 = ef\bar{v}(p_2)\gamma_\mu u(p_1)\,\frac{1}{s}\epsilon_{\mu\alpha\beta\gamma}k_{1\alpha}k_{2\beta}k_{3\gamma}$$

with

$$f = \frac{em_\omega^2 \sin\theta_m}{3^{1/2}(s-m_\omega^2)} \sum_{i=1}^{3} \frac{f_0}{s_i - m_\rho^2 + i\Gamma_\rho},$$

where $m_\omega$ is the $\omega$-meson mass, $\theta_m$ is the mixing angle for the vector nonet, $s_i = (p-k_i)^2$, and $f_0$ is determined from $\omega$ decay to be $f_0^2/4\pi \approx 2/\mu^2$. Here $\phi$ coupling to $3\pi$ is neglected. The region of interest for the analysis of asymmetry in this process corresponds to the positions of the $A_1$ and $A_2$ resonances. These resonances are produced through the two-photon intermediate state. They decay into the $\pi\rho$ state and then, subsequently, the $\rho$ meson decays into $2\pi$ mesons. We first consider the region around the $A_2$ resonance.

*$A_2$ region.* The two-photon amplitude is calculated in two parts, one for the absorptive part and one for the dispersive part.

For calculating the absorptive part, consider the helicity amplitudes for the process

$$e(p_1) + \bar{e}(p_2) \to \rho^+(k) + \pi^-(k_2), \quad (24)$$

where $k = k_1 + k_3$. The absorptive part of the amplitude for this process, with two photons in the intermediate state, is given by

$$\text{abs}T^f_{i,j}(k,k_2;p_1,p_2) = \frac{1}{16\pi^2}\sum_{m,n}\int \frac{d^3q_1}{2q_{10}}\frac{d^3q_2}{2q_{20}} A^{m,n}_{i,j}(q_1,q_2;p_1,p_2) B^f_{m,n}(k,k_2;q_1,q_2)\,\delta^4(p-q_1-q_2), \quad (25)$$

where the $A$'s are the helicity amplitudes for $e(p_1)+\bar{e}(p_2) \to \gamma_\mu(q_1) + \gamma_\nu(q_2)$, the $B$'s are the helicity amplitudes for $\gamma_\mu(q_1)+\gamma_\nu(q_2) \to \rho^+_\alpha(k) + \pi^-(k_2)$, and $i$, $j$, $m$, $n$, and $f$ are the helicities of $e(p_1)$, $\bar{e}(p_2)$, $\gamma_\mu(q_1)$, $\gamma_\nu(q_2)$, and $\rho^+_\alpha(k)$, respectively. The Born terms for the amplitude $A$ are from the lepton exchange diagrams,

$$A_{\mu\nu} = e^2\bar{v}(p_2)\left[\gamma_\nu \frac{1}{\not{p}_1 - \not{q}_1}\gamma_\mu + \begin{pmatrix} \mu \leftrightarrow \nu \\ q_1 \leftrightarrow q_2 \end{pmatrix}\right]u(p_1), \quad (26)$$

while those for $B$ are, as shown in Fig. 2, from the pion and $\rho$ exchanges and a seagull diagram needed for gauge invariance of the amplitude, i.e.,

$$B_{\mu\nu\alpha} = eg_\rho\left[\frac{\epsilon_{\mu\alpha\beta\delta}q_{1\beta}k_\delta(2k_2-q_2)_\nu}{-2q_2\cdot k_2} - \frac{\epsilon_{\alpha\nu\beta\delta}q_{2\beta}k_{2\delta}(2k-q_1)_\mu}{-2q_1\cdot k} + \begin{pmatrix} \mu \leftrightarrow \nu \\ q_1 \leftrightarrow q_2 \end{pmatrix} - \epsilon_{\mu\alpha\nu\delta}(q_1-q_2)_\delta\right], \quad (27)$$

$g_\rho$ being the $\gamma\pi^-\rho^+$ coupling constant, whose value is $f_0 e \sin\theta_m/(2(3)^{1/2}g_{\rho\pi\pi})$. We calculate the helicity amplitudes for $A$ and $B$ and project out the partial wave $J=2$ to get

$$
\begin{aligned}
&A^{1,-1}_{1/2,-1/2}=-\mathfrak{D}^{2*}_{1,2}(\theta_1,\phi_1)\left(\frac{5e^2}{m}\right), \quad A^{-1,1}_{1/2,-1/2}=-\mathfrak{D}^{2*}_{1,-2}(\theta_1,\phi_1)\left(\frac{5e^2}{3m}\right),\\
&A^{1,-1}_{-1/2,1/2}=\mathfrak{D}^{2*}_{-1,2}(\theta_1,\phi_1)\left(\frac{5e^2}{3m}\right), \quad A^{-1,1}_{-1/2,1/2}=\mathfrak{D}^{2*}_{-1,-2}(\theta_1,\phi_1)\left(\frac{5e^2}{m}\right),\\
&B^{1}_{1,1}=\mathfrak{D}^{2*}_{0,1}(\theta_2,\phi_2)[\tfrac{5}{2}(\tfrac{3}{2})^{1/2}t^{+}_{++}], \quad B^{0}_{1,1}=\mathfrak{D}^{2*}_{0,0}(\theta_2,\phi_2)(\tfrac{5}{4}t^{0}_{++}),\\
&B^{1}_{1,-1}=\mathfrak{D}^{2*}_{2,1}(\theta_2,\phi_2)(\tfrac{5}{4}t^{+}_{+-}), \quad B^{-1}_{1,-1}=-\mathfrak{D}^{2*}_{2,-1}(\theta_2,\phi_2)(\tfrac{5}{4}t^{+}_{+-}),\\
&B^{1}_{-1,1}=\mathfrak{D}^{2*}_{-2,1}(\theta_2,\phi_2)(\tfrac{5}{4}t^{+}_{+-}), \quad B^{-1}_{-1,1}=-\mathfrak{D}^{2*}_{-2,-1}(\theta_2,\phi_2)(\tfrac{5}{4}t^{+}_{+-}),\\
&B^{0}_{-1,-1}=-\mathfrak{D}^{2*}_{0,0}(\theta_2,\phi_2)(\tfrac{5}{4}t^{0}_{++}), \quad B^{-1}_{-1,-1}=-\mathfrak{D}^{2*}_{0,-1}(\theta_2,\phi_2)(\tfrac{5}{2}\sqrt{3/2}\,t^{+}_{++}),
\end{aligned}
\tag{28}
$$

the remaining amplitudes being zero (we have neglected the lepton and pion masses, which is justified at the energies under consideration). In the above expressions, the $\mathfrak{D}$'s are the Wigner $\mathfrak{D}$ functions, $\theta_1,\phi_1$ are the scattering and azimuthal angles in the center-of-mass frame for the process $e\bar{e}\to\gamma\gamma$, $\theta_2,\phi_2$ are the corresponding angles for $\gamma\gamma\to\rho^+\pi^-$, and

$$
\begin{aligned}
t^{+}_{++}&=-\frac{eg_\rho i m_\rho^2}{2s^{1/2}}\left\{\frac{4}{3}+\left[\frac{8}{3}+x\left(-4x+2(1-x^2)\ln\frac{x-1}{x+1}\right)\right]\right\},\\
t^{0}_{++}&=\frac{eg_\rho i m_\rho x}{x-1}\left(4-2(3x^2-1)+\frac{(3x^2-1)(x^2-1)}{x}\ln\frac{x-1}{x+1}\right),\\
t^{+}_{+-}&=-\frac{eg_\rho i}{2(2s)^{1/2}}\left[-\frac{4s}{3}+4m_\rho^2+(s-m_\rho^2)\left(-\tfrac{4}{3}(2-x)-2x(1-x)^2+(1-x)^3(1+x)\ln\frac{x-1}{x+1}\right)\right],
\end{aligned}
\tag{29}
$$

with $x=(s+m_\rho^2)/(s-m_\rho^2)$. The final-state interaction is taken into account in a way similar to that for $\gamma+\gamma\to\pi^++\pi^-$, which leads to a multiplication of the $B$'s by a factor $m_{A_2}^2/(m_{A_2}^2-s-i\Gamma_{A_2})$, where $m_{A_2}$ is the mass of the $A_2$ meson and $\Gamma_{A_2}$ is related to its width. We substitute these $A$ and $B$ functions into (25), express the functions of $\theta_2,\phi_2$ in terms of functions of $\theta_1,\phi_1$ and of $\theta',\phi'$ (where $\theta',\phi'$ are the center-of-mass scattering and azimuthal angles for the process $e\bar{e}\to\rho^+\pi^-$) by using the relations

$$
\mathfrak{D}^{J*}_{\mu\mu'}(\theta_2,\phi_2)=\sum_{M=-J}^{J}\mathfrak{D}^{J*}_{M\mu'}(\theta',\phi')\mathfrak{D}^{J}_{M\mu}(\theta_1,\phi_1),
\tag{30}
$$

and then carry out the integrations in (25). From the orthogonality of the $\mathfrak{D}$ functions for the integration over the angles $\theta_1,\phi_1$ of $\vec{q}_1$, we find that the only nonvanishing absorptive parts are

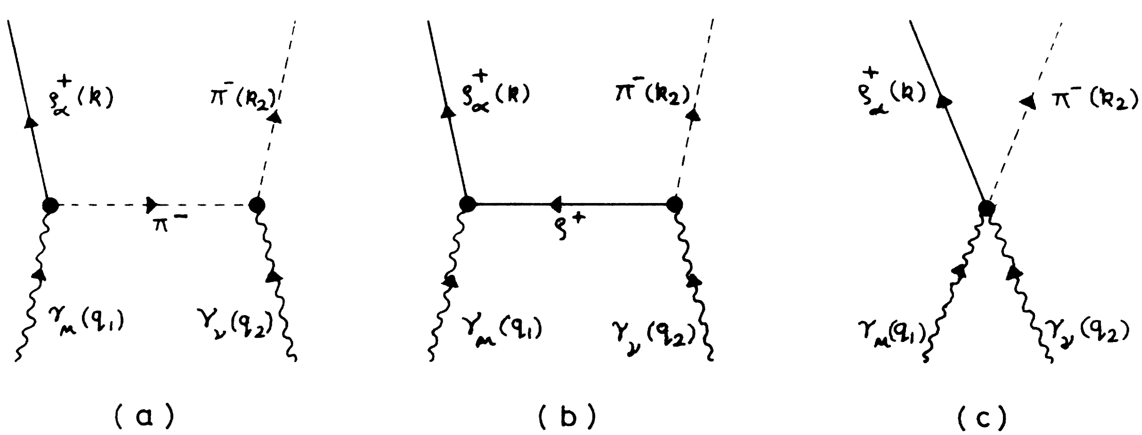


FIG. 2. Born diagrams for $\gamma_\mu(q_1)+\gamma_\nu(q_2)\to\rho^+_\alpha(k)+\pi^-(k_2)$. Diagrams obtained from (a) and (b) by interchanging the two photons should also be included.

$$\begin{aligned}\text{abs}T^{1}_{1/2,-1/2} &= \text{abs}T^{-1}_{-1/2,1/2}\\ &= t(s)\mathfrak{D}^{2*}_{1,1}(\theta',\phi'),\\ \text{abs}T^{-1}_{1/2,-1/2} &= -t(s)\mathfrak{D}^{2*}_{1,-1}(\theta',\phi'),\\ \text{abs}T^{1}_{-1/2,1/2} &= -t(s)\mathfrak{D}^{2*}_{-1,1}(\theta',\phi'),\end{aligned} \tag{31}$$

where

$$t(s) = -\frac{5e^2}{96\pi m}t^{+}_{+-}\left(\frac{m_{A_2}^{\ 2}}{m_{A_2}^{\ 2}-s-i\Gamma_{A_2}}\right) \tag{32}$$

and the $\mathfrak{D}$ functions are given by

$$\begin{aligned}\mathfrak{D}^{2}_{1,1}(\theta',\phi') &= \tfrac{1}{2}(1+\cos\theta')(2\cos\theta'-1),\\ \mathfrak{D}^{2}_{1,-1}(\theta',\phi') &= \mathfrak{D}^{2*}_{-1,1}(\theta',\phi')\\ &= \tfrac{1}{2}(1-\cos\theta')(2\cos\theta'+1)e^{2i\phi'}.\end{aligned} \tag{33}$$

Henceforth, we choose $\phi'=0$, since we will not be interested in the reactions with polarized $e\bar{e}$ beams.

Finally, the amplitude for annihilation into $\pi^+\pi^-\pi^0$ is obtained by multiplying the amplitudes in (31) by the $\rho$-meson propagator and the proper helicity projections of the amplitude for $\rho_\alpha(k)\to\pi^+(k_1)+\pi^0(k_3)$, i.e., by

$$\epsilon^\alpha T_\alpha = \frac{g_{\rho\pi\pi}\epsilon^\alpha(k_3-k_1)_\alpha}{s_2-m_\rho^{\ 2}+i\Gamma_\rho}, \tag{34}$$

and then symmetrizing with respect to $\pi^+$ and $\pi^-$. Note that we are using the amplitudes for the $\rho$ meson on the mass shell. This is justified, since the dominant contribution comes from the kinematic region where $s_1\approx s_2\approx m_\rho^{\ 2}$.

At this stage, it is convenient to choose a frame, which we call the symmetric center-of-mass frame, in which the $z$ axis is along the direction of $\vec{k}_1\times\vec{k}_2$, and the $x$ axis is along the direction of $\vec{k}_3$. Defining, in this frame, $\theta$ and $\phi$ as the polar and azimuthal angles of $\vec{p}_1$, and $\phi_1,\phi_2$ as the azimuthal angles of $\vec{k}_1,\vec{k}_2$, we have

$$\cos\theta' = -\sin\theta\cos(\phi-\phi_2), \tag{35}$$

and corresponding to helicities $\pm 1$ for $\rho^+$,

$$\epsilon_{\pm}\circ(k_3-k_1) = \frac{2^{1/2}|\vec{k}_1|\sin(\phi_2-\phi_1)}{\sin\theta'}\times[\mp\sin\theta\sin(\phi_2-\phi)+i\cos\theta]. \tag{36}$$

Using these equations in (34) we carry out the substitution and symmetrization described in the previous paragraph. This would give us the two-photon absorptive parts of the helicity amplitudes for $e\bar{e}\to 3\pi$.

For estimating the dispersive part of the two-photon amplitude, consider the amplitude for the process (24) but without strong interaction, which we call $(T^0_2)_\alpha$:

$$(T^0_2)_\alpha = \frac{-i}{2(2\pi)^4}\int\frac{d^4q_1}{q_1^{\ 2}}\frac{d^4q_2}{q_2^{\ 2}}\delta^4(q_1+q_2-p)A_{\mu\nu}B_{\mu\nu\alpha}. \tag{37}$$

The leading contribution to this integral comes from the soft-photon region, i.e., $q_1\approx 0$ or $q_2\approx 0$. On multiplying the leading contribution, including the soft-photon bremsstrahlung processes, by the form factor for the $\gamma\pi\rho$ vertex, we obtain the dispersive part of the amplitude for process (24) without final-state interaction:

$$\text{disp}(T^0_2)_\alpha = eg_\rho\bar{v}(p_2)\gamma_\mu u(p_1)\frac{1}{s}\epsilon_{\mu\alpha\beta\delta}k_{2\beta}k_\delta\frac{m_\omega^{\ 2}}{(m_\omega^{\ 2}-s)}\left[-\frac{e^2}{4\pi^2}\ln\left(\frac{s}{\lambda}\right)\ln\left(\frac{x-\cos\theta'}{y+\cos\theta'}\right)\right], \tag{38}$$

where $x\approx(s+m_\rho^2)/(s-m_\rho^2)$, $y\approx 1$, $\theta'$ is defined in (30), and $\lambda=4(\Delta E)^2$; $\Delta E$ is the maximum energy of the soft photon emitted and is taken to be about 1% of $s^{1/2}/2$. The final-state interaction in the $J=2$ positive-parity state is introduced by replacing the $J=2$, positive-parity parts of the helicity amplitudes by the resonant forms, analogous to the $f$-meson case. Near the $A_2$ resonance, the resulting helicity amplitudes which are nonzero are

$$\begin{aligned}\text{disp}T^{1}_{1/2,-1/2} &= \text{disp}T^{-1}_{-1/2,1/2}\\ &= i\tau(s)(1+\cos\theta')\left[\ln\left(\frac{1+\cos\theta'}{2-\cos\theta'}\right)-1.1(2\cos\theta'-1)+1.1(2\cos\theta'-1)\frac{m_{A_2}^{\ 2}}{m_{A_2}^{\ 2}-s-i\Gamma_{A_2}}\right],\\ \text{disp}T^{-1}_{1/2,-1/2} &= \text{disp}T^{1}_{-1/2,1/2}\\ &= -i\tau(s)(1-\cos\theta')\left[\ln\left(\frac{1+\cos\theta'}{2-\cos\theta'}\right)-1.1(2\cos\theta'+1)+1.1(2\cos\theta'+1)\frac{m_{A_2}^{\ 2}}{m_{A_2}^{\ 2}-s-i\Gamma_{A_2}}\right],\end{aligned} \tag{39}$$

where

$$\tau(s)=\frac{e^3 g_\rho|\vec{k}_2|}{8\sqrt{2}\,\pi^2 m}\left(\frac{m_\omega^2}{m_\omega^2-s}\right)\ln\left(\frac{s}{\lambda}\right). \tag{40}$$

Finally, the amplitudes for the decay into $\pi^+\pi^-\pi^0$ are obtained by multiplying these amplitudes by the $\rho$-meson propagator and the proper helicity projections for the amplitudes for $\rho_\alpha(k)\to\pi^+(k_1)+\pi^0(k_3)$, as was done for the absorptive parts.

The spin-averaged differential cross section for process (2), in the $A_2$ region, is then given by

$$\frac{d^2\sigma}{d\cos\theta\, d\phi}=\frac{m^2}{64s^2(2\pi)^4}\int ds_1\int ds_2\sum_{\text{pol}}|T_1+T_2|^2, \tag{41}$$

where $T_1$ and $T_2$ are the one-photon and two-photon amplitudes, and the ranges of integration are $(\sqrt{s}-\mu)^2$ and $4\mu^2$ for $s_1$, and $2\mu^2+[\sqrt{s_1}k_{10}\pm|\vec{k}_1|(s_1-4\mu^2)^{1/2}]$ for $s_2$, where $|\vec{k}_1|=(s-s_1-\mu^2)/2s_1^{1/2}$, $k_{10}=(|\vec{k}_1|^2+\mu^2)^{1/2}$. In (41), the relevant terms up to order $\alpha^3$ are

$$\sum_{\text{pol}}|T_1|^2=\frac{e^2|f|^2}{2m^2}|\vec{k}_1|^2|\vec{k}_2|^2\sin^2(\phi_1-\phi_2)\sin^2\theta, \tag{42}$$

$$2\sum_{\text{pol}}\text{Re}(T_1^*\text{abs}T_2)=-2\text{Re}\bigg[\frac{5e\alpha}{12(2^{1/2})m^2}f^*g_{\rho\pi\pi}t^+_{+-}|\vec{k}_1|\,|\vec{k}_2|$$
$$\times\sin^2(\phi_1-\phi_2)\sin\theta\cos2\theta\left(\frac{|\vec{k}_1|\cos(\phi-\phi_2)}{s_2-m_\rho^2+i\Gamma_\rho}-\frac{|\vec{k}_2|\cos(\phi-\phi_1)}{s_1-m_\rho^2+i\Gamma_\rho}\right)\frac{m_{A_2}^2}{m_{A_2}^2-s-i\Gamma_{A_2}}\bigg], \tag{43}$$

$$2\sum_{\text{pol}}\text{Re}(T_1^*\text{disp}T_2)=\frac{8g_{\rho\pi\pi}ef^*}{\sqrt{2}m}\tau(s)|\vec{k}_1|^2|\vec{k}_2|\sin^2(\phi_1-\phi_2)$$
$$\times\bigg[\frac{\sin^2\theta}{s_2-m_\rho^2+i\Gamma_\rho}\ln\left(\frac{1-\sin\theta\cos(\phi-\phi_2)}{2+\sin\theta\cos(\phi-\phi_2)}\right)-\frac{\sin^2\theta}{s_1-m_\rho^2+i\Gamma_\rho}\ln\left(\frac{1-\sin\theta\cos(\phi-\phi_1)}{2+\sin\theta\cos(\phi-\phi_1)}\right)$$
$$+1.1\left(\frac{m_{A_2}^2}{m_{A_2}^2-s-i\Gamma_{A_2}}-1\right)(\sin\theta)(\cos2\theta)\left(\frac{\cos(\phi-\phi_2)}{s_2-m_\rho^2+i\Gamma_\rho}-\frac{\cos(\phi-\phi_1)}{s_1-m_\rho^2+i\Gamma_\rho}\right)\bigg]. \tag{44}$$

Evaluating the above expressions near the resonance, i.e., at $s^{1/2}\approx m_{A_2}=1310$ MeV, the differential cross section is of the form

$$\frac{d^2\sigma}{d\cos\theta\, d\phi}\approx a(\sin^2\theta+\alpha b\sin\phi\sin^3\theta+\alpha c\sin\phi\sin\theta\cos2\theta), \tag{45}$$

where

$$a\approx 2\times10^{-33}\ \text{cm}^2,$$
$$b\approx 3.8, \tag{46}$$
$$c\approx\frac{0.35\,m_{A_2}^2\Gamma_{A_2}}{(m_{A_2}^2-s)^2+\Gamma_{A_2}^2}-2.8\left(\frac{m_{A_2}^2(m_{A_2}^2-s)}{(m_{A_2}^2-s)^2+\Gamma_{A_2}^2}-1\right).$$

The asymmetry for this cross section is conveniently described in terms of the asymmetry parameter $\bar{A}$ defined as

$$\bar{A}=\frac{\int_0^\pi(d\sigma/d\phi)\,d\phi-\int_\pi^{2\pi}(d\sigma/d\phi)\,d\phi}{\int_0^\pi(d\sigma/d\phi)\,d\phi+\int_\pi^{2\pi}(d\sigma/d\phi)\,d\phi}. \tag{47}$$

This parameter is plotted in Fig. 3 as a function of $s^{1/2}$ for two cases, one for the region $\theta<\pi/6$ or $\theta>5\pi/6$ and the other for all allowed values of $\theta$, with $\Gamma_{A_2}\approx\frac{5}{7}\mu m_{A_2}$. Large asymmetries, of the order of 20%, with a zero in them near the $A_2$ resonance, are the striking results predicted. Observation of these effects would give us a better understanding of the $A_2$ resonance and the final-state interaction.

*$A_1$ region.* The region near $A_1$ is of special importance for the analysis of the asymmetry for process (2), because of the possible enhancement[1] of the effect of the weak neutral currents.

We first note that Bose statistics and angular-momentum conservation imply that the coupling of $A_1$ to two on-shell photons is zero. Therefore the absorptive part of the two-photon amplitude for process (2) through the $A_1$ meson is zero. For calculating the dispersive part, we start with the leading soft-photon expression (38) for the two-photon amplitude for process (24). The final-state interaction near the $A_1$ region is introduced by replacing the $J=1$ positive-parity parts of the helicity amplitudes by their resonant forms, analogous to the $f$ meson case. Near the $A_1$ resonance, the resulting helicity amplitudes which are nonzero are

$$\begin{aligned}\operatorname{disp} T^{1}_{1/2,-1/2} &= \operatorname{disp} T^{-1}_{-1/2,1/2} \\ &= i\tau(s)(1+\cos\theta')\left(\ln\frac{1+\cos\theta'}{2-\cos\theta'} + \frac{m_{A_1}{}^2}{m_{A_1}{}^2 - s - i\Gamma_{A_1}} - 1\right) \\ \operatorname{disp} T^{-1}_{1/2,-1/2} &= \operatorname{disp} T^{1}_{-1/2,1/2} \\ &= -i\tau(s)(1-\cos\theta')\left(\ln\frac{1+\cos\theta'}{2-\cos\theta'} - \frac{m_{A_1}{}^2}{m_{A_1}{}^2 - s - i\Gamma_{A_1}} + 1\right),\end{aligned} \tag{48}$$

where $m_{A_1}$ is the mass of $A_1$ meson and $\Gamma_{A_1}$ is related to its width, and $\theta'$ and $\tau(s)$ are defined in (30) and (40), respectively. As discussed earlier, the final amplitudes for the decay into $\pi^+\pi^-\pi^0$ are obtained by multiplying these helicity amplitudes by the $\rho$-meson propagator and the proper helicity projections for the amplitudes for $\rho^{+}_{\alpha}(k)\to\pi^+(k_1)+\pi^0(k_3)$. The differential cross section for process (2), in the $A_1$ region, is given to order $\alpha^3$ by

$$\frac{d^2\sigma}{d\cos\theta\, d\phi} = \frac{m^2}{64 s^2 (2\pi)^4}\int ds_1 \int ds_2 \sum_{\text{pol}} \left[|T_1|^2 + 2\operatorname{Re}(T_1^* \operatorname{disp} T_2)\right], \tag{49}$$

where $|T_1|^2$ is given in (42) and

$$\begin{aligned}\sum_{\text{pol}} 2\operatorname{Re}(T_1^*\operatorname{disp} T_2) = {}& \frac{8 g_{\rho\pi\pi} e f^*}{\sqrt{2}\, m}\tau(s)|\vec{k}_1|^2|\vec{k}_2|\sin^2(\phi_2-\phi_1) \\ &\times\left[\frac{\sin^2\theta}{s_2 - m_\rho{}^2 + i\Gamma_\rho}\ln\left(\frac{1-\sin\theta\cos(\phi-\phi_2)}{2+\sin\theta\cos(\phi-\phi_2)}\right) - \frac{\sin^2\theta}{s_1 - m_\rho{}^2 + i\Gamma_\rho}\ln\left(\frac{1-\sin\theta\cos(\phi-\phi_1)}{2+\sin\theta\cos(\phi-\phi_1)}\right)\right. \\ &\left. -\left(\frac{m_{A_1}{}^2}{m_{A_1}{}^2 - s - i\Gamma_{A_1}} - 1\right)(\sin\theta)\left(\frac{\cos(\phi-\phi_2)}{s_2 - m_\rho{}^2 + i\Gamma_\rho} - \frac{\cos(\phi-\phi_1)}{s_1 - m_\rho{}^2 + i\Gamma_\rho}\right)\right].\end{aligned} \tag{50}$$

Evaluating near the resonance, i.e., at $s^{1/2}\approx 1100$ MeV, the differential cross section is of the form

$$\frac{d^2\sigma}{d\cos\theta\, d\phi} \approx a'(\sin^2\theta + \alpha b' \sin\phi\sin^3\theta + \alpha c' \sin\phi\sin\theta), \tag{51}$$

where

$$\begin{aligned} a' &\approx 10^{-33}\ \text{cm}^2, \\ b' &\approx 3.8, \\ c' &\approx 2.5\left(\frac{m_{A_1}{}^2(m_{A_1}{}^2 - s)}{(m_{A_1}{}^2 - s)^2 + \Gamma_{A_1}{}^2} - 1\right). \end{aligned} \tag{52}$$

The asymmetry parameter $\bar{A}$, defined in (47), resulting from this cross section in the $A_1$ region, is plotted in Fig. 4 as a function of $s^{1/2}$, for two cases, one for the region $\theta<\pi/6$ or $\theta>5\pi/6$, and the other for all allowed values of $\theta$, with $\Gamma_{A_1} \approx \mu m_{A_1}$. Once again the asymmetry is quite large and goes through a zero near the $A_1$-resonance position. It may be noted that the experimental value of $\Gamma_{A_1}$ is still uncertain.[11] The detailed observation of $\bar{A}$ in this region may be a good

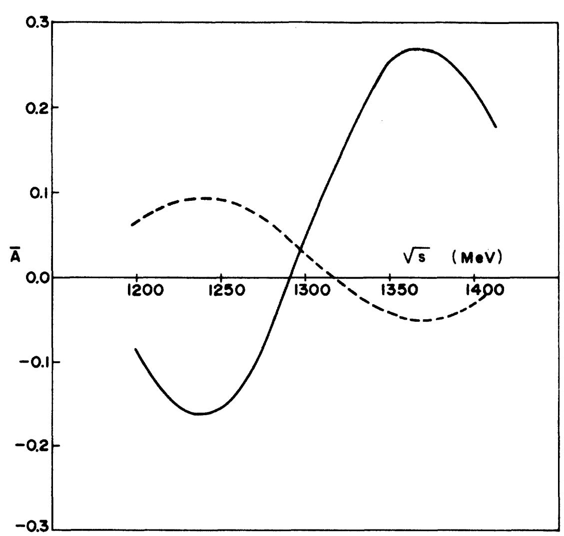


FIG. 3. Asymmetry $\bar{A}$ in $e^+ + e^- \to \pi^+ + \pi^- + \pi^0$ near the $A_2$-resonance region. The dashed line is for all allowed values of $\theta$, while the solid line is for $\theta<\pi/6$ or $\theta>5\pi/6$.

way of establishing the value of $\Gamma_{A_1}$ as well as other characteristics of the $A_1$ resonance.

A region of considerable interest for the analysis of the asymmetry is the region between the positions of $A_1$ and $A_2$ resonances. Since the positions of these two resonances are fairly close to each other, their production amplitudes will interfere in the region in between. Consequently, there will be another zero in $\bar{A}$, at about $s^{1/2} \approx 1200$ MeV, for the case of $\theta$ taking all allowed values. Needless to say, the analysis of asymmetry $\bar{A}$ for process (2), around and in between the positions of $A_1$ and $A_2$ resonances, will provide very valuable information about the resonances.

It may be noted that there is an asymmetry of about 5% introduced by the interference[1] with the possible weak neutral currents, but this interference is proportional to $\cos\theta$ and can be easily isolated by its parity violation, i.e., it changes sign under $\theta \to \pi - \theta$.

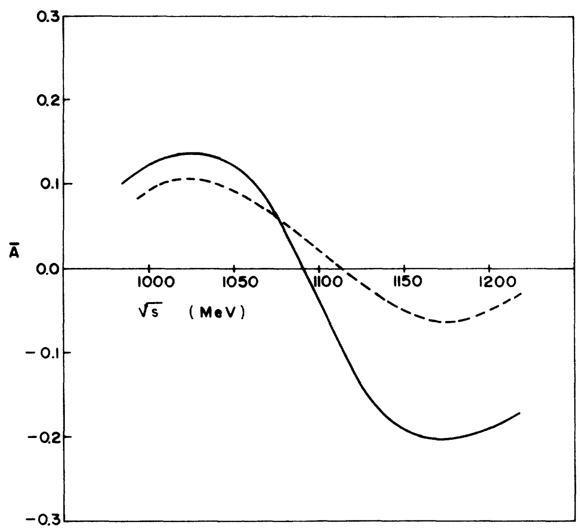


FIG. 4. Asymmetry $\bar{A}$ in $e^+ e^- \to \pi^+ + \pi^- + \pi^0$ near the $A_1$-resonance region. The dashed line is for all allowed values of $\theta$, while the solid line is for $\theta < \pi/6$ or $\theta > 5\pi/6$.

## IV. CONCLUSIONS

In conclusion, we feel that the asymmetry in lepton annihilation into hadrons, at or near $C = +1$ resonance energies, will be large but will go through a zero in the nearby region. Its analysis will give important information about the resonances and the framework of their analysis.

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