Dual four-point functions

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(Received 9 June 1975)

The properties of a phenomenologically useful dual amplitude and its similarity with composite-particle scattering amplitudes are discussed.

I. INTRODUCTION

In this paper we present a dual model for the elastic scattering of two scalar particles. This model gives a very good description of the differential cross section for pp and $p\bar{p}$ at high energies and large momentum transfers.[1] The scattering amplitude of this model has many of the properties expected of an amplitude describing the scattering of composite particles. The underlying dynamical basis for this model is not fully understood and in this paper we will describe how it was arrived at using considerations of duality. In order to do this we first briefly review the properties of a factorizable dual Born term B_N (see Ref. 2) containing nonlinear trajectories $\alpha(t)$ of the form

$$\alpha(t)=\frac{\log\tau'}{\log(q^{-1})}, \tag{1.1}$$

where

$$\tau'=a't+b, \tag{1.2}$$

and a', b, and q are parameters where $0<q<1$. In the limit $q\to 1$ the N-point function B_N of this model becomes the Veneziano N-point function V_N. Although this model contains all the desirable formal properties of the Veneziano model, the form (1.1) of the trajectory function gives rise to a number of unacceptable properties except in the limit $q\to 1$. These properties of B_4 are reviewed in Sec. II.

In Sec. III we note that if we let the parameter q in Eq. (1.1) become greater than unity some qualitative changes must occur in the model and the physically unacceptable properties disappear. The resulting $\alpha(t)$ then becomes a physically possible trajectory function with properties similar to those determining relativistic bound states of two massive particles interacting via the exchange of massless particles. However, since the N-point dual Born term B_N with $q<1$ has a natural boundary at $|q|=1$, we cannot construct an N-point function for $q>1$ by simple continuation in q of the $q<1$ amplitude. For this reason we can only use the $q<1$ amplitude B_4 as a guide in postulating a four-point function D_4 having trajectory functions of the form (1.1) with $q>1$. Such a dual amplitude D_4 is written down in Sec. III.

In Sec. IV the mathematical and physical properties of D_4 are discussed. D_4, unlike B_4, has cuts in s and t and therefore cannot be regarded as a Born term in a field theory with an infinite number of particles. Instead it can be regarded as the first approximation to the scattering amplitude of a composite system. However, this amplitude, just like V_4 and B_4, does contain an infinite number of poles on the real axis and in order to compare it with experiments in the region of these poles one must also find a smoothing procedure which replaces f_4 by a smooth function $\bar{f}_4$. This procedure is carried out and the physical properties of the resulting smoothed amplitude are discussed.

The problem of constructing an N-point function is not dealt with in this paper. This problem is more complicated than in the case of pure pole models with no constituent particle thresholds.

In Sec. V the physical significance of this model is discussed.

II. THE DUAL MODEL WITH $q<1$

The dual four-point function B_4 with trajectories (1.1) is given by

$$B_4(s,t)=\frac{G(q)G(\sigma'\tau')}{G(\tau')G(\sigma')}, \tag{2.1}$$

where

$$G(x)=\prod_{n=0}^{\infty}(1-q^n x) \tag{2.2}$$

and

$$\sigma'=a's+b. \tag{2.3}$$

The only singularities of B_4 are poles in s and t. The poles in s are at $\sigma'\equiv a's+b=q^{-j}$ for $j=0,1,2,\ldots$

with residues which are polynomials in t of order j. As $s\to\infty$ in a suitable direction in the complex plane $B_4(s,t)$ behaves essentially like $s^{\alpha(t)}$. Furthermore, the N-point function B_N is a meromorphic function of all the $N(N-3)/2$ invariants whose residues are polynomials in the overlapping variables.[2] B_N factorizes[3] and possesses multi-Regge behavior.[4] The explicit form of the triple Regge vertex has been calculated.[4]

The difficulties with the amplitude (2.1) arise from the form of the trajectory function (1.1) which is plotted in Fig. 1. The branch cut in the logarithmic trajectory function is taken between $\tau=0$ and $\tau=-\infty$, and from $t=-b/a'$ to $t=-\infty$. $\alpha(t)$ is then real for $\tau>0$ and the equations $\alpha(t_n)=n$ determine the energies of the lowest-lying states with angular momentum n. Since $\mathrm{Re}\,\alpha(t)$ increases as t becomes large and negative $B_4(s,t)$ increases as $s\to\infty$ for t sufficiently large and negative.[2,5] Now although there is no compelling reason that a dual Born term should give a good quantitative fit in the large-s, large-t region, it should at least have a reasonable extrapolation in this region. This property is particularly important in calculating loop diagrams from the Born term. In the limit $q\to 1$, $B_4\sim V_4$ does not rise for large negative t. On the contrary it falls off exponentially in t which is a much too rapid decay to describe the large-angle data.

A second problem with the trajectories (1.1) is the fact that they do not rise sufficiently rapidly as $t\to+\infty$. One can show from elementary kinematic considerations that if $\alpha(t)<c\sqrt{t}$ as $t\to\infty$, then there must be ghosts on some daughter trajectories.[6] In fact we have explicitly verified that although the amplitude (2.1) gives rise to no ghosts on the leading trajectory, there are ghosts on the first daughter trajectory for t sufficiently large.

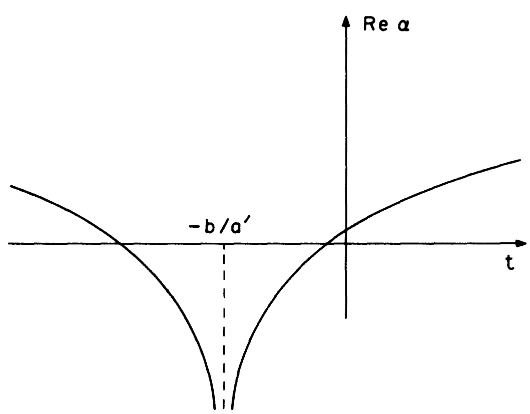

FIG. 1. Logarithmic trajectory associated with an unbounded mass spectrum.

III. THE DUAL MODEL WITH $q>1$

The above two difficulties arise from the behavior of the trajectory function (1.1) for large positive and negative t. Both of these difficulties disappear if we have a trajectory function of the same form (1.1) but with $q^{-1}>1$. Furthermore, we take the parameter a' to be less than zero, which moves the singularity at $\tau'=0$ to positive values of t. We thus consider a trajectory function

$$\alpha(t)=\frac{\log\tau}{\log q}, \tag{3.1}$$

where

$$\tau=b-at. \tag{3.2}$$

Before constructing a scattering amplitude having the trajectory function (3.1), we will first just examine the features of the trajectory itself. In Fig. 2 we plot the function $\mathrm{Re}\,\alpha(t)$ determined from Eq. (3.1). The branch cut in $\log\tau$ from $\tau=0$ to $\tau=-\infty$ now corresponds to a branch cut from $t=b/a$ to $t=+\infty$. For $t<b/a$, $\alpha(t)$ is real. The equations $\alpha(t_n)=n$ or equivalently

$$b-at_n=q^n,\quad n=0,1,\dots \tag{3.3}$$

determine the energies t_n of the infinite number of bound states lying between $t_0=(b/a)(1-1/b)$ and $t_\infty=b/a$. Since $t_n=(b/a)(1-q^n/b)$, there are clearly an infinite number of bound states lying in any neighborhood to the left of $t_\infty=b/a$. Thus the branch point in $\alpha(t)$ at $t=b/a$ is an accumulation point of poles. Furthermore, for $t>b/a$ the trajectory function $\alpha(t)$ is complex and $\mathrm{Re}\,\alpha(t)$ is decreasing. Since $\mathrm{Re}\,\alpha(t)$ is decreasing, we do not expect the amplitude to have any resonances for $t>b/a$. The trajectory function (3.1) thus gives rise to an infinite number of bound states of arbitrarily high spin below the accumulation point at $t_\infty=b/a$ and a nonresonant amplitude for $t>t_\infty$. Such a spectrum is similar to that produced by the binding of two particles of mass $M=\frac{1}{2}(b/a)^{1/2}$ via

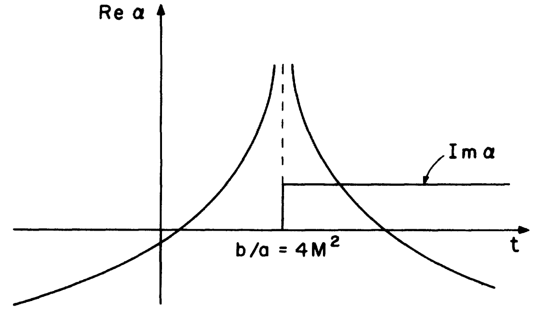

FIG. 2. Logarithmic trajectory associated with a bounded mass spectrum.

the exchange of massless quanta. The trajectory function (3.1) thus suggests a composite model for the hadron and there is no problem with ghosts in B_4 produced by the fact that the trajectory function of Fig. 1 rises too slowly as $t \to +\infty$. Instead $\alpha(t) \to \infty$ for t near b/a, the limiting mass on the trajectory.

In the scattering region $t<0$ the trajectory function $\alpha(t)$ of Eq. (3.1) is real and decreases to $-\infty$ like $-\log(-t)$ at $t \to -\infty$. Thus an amplitude with Regge behavior $s^{\alpha(t)}$ would have acceptable behavior for large negative t. In fact the amplitude would fall off for large t much more like the elastic scattering data than the Veneziano $q=1$ limit which drops off exponentially in t and hence falls orders of magnitude below the data. The high-s behavior $s^{\alpha(t)}$, where $\alpha(t) \sim -\log(-t)$ for $t \to -\infty$, also turns out to be the behavior of the amplitude for the scattering of two massive particles interacting by the exchange of massless particles in the ladder approximation.[7] Thus the same physical picture which explains a possible mechanism for the accumulation point at $t=+b/a=4M^2$ also can account for the large-t behavior of the high-energy scattering amplitude.

We have thus seen that letting $q \to q^{-1}$ in Eq. (1.1) produces a trajectory function (3.1) which not only no longer has the undesirable behavior of (1.1) for $t \to +\infty$ and $t \to -\infty$, but also suggests a simple composite-particle model for the hadron. We now want to construct a dual amplitude with trajectory (3.1). We cannot simply analytically continue the amplitude (2.1) to values of $q>1$ since the circle $|q|=1$ is a natural boundary for B_4. This is because there are poles in B_4 at $\tau'=q^{-n}$ or at $q=(\tau')^{-1/n}$ for arbitrarily large n. Thus for any fixed value of τ' the unit circle $|q|=1$ contains a dense set of poles and analytic continuation to $|q|>1$ is impossible. This is not surprising since, as we have seen, there is a great difference between the physics of trajectories of Figs. 1 and 2. We can thus only use the amplitude B_4 as a guide in constructing a D_4 with the trajectories of Fig. 2. We will also be guided by our composite-particle picture for the trajectories (3.1).

We begin with the infinite-product representation (2.1) and (2.2) for B_4 and rearrange the infinite products so that they converge when $\sigma' \to \sigma$, $\tau' \to \tau$, and $q \to q^{-1}$. That is, the individual infinite products

$$\prod_{l=0}^{\infty} \frac{(1-\sigma\tau q^{-l})(1-q^{-l-1})}{(1-\sigma q^{-l})(1-\tau q^{-l})} = \frac{(1-\sigma\tau)}{(1-\sigma)(1-\tau)} \prod_{l=1}^{\infty} \frac{(1-\sigma\tau q^{-l})(1-q^{-l})}{(1-\sigma q^{-l})(1-\tau q^{-l})} \quad (3.4)$$

do not converge when $q<1$. However, by factoring out factors $\sigma\tau q^{-2l}$ from the numerator and denominator of each factor in Eq. (3.4) we are led to consider the infinite product

$$-\frac{(1-\sigma^{-1}\tau^{-1})}{(1-\sigma^{-1})(1-\tau^{-1})} \prod_{l=1}^{\infty} \frac{(1-\sigma^{-1}\tau^{-1}q^{l})(1-q^{l})}{(1-\sigma^{-1}q^{l})(1-\tau^{-1}q^{l})} = -B_4(\sigma^{-1}, \tau^{-1}) \ . \quad (3.5)$$

$-B_4(\sigma^{-1}, \tau^{-1})$ has poles in σ at $\sigma=q^j$ and near $\sigma \sim q^j$ behaves like

$$\frac{1}{1-q^{-j}\sigma} \frac{1}{\tau^j} \prod_{l=1}^{j} \left[\frac{(\tau-q^{-l})}{(1-q^{-l})}\right] \ . \quad (3.6)$$

The residue (3.6) of the pole at $\sigma=q^j$ can be written as $(1/\tau^j)\, P_j(\tau, q^{-1})$, where $P_j(\tau, q^{-1})$ is a polynomial in τ of order j obtained from the residue of $B_4(\sigma', \tau', q)$ [Eq. (2.1)] at $\sigma'=q^{-j}$ by replacing q by q^{-1}. We can construct a function which has polynomial residues at $\sigma=q^j$ by multiplying $B_4(\sigma^{-1}, \tau^{-1})$ by a factor $\tau^{\alpha(s)}$, where $\alpha(s)$ is the trajectory function (3.1). This factor cancels the τ^{-j} factor in the residue (3.6) where $\sigma=q^j$, or equivalently $\alpha(s)=j$, and at the same time does not destroy the st crossing symmetry of the amplitude, since $\tau^{\alpha(s)}=e^{\alpha(s)\ln\tau}=q^{\alpha(s)\alpha(t)}=\sigma^{\alpha(t)}$. Finally, we note that if we want the $\alpha(0)=\log b/\log q$ to be positive we must have $b<1$. From (3.2) this means that the lowest-lying energy t_0 is negative. To eliminate this negative $(\text{mass})^2$ particle we must replace $G(x)$ by $G(qx)$ everywhere in the expression for B_4. The first zero in $G(qx)$ is at $x=q^{-1}$ instead of $x=1$. The poles of $G(q/\sigma)$ are then at

$$\frac{q}{\sigma}=1, q^{-1}, q^{-2}, \ldots \quad (3.7)$$

The $(\text{energy})^2$ s_n of the nth state is then

$$s_n=\frac{b}{a}\left(1-\frac{q^n}{b}\right), \quad n=1,2,\ldots \quad (3.8)$$

which is positive provided $q<b$. We are thus led to consider the amplitude $D_4(s,t)$ defined by

$$D_4(s,t)=\frac{G(q/\sigma\tau)\, q^{\alpha(s)\alpha(t)}}{G(q/\tau)\, G(q/\sigma)} \ , \quad (3.9)$$

where

$$\sigma=b-as, \quad \tau=b-at \ , \quad (3.10)$$

and

$$a>0, \quad 0<q<b<1$$

and

$$\alpha(s)=\frac{\log\sigma}{\log q}, \quad \alpha(t)=\frac{\log\tau}{\log q} \ .$$

The arguments leading to the expression (3.9) were motivated by the desire to incorporate as many as possible of the dual features of B_4 while changing the trajectory function from the expression (1.1) depicted in Fig. 1 to the expression (3.1) depicted in Fig. 2. The simplest such modification (3.5) which accomplishes this has the nonpolynomial residues (3.6). These nonpolynomial residues are then eliminated by the crossing symmetric factor $q^{\alpha(s)\alpha(t)} = \sigma^{\alpha(t)}$, which at the same time introduces a cut into D_4 for $\sigma < 0$ or for

$$s > \frac{b}{a} \equiv 4M^2 . \tag{3.11}$$

This means that our amplitude D_4, unlike B_4, is not meromorphic but has a branch cut for $4M^2 < s < \infty$. This threshold branch cut is what would be expected if our previous bound-state interpretation of the trajectory function (3.1) is correct. Scattering amplitudes which involve bound states produced by the exchange of massless particles always have an accumulation of poles and then a branch cut. (The usual nonrelativistic Coulomb scattering amplitude is an example of such behavior.) The motivation for postulating (3.9) as a dual model for the scattering of composite particles follows the reasoning by which (3.9) was originally obtained. We could just as well have postulated (3.9) and then worked out all its predictions for scattering. In the next section of this paper we discuss what modifications must be introduced when we take account of the fact that our bound states are really resonances which decay with a finite lifetime.

IV. THE SMOOTHED AMPLITUDE IN THE RESONANCE REGION

The amplitude D_4 of Eq. (3.9) is a smooth function of s except in the resonance region, $4M^2(1-q/b) < s < 4M^2$. The individual factors in (3.9) are singular as $s \to 4M^2$ from above or equivalently when $\sigma \to 0^-$. However, the resulting function $D_4(s,t)$ is smoothly behaved in this limit and one can use (3.9) to calculate D_4 for all values of s greater than $4M^2$.[8] For values of $s < 4M^2(1-q/b)$, $D_4(s,t)$ is clearly smooth. We now examine $D_4(s,t)$ in the resonance region.

The poles in s in D_4 arise from the vanishing of $G(q/\sigma)$ at $\sigma = q^n$, $n = 1, 2, 3, \ldots$. Using Eqs. (2.2) and (3.9), we find that near the nth pole at $\sigma = \sigma_n \equiv q^n$, D_4 behaves as

$$D_4(s,t) \xrightarrow[\sigma \to q^n]{} \frac{G(\tau)G(q^n)q^n}{G^2(q)G(\tau q^n)(\sigma - q^n)} . \tag{4.1}$$

Thus as $\sigma_n \to 0$, or as $s_n \to 4M^2$, the residue of $D_4(s,t)$ at s_n vanishes like q^n. The case in which near the accumulation point the poles become denser while the contribution of an individual pole to the amplitude becomes smaller is very similar to the situation in which γ rays are scattered by an atom near the ionization threshold. In this case the averaged amplitude below the ionization energy connects smoothly to the amplitude in the continuum above the ionization energy. We will now show that the amplitude (3.9) possesses this same property.

From (4.1) we conclude that

$$\mathrm{Im} D_4(s,t) = \sum_{n=1}^{\infty} \frac{\pi}{G(q)} \frac{G(\tau)}{G(q)} \frac{G(\sigma)\delta(\sigma - q^n)q^n}{G(\tau\sigma)} \tag{4.2}$$

for $4M^2(1-q/b) < s < 4M^2$. Now

$$\delta(\sigma - q^n) = \frac{\delta(n - \alpha(s))}{(\ln q) q^n} .$$

Hence

$$\mathrm{Im} D_4(s,t) = +\frac{\pi}{\ln q} \sum_{n=1}^{\infty} \frac{G(\tau)}{G^2(q)} \frac{G(q^n)\delta(n-\alpha(\sigma))}{G(\tau q^n)} , \quad (1-q/b)4M^2 < s < 4M^2 . \tag{4.3}$$

To define an averaged amplitude we replace the function $\delta(n - \alpha(t))$ in (4.2) by the function which is 1 in the interval $n < \alpha(\sigma) < n+1$ and zero otherwise. This spreading of the δ function replaces (4.2) by

$$+\frac{\pi}{\ln q} \frac{G(\tau)G(q^n)}{G^2(q)G(\tau q^n)} , \tag{4.4}$$

where n is the smallest integer greater than $\alpha(\sigma)$.[9] As long as $q^n \ll 1$ the step function (4.4) does not change appreciably as $n \to n+1$ and hence (4.4) differs negligibly from the smooth function

$$\mathrm{Im}\bar{D}(s,t) = -\frac{\pi}{|\ln q|} \frac{G(\sigma)G(\tau)}{G^2(q)G(\sigma\tau)} . \tag{4.5}$$

We take (4.5) as our averaged amplitude. Physically the replacement of (4.3) by (4.5) accounts for the fact that the resonances can decay and hence have finite width. We expect that the amplitude (4.5) will give an accurate representation of the physical amplitude except in the region of the low-lying resonances $n = 1, 2$ where the spacing $(1/a)q^n(1-q)$ is not small. We will determine the real part of the averaged amplitude by requiring continuity at the accumulation point.[10]

We now look at the amplitude (3.9) for $s > 4M^2$, i.e., $\sigma < 0$. In this region $D_4(st)$ can be written as

$$\frac{G(q/\sigma\tau)(|\sigma|)^{\alpha(t)}}{G(q/\tau)G(q/\sigma)} e^{-i\pi\alpha(t)} . \tag{4.6}$$

Thus $D_4(s,t)$ possesses the Regge asymptotic phase $e^{-i\pi\alpha(t)}$ for all $s > 4M^2$. In the resonance region we use the average amplitude $\bar{D}$ which has the same phase

$$\bar{D}(s,t)=\frac{\pi}{\ln q}\,\frac{G(\sigma)G(\tau)}{G^2(q)G(\sigma\tau)}\,\frac{e^{-i\pi\alpha(t)}}{\sin\pi\alpha(t)}\,. \tag{4.7}$$

The behavior of the amplitude (4.6) as $\sigma\to 0^+$ looks quite different from that of the amplitude (4.7) as $\sigma\to 0^-$. However, one can see that the amplitudes do not differ appreciably by using the following identity which is easily proved using Jacobi elliptic functions[11]:

$$G(\sigma)G(q/\sigma)=C(q)\sin\pi\alpha(s)\,q^{-1/2[\alpha(\alpha-1)]}I(\alpha)\,, \tag{4.8}$$

where

$$C(q)=\frac{-2G(e^{4\pi^2/\ln q})e^{\pi^2/2\ln q}}{G(q)q^{1/8}(-\ln q/2\pi)^{1/2}}\,, \tag{4.9}$$

$$I(\alpha)=\prod_{n=1}^{\infty}(1-e^{-2\pi i\alpha_n(s)})(1-e^{+2\pi i\alpha_{-n}(s)})\,, \tag{4.10}$$

and $\alpha_n(s)$ is the trajectory function on the nth sheet, i.e.,

$$\alpha_n(s)=\frac{\ln\sigma}{\ln q}+\frac{2\pi i n}{\ln q}\,,\quad n=0,\pm1,\pm2,\ldots\,. \tag{4.11}$$

If we use Eq. (4.8) for each of the factors in (3.9) we arrive at the expression

$$D_4(s,t)=\frac{G(\sigma)G(\tau)}{C(q)G(\sigma\tau)}\,\frac{\sin\pi[\alpha(s)+\alpha(t)]}{\sin\pi\alpha(s)\sin\pi\alpha(t)}\times\frac{I(\alpha(s)+\alpha(t))}{I(\alpha(s))I(\alpha(t))}\,. \tag{4.12}$$

Now above the accumulation point $\alpha(s)$ is complex and equal to $\ln(|\sigma|-\pi i)/\ln q$. Thus the factor $\sin\pi[\alpha(s)+\alpha(t)]/\sin\pi\alpha(s)$ can be replaced by $e^{-\pi i\alpha(t)}$ if we neglect terms of order $e^{+2\pi^2/\ln q}$, which is very small unless q is near zero. Furthermore, under the same conditions the factors $I(\alpha)$ are essentially equal to unity and hence (4.12) takes on the same form as the averaged amplitude in the resonance region.

The above discussion was only used to show that the structure that enters in the averaged amplitude (4.7) also appears in the amplitude (3.9) when use is made of the identity (4.8). The identity could also be used to define an alternate averaging procedure in the resonance region which up to a factor $I[\alpha(t)]$ yields the same amplitude (4.7) with the right phase. In actual practice one calculates directly with the original amplitude (3.9) which behaves very smoothly as s approaches $4M^2$ from above. Thus for q not near zero, the amplitude then smoothly joins to the averaged amplitude (4.7). The accumulation point and the branch point at $4M^2$ have compensating effect and we are left with no singularity at $s=4M^2$ except for the numerically negligible effect of the function $I(\alpha)$.

V. DISCUSSION

We conjecture that the dual amplitude (3.9) is an approximate representation of the scattering amplitude for particles which are bound states whose constituents interact via the exchange of massless gluons. The reasons for this conjecture are the following:

(a) The fact the amplitude contains an infinite number of resonances near $4M^2$ with arbitrarily high spin means that the theory must contain a long-range force, i.e., mass-zero particles.

(b) The nonrelativistic Coulomb trajectories are an example of trajectories having the property (a). They are quite different, however, from the trajectories $\alpha(t)$ of Fig. 2. The exact solution of the Bethe-Salpeter equation for the scattering of massive particles via the exchange of massless scalar[7] particles yields trajectory functions which for large t behave like $\ln|t|$. Thus at least in the large-$|t|$ region the trajectory functions of Fig. 2 can be obtained from a bound-state picture.

(c) The dual structure of the amplitude (3.9) suggests that it might be related to a Feynman amplitude corresponding to duality diagrams of the Harari-Rosner type such as that depicted in Fig. 3. The amplitude of Fig. 3 is an eight-point function for the scattering of constituents via the exchange of massless gluons. The amplitude corresponding to the diagrams of Fig. 3 can be constructed from the solution to Bethe-Salpeter equations for the elastic scattering of the constituents which corresponds to summing the ladder diagrams of Fig. 4. The bound-state scattering amplitude is then obtained from the residue of this eight-point function at the spin-zero poles in the variables $(p_1+p_2)^2$, $(p_3+p_4)^2$, $(p_5+p_6)^2$, $(p_7+p_8)^2$. This amplitude is obviously a crossing-symmetric amplitude containing the infinite set of poles in s and t at the positions determined by the position of the poles of the Bethe-Salpeter amplitude of Fig. 4. But as mentioned above, these poles lie on a trajectory which is depicted qualitatively in Fig. 2 and for $t\to\pm\infty$ has logarithmic behavior.

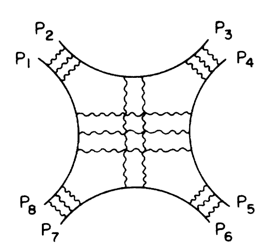

FIG. 3. Typical Feynman diagram involving the scattering of constituent particles. Such diagrams are associated with Harari-Rosner duality graphs.

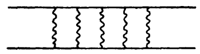

FIG. 4. Ladder graph approximation for the elastic scattering of constituents.

(d) The model of Fig. 3 possesses the usual difficulty that its components, quarks and mass-zero gluons, are not observed in nature and therefore one must explain why physical particle states with such properties do not appear. We have no new ideas about the solution to this problem but we note that in our smoothed amplitude both the accumulation point (i.e., the mass-zero particle) and the branch point (quark-mass threshold) have disappeared. It is not inconceivable that inclusion of higher-order corrections to Fig. 3 involving quark-antiquark pairs could produce an amplitude which has some of the properties of our smoothed model amplitude (4.6), and hence would not make manifest the presence of quarks or zero-mass gluons. This of course is all very speculative and we have no suggestions of how this could come about in detail.

(e) Since calculation of diagrams of the type of Fig. 3 are rather involved, it is desirable to see if the dual amplitude (3.8) which motivated this bound-state picture can give a reasonable description of elastic scattering. Such an analysis of large-angle *pp* elastic scattering where the Pomeron contribution is negligible has been carried out by Coon, Sukhatme, and Tran,[1] and quite a satisfactory fit was obtained.

In conclusion, we have constructed a dual amplitude (3.8) which has many features of a composite-particle model of scattering. It gives a reasonable description of elastic scattering data and suggests that processes such as those depicted in Fig. 3 may be the fundamental mechanics contributing to strong interactions.

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*Work supported in part by the U. S. Atomic Energy Commission.

†Work supported in part by the National Science Foundation.

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