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Impact-parameter expansion of the Veneziano amplitude*

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An impact-parameter expansion of the Veneziano scattering amplitude is obtained and some analytic properties are briefly discussed. A simple asymptotic limit to the expansion is also furnished for large impact parameter.

I. INTRODUCTION

In strong-interaction dynamics, the high-energy behaviors of scattering amplitudes are of interest and have been the objects of continuing investigation. As an extension of the eikonal approach in potential theory,[1] the impact-parameter expansion[1-5] has provided insight into high-energy scattering processes and serves as a useful integral alternative to the usual partial-wave series summation of the amplitude.

The Veneziano scattering amplitude,[6] which exhibits complete crossing symmetry and Regge asymptotic behavior in all channels, has also stimulated studies[7-11] of the dynamics of duality and rising linear trajectories, though unfortunately violating unitarity in its unexpurgated form.

In this paper we examine an impact-parameter expansion of the Veneziano amplitude. More particularly, we obtain an expansion and describe its analytic properties, compute the discontinuity across the branch cut, and finally give the asymptotic expansion in the large-impact-parameter limit.

II. VENEZIANO AMPLITUDE AND IMPACT REPRESENTATION

The Veneziano scattering amplitude, A, is given in terms of the usual Mandelstam variables, $s=4(q^2+m^2)$ and $t=-2q^2(1-\cos\theta)$, for q the center-of-mass momentum and θ the scattering angle, by

$$A(s,t)=\frac{\Gamma(1-\alpha(s))\Gamma(1-\alpha(t))}{\Gamma(1-\alpha(s)-\alpha(t))}. \tag{2.1}$$

The Γ's are the Euler functions,[12]

$$\Gamma(z)=\int_0^\infty dx\, x^{z-1}e^{-x}, \tag{2.2}$$

analytic in z except for poles at negative real integer z, or zero. The α are Regge trajectories, taken to be real and linearly rising,

$$\alpha(u)=a+cu, \quad c>0. \tag{2.3}$$

Obviously, $A(s,t)$ possesses poles in s when

$$\alpha(s_j)=j, \tag{2.4}$$

with j a positive integer. These poles are equally spaced, since, from Eq. (2.3),

$$s_j=\frac{(j-a)}{c}. \tag{2.5}$$

Similarly, for all t such that

$$\alpha(t_j)=j, \tag{2.6}$$

$A(s,t)$ contains an infinite series of poles in t. The denominator of Eq. (2.1) cancels one of the poles when s and t coincide at different values of j, ensuring that only single poles occur. As $z\to\infty$, $\Gamma(z)\to z^z$, and the asymptotic form of $A(s,t)$ is seen to be

$$\lim_{s\to\infty}A(s,t)\cong\Gamma(1-\alpha(t))(-cs)^{\alpha(t)},$$
$$\lim_{t\to\infty}A(s,t)\cong\Gamma(1-\alpha(s))(-ct)^{\alpha(s)}, \tag{2.7}$$

which displays the characteristic Regge asymptotic behavior. Crossing symmetry under the interchange $s\leftrightarrow t$ is also obvious in Eqs. (2.1) and (2.7). The residue of the jth pole is proportional to a t polynomial of order j, giving rise to the interpretation that each pole corresponds to a multiplet of particles of the same mass and of spins $0,1,2,3,\ldots,j$. Because the series of resonances is equivalent to a Regge asymptotic form, the amplitude is dual. Since the poles are real, however, there is no discontinuity across the axis and unitarity is violated.

Considering a collision with orbital momentum l in the center-of-mass system, we define the impact parameter, b, by

$$l+\tfrac{1}{2}=qb. \tag{2.8}$$

Semiclassically, b is the closest distance of approach. The impact representation of the amplitude is written[1] as an integral high-energy complement to the partial-wave sum,

$$A(s,t)=\int_0^\infty d\beta\,\beta J_0(\beta y)h(s,\beta/2q), \tag{2.9}$$

with h the impact coefficients, J_0 the Bessel function of order zero, and

$$\beta = 2qb, \qquad y^2 = -t/4q^2. \tag{2.10}$$

The impact coefficients are formally obtained from Eq. (2.9) by the following inversion:

$$h(s,b) = \int_0^\infty dy\, y J_0(\beta y) A(s, -4q^2y^2). \tag{2.11}$$

III. FORMAL EXPANSION AND ASYMPTOTIC LIMITS

Equation (2.1) can be recast as

$$\frac{\Gamma(1-\alpha(s))\Gamma(1-\alpha(t))}{\Gamma(1-\alpha(s)-\alpha(t))} = \frac{-\alpha(s)\Gamma(-\alpha(s))\Gamma(1-\alpha(t))}{\Gamma(1-\alpha(s)-\alpha(t))} = -\alpha(s)B(-\alpha(s), 1-\alpha(t)), \tag{3.1}$$

with B the beta function written in the form[12]

$$B(v,w) = \int_0^\infty dx\, e^{-xw}(1-e^{-x})^{v-1}, \tag{3.2}$$

for $\mathrm{Re}v > 0$ and $\mathrm{Re}w > 0$. For the linear trajectory,

$$\alpha(t) = a + ct, \quad c > 0 \tag{3.3}$$

one obtains the impact coefficients of the Veneziano scattering amplitude from Eqs. (2.11), (3.1), (3.2), and (3.3),

$$h(s,b) = -\alpha(s)\int_0^\infty dx\, e^{-x(1-a)}(1-e^{-x})^{-\alpha(s)-1} \times \int_0^\infty dy\, y J_0(\beta y) e^{-4cq^2y^2x^2}. \tag{3.4}$$

Performing the integration over y,[13] it follows that

$$h(s,b) = \frac{-\alpha(s)b}{2qc}\int_0^\infty dx\, e^{-x(1-a)-\gamma x^{-1}} \times \left(\frac{1-e^{-x}}{x}\right)^{-\alpha(s)-1} x^{-\alpha(s)-2}, \tag{3.5}$$

$$\gamma = \frac{b^2}{4c},$$

which is valid for $a+ct<1$, $\alpha(s)<0$. The integral is well behaved for $\mathrm{Re}\gamma > 0$. For $\mathrm{Re}\gamma < 0$, trouble seems to occur at $x \to 0$. This is circumvented by expanding the exponential in a Taylor series at $x = 0$,

$$\left(\frac{1-e^{-x}}{x}\right)^{-\alpha(s)-1} = \sum_{n=0}^\infty f_n x^n, \qquad f_n = \frac{1}{n!}\frac{\partial^n}{\partial x^n}\left(\frac{1-e^{-x}}{x}\right)^{-\alpha(s)-1}\Bigg|_{x=0}, \tag{3.6}$$

since

$$\lim_{x\to 0}\left(\frac{1-e^{-x}}{x}\right)^{-\alpha(s)-1} = 1. \tag{3.7}$$

Then employing Eq. (3.7) in Eq. (3.5), there results

$$h(s,b) = \frac{-\alpha(s)b}{2qc}\sum_{n=0}^\infty \int_0^\infty dx\, e^{-x(1-a)-\gamma x^{-1}} \times f_n x^{n-\alpha(s)-2}, \tag{3.8}$$

which is a series well behaved at the origin and convergent. Performing the integrations[13] one obtains an expression

$$h(s,b) = \frac{-\alpha(s)b}{qc}\sum_{n=0}^\infty f_n\left(\frac{\gamma}{1-a}\right)^{[\alpha(s)+1-n]/2} \times K_{\alpha(s)+1-n}(2\gamma^{1/2}(1-a)^{1/2}), \tag{3.9}$$

for K the modified Bessel function, which is equally applicable for $\mathrm{Re}\gamma < 0$.

From Eq. (3.9) and the structure of K,[12] it is evident that h has a branch point at $b=0$ and branch cut along the real axis from $-\infty < b < 0$. We use the identity[14]

$$K_\nu(z) = \tfrac{1}{2}\pi(\sin\pi\nu)^{-1}[I_{-\nu}(z) - I_\nu(z)], \tag{3.10}$$

and contour representation of the Bessel function of the second kind,

$$I_\nu(z) = (2\pi i)^{-1}\left(\frac{z}{2}\right)^\nu \int_C dx\, e^{-(x+z^2/4x)}(-x)^{-\nu-1}, \tag{3.11}$$

where the contour C starts at infinity, encloses the origin, and returns to infinity counterclockwise and along the real axis. One finds the discontinuity in h across the cut,

$$\lim_{\epsilon\to 0}[h(s,b+i\epsilon) - h(s,b-i\epsilon)] = -\frac{\alpha(s)b}{4iqc}\sum_{n=0}^\infty f_n(-\gamma)^{\alpha(s)+1-n}\int_C dx(-x)^{n-\alpha(s)-2}e^{-x-\gamma(1-a)x^{-1}}. \tag{3.12}$$

For large b an asymptotic expression for h can be obtained from Eq. (3.9) by taking the low-order terms in the *large-z* expansion of $K_\nu(z)$,

$$K_\nu(z) \cong \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}. \tag{3.13}$$

From Eq. (3.9), we obtain the diffraction-like result for the asymptotic impact coefficients,

$$\lim_{b\to\infty} h(s,b) \cong \frac{-\alpha(s)b^{1/2}}{qc}\sum_{n=0}^{\infty} g_n\left(\frac{\pi}{2\xi}\right)^{1/2} e^{-\xi b}, \quad \xi=\left(\frac{1-a}{c}\right)^{1/2}, \quad g_n=f_n\left[\frac{b}{2c^{1/2}(1-a)}\right]^{[\alpha(s)+1-n]/2}. \tag{3.14}$$

For high energies, $\alpha(s)\to 4q^2c$, and the leading term ($n=0$) gives

$$\lim_{b\to\infty} h(s,b) \sim -q\left(\frac{8\pi b}{\xi}\right)^{1/2}\left[\frac{b}{2c^{1/2}(1-a)}\right]^{(4q^2c+1)/2} e^{-\xi b}. \tag{3.15}$$

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