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Bounds on the imaginary parts of spin-0–spin-1/2 particle scattering amplitudes from experimental data and limited use of unitarity

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Bounds on the moduli and relative phase of the spin-0–spin-1/2 particle scattering amplitudes are used in the unitarity relation to find upper bounds for the imaginary parts of the spin-nonflip and spin-flip amplitudes.

I. INTRODUCTION

Construction of the scattering amplitudes from the experimental data and their rigorous (model-independent) properties is one of the central problems in particle physics and scattering theory today. Several authors have studied different aspects of this problem.[1-11]

For the scalar-scalar case the modulus of the amplitude is the only measurable quantity, its square being the differential cross section. The phase is connected to the modulus through the unitarity equation in the elastic scattering case. The existence and uniqueness of its solution and question of convergence of iteration are discussed in Refs. 1–3. The same problem with the additional assumption of analyticity in z within the Lehmann-Martin ellipse is studied by the authors of Ref. 9.

For the spin-0–spin-$\frac{1}{2}$ scattering case there are two amplitudes and two coupled unitarity integrals. However, the moduli of the spin-nonflip amplitude $f(z)$ and the spin-flip amplitude $g(z)$ are not determined with a knowledge of the differential cross section $D(z)$ and the polarization $P(z)$, the presently available experimental quantities. Only upper and lower bounds[12-14] for the moduli and relative phase of these amplitudes are determined by D and P. The additional knowledge of the rotation parameter R would determine the relative phase uniquely (modulo 2π) as well as the moduli. But even then one could not distinguish between $|f|$ and $|g|$ even though the values of the moduli would be known. Only when the other rotation parameter A is known can we tell which value corresponds to which modulus.

If one wanted to use unitarity to find the phases together with a knowledge of D and P at all angles at a fixed energy one could not do this because the moduli of the amplitudes are not known. If, however, we introduce the transversity amplitudes

$$t_\pm = f \pm ig,$$

their moduli are determined by D and P and the corresponding unitarity equations can be used to determine the phases.[7] One can then go back to the above relations between $t_\pm$ and f and g and determine the phases of f and g.

The reader may wonder how one can find these phases by simply making a transformation if one was not able to find them in the first place directly from the unitarity equations of f and g. The answer to this is that the experimental information D and P is distributed between the three quantities $|f|$, $|g|$, and the relative phase ($\sin\alpha$) in terms of their combinations or as bounds on each of them. Thus there is too little information on $|f|$ and $|g|$ which are needed in the unitarity equations and too much information on the phases (relative phase bounded) which will be determined by those equations. On the other hand, D and P give complete information on $|t_+|$ and $|t_-|$ (uniquely determined) but say nothing at all on their phases. Hence t_+ and t_- are the suitable amplitudes for the unitarity equations if only D and P are known. For a completely new introduction of the transversity amplitudes see Refs. 15 and 10.

Finally for the spin-$(\frac{1}{2},\frac{1}{2})$ cases we refer the reader to the recent works, Refs. 8 and 11.

In this paper we use the two unitarity relations for the f and g amplitudes to establish bounds on their imaginary parts below the inelastic threshold with a knowledge of D and P. The reader may ask why one needs bounds if the unitarity equations can be solved for the phases. The reason for this is that the solvability conditions are too restrictive and in most practical cases not fulfilled.

Finally the justification for the use of f and g amplitudes rather than the transversity amplitudes in the derivation of bounds lies in the simplicity of their unitarity equations even though the distribution of information between moduli and phases is not as neat as in the case of t_+, t_- amplitudes. We try to exploit the information contained in the bounds of moduli and the relative phase to majorize the unitarity integrals.

In Sec. II we consider the spin-nonflip amplitude. Different types of bounds are obtained. In the first type [Eq. (2)] only the largest value of expression (1) in the entire angle region is used. In the second type several improvements are carried out in the majorization of the phase factors containing both the absolute and relative phases. In these, both D and P data are used without revoking the analyticity in energy [Eq. (10)]. Finally maximizing both amplitudes simultaneously a third bound [Eq. (13)] is obtained which is improved by maximizing the amplitudes subject to constraints imposed by the polarization.

In Sec. III we consider the spin-flip amplitude. In the unitarity relations of $f(z)$ and $g(z)$ there is an inherent difference in the way the phases enter those relations. Therefore, the way the majorization of the phase factors can be improved is different for both cases. The best result for $\mathrm{Im}g(z)$ is represented here by Eq. (23). Simultaneous maximization of the amplitudes with and without the constraints imposed by the polarization is more complicated for $\mathrm{Im}g(z)$ but the principle is the same.

In Sec. IV we compare our inequalities with the experimental data and give numerical results in Tables I–V. In Sec. V we summarize the results and give our conclusions. The details of the majorizations of some expressions used in derivations are given in the Appendix.

II. SPIN-NONFLIP AMPLITUDE

With best upper bounds obtained from differential cross section $D(z)$ and polarization $P(z)$ for the moduli of the spin-nonflip and spin-flip amplitudes $|f|$ and $|g|$ (see Refs. 12–14),

$$|f|,|g\sin\theta| \leq \{\tfrac{1}{2}D[1+(1-P^2)^{1/2}]\}^{1/2}, \tag{1}$$

we try to exploit the elastic unitarity integral to find upper bounds on $\mathrm{Im}f(z)$ and $\mathrm{Im}g(z)$. Consider first

$$\mathrm{Im}f(z) = \frac{q}{2\pi}\int\int dx\,dy\,\frac{\theta(K)}{\sqrt{K}}\,[f^*(x)f(y) + (z-xy)g^*(x)g(y)],$$

where $K = 1 - x^2 - y^2 - z^2 + 2xyz$. $\theta(K)$ ensures that the integration is over the ellipse defined by $K=0$. q is the center-of-mass momentum. We first majorize

$$|f(x)|,\ |f(y)|,\ |g(x)(1-x^2)^{1/2}|,\ \text{and}\ |g(y)(1-y^2)^{1/2}|$$

by taking the largest value (sup) of

$$\{\tfrac{1}{2}D[1+(1-P^2)^{1/2}]\}$$

in the entire angle region with the understanding that each term under the integral will be majorized by positive quantities:

$$\mathrm{Im}f(z) \leq \frac{q}{2\pi}\{\tfrac{1}{2}D[1+(1-P^2)^{1/2}]\}_{\mathrm{sup}}\int\int dx\,dy\,\frac{\theta(K)}{\sqrt{K}}\left[\cos(\phi_f(y)-\phi_f(x)) + \frac{z-xy}{(1-x^2)^{1/2}(1-y^2)^{1/2}}\cos(\phi_g(y)-\phi_g(x))\right].$$

We note that the spin-flip amplitude $g(z)$ appearing in the unitarity relation contains a $\sin\theta$ factor implicitly. That is, the differential cross section $D(z) = |f|^2 + (1-z^2)|g|^2$. Here ϕ_f and ϕ_g are the phases of f and g. The extremum of

$$\frac{z-xy}{(1-x^2)^{1/2}(1-y^2)^{1/2}} \equiv \psi(x,y) \quad (z = \text{fixed})$$

is at $x=y=0$ with $\psi = z$. For this case $|\psi_{\mathrm{sup}}| = 1$. Majorizing now cosine terms by one, we find

$$\mathrm{Im}f(z) \leq \frac{q}{2\pi}\{\tfrac{1}{2}D[1+(1-P^2)^{1/2}]\}_{\mathrm{sup}}\,2\int\int dx\,dy\,\frac{\theta(K)}{\sqrt{K}}.$$

With

$$\frac{\theta(K)}{\sqrt{K}} = \frac{\pi}{2}\sum_{l=0}^{\infty}(2l+1)P_l(y)P_l(z)P_l(x),$$

the integral becomes

$$\int\int dx\,dy\,\frac{\theta(K)}{\sqrt{K}} = \int\frac{\pi}{2}\sum_{l=0}^{\infty}(2l+1)\frac{2}{(2l+1)}\delta_{l0}P_l(y)P_l(z)\,dy = 2\pi.$$

Hence

$$\mathrm{Im}f(z) \leq 2q\{\tfrac{1}{2}D[1+(1-P^2)^{1/2}]\}_{\mathrm{sup}}. \tag{2}$$

The noteworthy aspect of this result is the explicit q dependence as opposed to

$$\mathrm{Im} f(z) \leq \sqrt{D} \text{ or } \{\tfrac{1}{2}D[1+(1-P^2)^{1/2}]\}^{1/2}.$$

We also remark that the full experimental information is not used in the sense that the only piece of data used is the sup of $\frac{1}{2}D[1+(1-P^2)^{1/2}]$ in the entire angle region.

Next we try to improve this bound by better majorizations. Using only D and P we cannot improve $|f|$ or $|g|$. Therefore, the implication is clear: We must improve the phase factors. There is the relation between the sine of the relative phase α of f and g and the polarization[12]:

$$\frac{\sin\alpha}{P} \geq 1, \quad \alpha \equiv \phi_f - \phi_g . \tag{3}$$

We write now $\mathrm{Im} f(z)$ in the form

$$\mathrm{Im} f(z) \leq \frac{q}{2\pi} \int\int dx\, dy\, \frac{\theta(K)}{\sqrt{K}} |f(x)|_{\max} |f(y)|_{\max} \left[\cos(\Delta\alpha+\beta) + \frac{z-xy}{(1-x^2)^{1/2}(1-y^2)^{1/2}} \cos\beta\right], \tag{4}$$

where

$$\Delta\alpha \equiv (\phi_f(y)-\phi_g(y)) - (\phi_f(x)-\phi_g(x)) \equiv \alpha(y)-\alpha(x),$$

$$\beta \equiv \phi_g(y)-\phi_g(x).$$

We already saw that the extremum of

$$\frac{z-xy}{(1-x^2)^{1/2}(1-y^2)^{1/2}}$$

within the unitarity ellipse is z, the largest value of which is one. The above majorization is justified only if the quantities in the bracket are positive. We now find the maximum of the bracket for a fixed $\Delta\alpha$ as a function of β.

Call

$$F(\beta) \equiv \cos(\Delta\alpha+\beta) + \cos\beta. \tag{5}$$

Then

$$\frac{dF}{d\beta} = -\sin(\Delta\alpha+\beta) \pm \sin\beta = 0$$

(see the discussion below for different signs) gives

$$\tan\beta = -\frac{\sin\Delta\alpha}{1 \mp \cos\Delta\alpha}.$$

With this value of β, $F(\beta)$ becomes

$$F(\beta) = [2(1\mp\cos\Delta\alpha)]^{1/2}.$$

There is one point in the majorization of $F(\beta)$ about which we must be careful. Actually we are majorizing $|\cos(\Delta\alpha+\beta)| + |\cos\beta|$. But since $\cos(\Delta\alpha+\beta)$ is just $\cos\beta$ shifted by $\Delta\alpha$, $|F(\beta)|$ will be the larger of the two cases. If in relation (4)

$$\left|\frac{z-xy}{(1-x^2)^{1/2}(1-y^2)^{1/2}}\right| = a$$

is kept rather than majorized by 1, one has to find the maximum of an expression such as

$$|\cos(\Delta\alpha+\beta)| + a|\cos\beta|,$$

where a is positive and less than 1.

In this case further possibilities arise depending on whether a is larger or smaller than $|\cos\Delta\alpha|$. The result here is

$$|F(\beta)| = (1+a^2+2a|\cos\Delta\alpha|)^{1/2} \text{ (see Appendix)}.$$

The imaginary part of f now satisfies

$$\mathrm{Im} f(z) \leq \frac{q}{2\pi}\{\tfrac{1}{2}D[1+(1-P^2)^{1/2}]\}_{\sup} \times \int\int dx\, dy\, \frac{\theta(K)}{\sqrt{K}} (1+a^2+2a|\cos\Delta\alpha|)^{1/2}. \tag{6}$$

We now write $\cos\Delta\alpha$ in the following form:

$$\cos(\alpha(y)-\alpha(x)) = \cos\alpha(y)\cos\alpha(x) + \sin\alpha(y)\sin\alpha(x). \tag{7}$$

We would hope to use (3) in the majorization of expression (7). However, the inequality (3) is in the opposite sense. That is, $|P|$ is not an upper bound for $|\sin\alpha|$ but a lower bound. Still there is information in the inequality (3) which can be used in the following manner.

First of all, (3) tells us that $\sin\alpha$ and P have the same sign. Hence a knowledge of experimental P gives us the sign of the second term in (7). When $\sin\alpha(y)\sin\alpha(x)$ has the same sign as the first term for a pair of points y, x we majorize $\cos\Delta\alpha$ by 1.

When it has the opposite sign we write (7) in the following form:

$$\cos|\alpha(y)-\alpha(x)| = [1-\sin^2\alpha(y)]^{1/2}[1-\sin^2\alpha(x)]^{1/2} - |\sin\alpha(y)||\sin\alpha(x)|. \tag{8}$$

To determine the relative sign of the two terms in (7) one either needs the sign of the R parameter[15] ($R = D^{-1}2|f||g|\cos\alpha\sin\theta$) or cases where $P = 1$.

Obviously we have to know where R (or $\cos\alpha$) changes its sign. From continuity arguments $\cos\alpha$ cannot change its sign unless $\sin\alpha=1$. But at places where $\sin\alpha=1$, P must be 1 as can be seen from relation (3).

We shall study the situation below more in detail. Let us now consider (8), when its two terms have opposite signs. To majorize this we see that in the first term $\sin\alpha$ must be replaced by its smallest value. Similarly the second term must be made as small as possible. Thus we find

$$\cos\Delta\alpha \leq [1-P^2(y)]^{1/2}[1-P^2(x)]^{1/2} - |P(y)||P(x)| . \quad (9)$$

Here inequality (3) has been used in the form

$$|\sin\alpha| \geq |P| .$$

Consider now a typical experimental polarization measurement.[16-20] (Even though the references are for the inelastic case our purpose here is just the discussion on how the unitarity ellipse is divided into different sections. We will not use the inelastic data in our calculations.) Suppose the polarization changes sign twice at points x_1 and x_2 and equals -1 at x_3 (Fig. 1). (Cases where P changes its sign less or more than twice are analyzed in a similar manner. See in particular the K^-p and π^-p data for the cases where P changes its sign three times.)

Since we are interested only in the sign of $\cos\alpha(y)\cos\alpha(x)$ it does not matter which one of the cosines is (+) or (−). For the sake of illustration we choose an arbitrary sign for it and draw $\cos\alpha$ as in Fig. 1. (We are also not considering the case where $\cos\alpha$ may be tangent to the z axis at the point where $\sin\alpha=\pm1$.) Also it does not matter what the value of $\cos\alpha$ is in any region, as long as it does not change sign. $\cos\alpha$ cannot do this without making $\sin\alpha=\pm1$. However, even though we can conclude from $|P|=1$ that $|\sin\alpha|=1$, the reverse is not true and we may lack information about such points where $|P|<1$, $|\sin\alpha|=1$ on whether $\cos\alpha$ changes sign or not. But even with this limited knowledge and possibly experimental

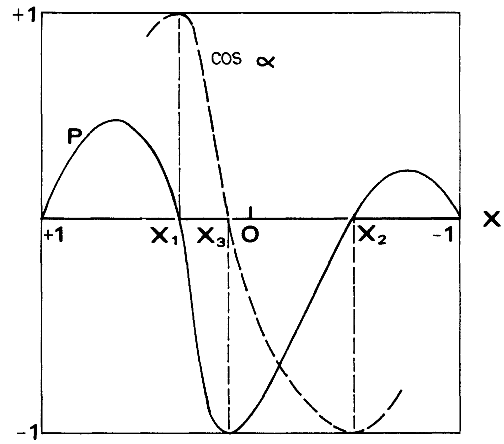

FIG. 1. An example for the polarization curve and the behavior of $\cos\alpha$.

knowledge on the sign of R alone we can do the following.

Let us divide the unitarity ellipse with a knowledge of the zeros of the experimental polarization into the domains shown in Fig. 2. Next we divide the same ellipse according to partial or complete knowledge of the zeros of R (or $\cos\alpha$). We may not know the sign of all domains in this second case. But for the regions in which the signs are known we can combine Fig. 2 with Fig. 3 and find the domains in which the signs are opposite. In those domains we majorize $\cos\Delta\alpha$ by (9); in all other domains we majorize it by 1. The results depend on the values of P as well as on the size of the domains which are determined by the zeros of the polarization and $\cos\alpha$. Here z plays the role of a fixed parameter and when z is changed the shape of the ellipse as well as the size of the domains change.

Before giving numerical results we would like to return to inequality (6). In this inequality only the sup of $\frac{1}{2}D[1+(1-P^2)^{1/2}]$ and the polarization data [through (9)] are being used. If we write (6) in the form

$$\mathrm{Im} f(z) \leq \frac{q}{2\pi}\int\int dx\,dy\,\frac{\theta(K)}{\sqrt{K}}\,(\tfrac{1}{2}D(y)\{1+[1-P^2(y)]^{1/2}\})^{1/2} \times(\tfrac{1}{2}D(x)\{1+[1-P^2(x)]^{1/2}\})^{1/2}(1+a^2+2a|\cos\Delta\alpha|)^{1/2} , \quad (10)$$

we have a better bound, with both D and P being used. Here it is understood that $|\cos\Delta\alpha|$ is majorized by either 1 or expression (9) depending on whether the product of the signs in the domains of the unitarity ellipses is (+) or (−).

To further improve inequality (10) we make the following observation:

Even though the bound (1) is the best one can find for $|f|$ and $|g\sin\theta|$ individually with a knowledge of P and D alone, we can do better when both

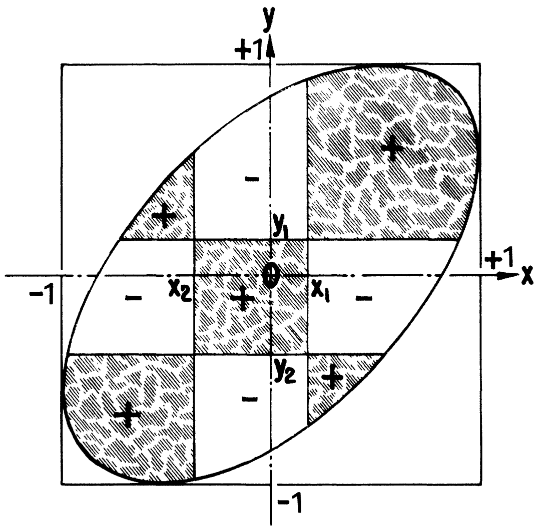

FIG. 2. Separation of the unitarity ellipse into domains of different signs for the term $\sin\alpha(y)\sin\alpha(x)$. .

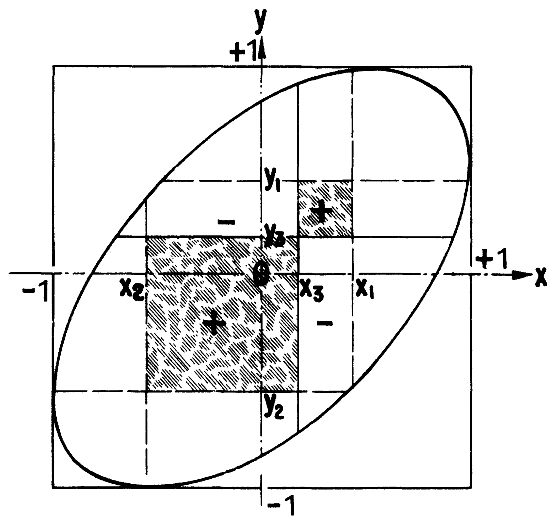

FIG. 3. Separation of the unitarity ellipse into domains of different signs for the term $\cos\alpha(y)\cos\alpha(x)$.

moduli are considered simultaneously. By this we mean that when one of the amplitudes takes its maximum value, the other cannot take its maximum.

Defining

$$|f(x)| \equiv a(x) \equiv x',$$

$$|f(y)| \equiv a(y) \equiv y',$$

we write the integrand of the unitarity integral in the form

$$F = x'y' + a(D - x'^2)^{1/2}(E - y'^2)^{1/2}. \tag{11}$$

Here

$$D \equiv D(x),$$

$$E \equiv D(y),$$

$$a \equiv \left| \frac{z - xy}{(1-x^2)^{1/2}(1-y^2)^{1/2}} \right|.$$

We now want to maximize F. $\partial F/\partial x' = 0$ and $\partial F/\partial y' = 0$ give

$$x'^2 = D,$$

$$y'^2 = E,$$

$$F_{\max} = \sqrt{D}\,\sqrt{E} \equiv [D(x)]^{1/2}[D(y)]^{1/2}. \tag{12}$$

With this result inequality (10) is replaced by

$$\operatorname{Im} f(z) \leq \frac{q}{2\pi} \int\int dx\, dy\, \frac{\theta(K)}{\sqrt{K}}\, [D(x)]^{1/2}[D(y)]^{1/2}. \tag{13}$$

Similarly inequality (2) is improved by almost a factor of 2:

$$\operatorname{Im} f(z) \leq q D_{\sup}. \tag{14}$$

However, we can do better than this, because when we maximized (11) we did not take into account the fact that x' and y' are individually bounded by (1). This means that a knowledge of D only, but not of P, has been used. Hence we should maximize (11) subject to constraints

$$\{\tfrac{1}{2}D[1 - (1 - P^2)^{1/2}]\}^{1/2} \leq |f|, |g \sin\theta| \leq \{\tfrac{1}{2}D[1 + (1 - P^2)^{1/2}]\}^{1/2}.$$

We have already found that the only extremum of (11) is at

$$x' = \sqrt{D} \text{ and } y' = \sqrt{E}.$$

Hence F has no maximum between the values

$$\begin{aligned} x'_l &\equiv \{\tfrac{1}{2}D[1 - (1 - P^2)^{1/2}]\}^{1/2} \leq x' \leq \{\tfrac{1}{2}D[1 + (1 - P^2)^{1/2}]\}^{1/2} \equiv x'_u, \\ y'_l &\equiv \{\tfrac{1}{2}E[1 - (1 - Q^2)^{1/2}]\}^{1/2} \leq y' \leq \{\tfrac{1}{2}E[1 + (1 - Q^2)^{1/2}]\}^{1/2} \equiv y'_u. \end{aligned} \tag{15}$$

Here $Q \equiv P(y)$.

Therefore, the largest value of F must be on the borders. We find that F has actually a maximum on the border plane. Its location is given for example for the plane $x' = x'_u$ by

$$y'^2 = E\frac{x'^2_u}{x'^2_u(1-a^2)+a^2D} .$$

For $Q^2 > P^2$ this falls outside of the limits (15). For $P^2 > Q^2$ its location depends on a.

Since the expressions are cumbersome we shall not pursue this further, but will give at the end of Sec. III numerical results obtained with inequalities (14) and (13).

III. SPIN-FLIP AMPLITUDE

We now turn to the spin-flip amplitude $g(z)$. The elastic unitarity condition is

$$\mathrm{Im}g(z) = \frac{q}{2\pi}\iint dx\,dy\,\frac{\theta(K)}{\sqrt{K}}\,\frac{1}{(1-z^2)}\,[K(x,y,z)\,g^*(x)\,g(y) + (x-zy)f^*(x)\,g(y) + (y-zx)\,g^*(x)f(y)] . \qquad (16)$$

Since we shall majorize $|f|$ and $|g\sin\theta|$ we write out the $\sin\theta$ factors explicitly. We also write the phase factors

$$\mathrm{Im}g(z) = \frac{q}{2\pi}\iint dx\,dy\,\frac{\theta(K)}{\sqrt{K}}\,\frac{1}{(1-z^2)} \times\left[\frac{K|g(x)|\,|g(y)|(1-x^2)^{1/2}(1-y^2)^{1/2}}{(1-x^2)^{1/2}(1-y^2)^{1/2}}\,e^{i[\phi_g(y)-\phi_g(x)]} + (x-zy)\,\frac{|f(x)|\,|g(y)|(1-y^2)^{1/2}}{(1-y^2)^{1/2}}\,e^{i[\phi_g(y)-\phi_f(x)]} + (y-zx)\,\frac{|g(x)|\,|f(y)|(1-x^2)^{1/2}}{(1-x^2)^{1/2}}\,e^{i[\phi_f(y)-\phi_g(x)]}\right]. \qquad (17)$$

Defining again

$$\phi_f(y)-\phi_g(y) \equiv \alpha(y), \quad \phi_g(y)-\phi_g(x) \equiv \beta(y,x) ,$$

this becomes

$$\mathrm{Im}g(z) \le \frac{q}{2\pi}\{\tfrac{1}{2}D[1+(1-P^2)^{1/2}]\}_{\sup}\iint dx\,dy\,\frac{\theta(K)}{\sqrt{K}}\,\frac{1}{(1-z^2)} \times\left\{\frac{K|\cos\beta|}{(1-x^2)^{1/2}(1-y^2)^{1/2}} + \left[\frac{|x-zy|}{(1-y^2)^{1/2}}\,|\cos(\beta-\alpha(x))| + \frac{|y-zx|}{(1-x^2)^{1/2}}\,|\cos(\alpha(y)+\beta)|\right]\right\}. \qquad (18)$$

Here we took $\{\frac{1}{2}D[1+(1-P^2)^{1/2}]\}_{\sup}$ outside of the integral not to clutter the expression. For numerical calculations it will be kept under the integral in its original form as a function of x and y:

$$(\tfrac{1}{2}D(x)\{1+[1-P^2(x)]^{1/2}\})^{1/2} \times(\tfrac{1}{2}D(y)\{1+[1-P^2(y)]^{1/2}\})^{1/2} .$$

Let us now majorize the integrand in (18). The moduli of the amplitudes are majorized and taken outside of the bracket with the understanding that the multiplying factors will be majorized by positive quantities.

Consider first

$$\psi = \frac{1-x^2-y^2-z^2+2xyz}{(1-x^2)^{1/2}(1-y^2)^{1/2}} .$$

$\partial\psi/\partial x = 0$ gives

$$2yz - x + x^3 - xy^2 - xz^2 = 0 .$$

Since ψ is symmetric with respect to x and y, $\partial\psi/\partial y = 0$ will give

$$2xz - y + y^3 - yx^2 - yz^2 = 0 .$$

The solution of these two equations is $x=0$ and $y=0$.

Hence $\psi = 1-z^2$, and the largest value of the first term in the large curly brackets in (18) is $(1-z^2)$. However, except for its simplicity we do not have to use this value, because ψ can be explicitly evaluated.

The second term which is the expression in the square bracket is symmetric in x and y. This can be seen from

$$\beta(y,x)=\phi_g(y)-\phi_g(x)=-\beta(x,y)$$

and

$$\cos(-\beta(y,x)-\alpha(y))=\cos(\beta(y,x)+\alpha(y)).$$

Thus an interchange $x \leftrightarrow y$ takes the first term in the large square brackets in (18) to the second term and the second term to the first term. Let us call the square bracket $\xi(x,y)$. Since $\xi(x,y)$ is symmetric in x and y its extremum is at $x=y$,

$$\xi(x,x)=\frac{x-zx}{(1-x^2)^{1/2}}\cos(\beta(x,x)-\alpha(x)) + \frac{x-zx}{(1-x^2)^{1/2}}\cos(\alpha(x)+\beta(x,x)).$$

Because $\beta(x,x)=0$ and cos is an even function we find

$$\xi=2\frac{|x|}{(1-x^2)^{1/2}}(1-z)\,|\cos\alpha(x)|\,. \qquad (19)$$

We can also find the largest value of

$$\frac{x}{(1-x^2)^{1/2}}$$

on or in the unitarity ellipse [for the time being $\cos\alpha(x)$ is majorized by 1]. As a larger x increases the numerator and decreases the denominator, x must be as large as possible, consistent with the condition to be in or on the ellipse. Since at the largest value $x=y$ it must be *on* the ellipse. Hence

$$1-x^2-y^2-z^2+2xyz=0\,,$$
$$1-2x^2-z^2+2x^2z=0$$

gives

$$x=\pm\left(\frac{1+z}{2}\right)^{1/2}$$

and

$$\frac{x}{(1-x^2)^{1/2}}=\pm\left(\frac{1+z}{1-z}\right)^{1/2}.$$

Finally $\xi=2(1-z^2)^{1/2}$ and the curly bracket in (18) is majorized by

$$[(1-z^2)+2(1-z^2)^{1/2}]\,. \qquad (20)$$

Again except for reasons of simplicity we do not have to use the largest value of $x/(1-x^2)^{1/2}$ since it can be explicitly evaluated.

The unitarity relation for $g(z)$ then becomes

$$\operatorname{Im}g(z)\le\frac{q}{2\pi}\{\tfrac{1}{2}D[1+(1-P^2)^{1/2}]\}_{\sup} \times\iint dx\,dy\,\frac{\theta(K)}{\sqrt{K}}\left[1+\frac{2}{(1-z^2)^{1/2}}\right]. \qquad (21)$$

The bound (20) can be improved if we replace in Eq. (19)

$$\cos\alpha(x)=[1-\sin^2\alpha(x)]^{1/2} \qquad (22)$$

and majorize this by

$$\cos\alpha(x)\le[1-P^2(x)]^{1/2}.$$

In this expression $|x|$ cannot be larger than $[(1+z)/2]^{1/2}$ since it is the coordinate of a point at which $x=y$.

We also replace $|f(x)|$ and $|g(x)(1-x^2)^{1/2}|$ by their bounds in Eq. (21) and find

$$\operatorname{Im}g(z)(1-z^2)^{1/2}\le\frac{q}{2\pi}\iint dx\,dy\,\frac{\theta(K)}{\sqrt{K}}(\tfrac{1}{2}D(x)\{1+[1-P^2(x)]^{1/2}\})^{1/2}(\tfrac{1}{2}D(y)\{1+[1-P^2(y)]^{1/2}\})^{1/2}A\,, \qquad (23)$$

where

$$A_{\max}=\{(1-z^2)^{1/2}+2[1-P^2(x_m)]^{1/2}\}.$$

Here x_m is the cosine of that scattering angle between zero and $\pm[(1+z)/2]^{1/2}$ at which the polarization is smallest.

In its general form the last factor of the integrand in (23) is

$$A=\frac{1}{[(1-z^2)]^{1/2}}\left\{\frac{K}{(1-x^2)^{1/2}(1-y^2)^{1/2}}+2\frac{|x|}{(1-x^2)^{1/2}}(1-z)[1-P^2(x)]^{1/2}\right\}. \qquad (24)$$

When integrating inequality (23) we should remember that the largest value of $|x|$ is $[(1+z)/2]^{1/2}$. Hence for

$$\left(\frac{1+z}{2}\right)^{1/2}\le x\le 1 \quad\text{and}\quad -\left(\frac{1+z}{2}\right)^{1/2}\ge x\ge -1$$

we should take $P(x)=0$.

Up to here we considered the maximum values of the first and second terms of (24) separately, but these terms do not reach their maximum values simultaneously. When we majorize all terms simultaneously we can get a better result. Calling all three terms $A(x)$ and setting $\partial A/\partial x = 0$ leads to

$$x^2 = \frac{1}{1+(1-z)^2} .$$

This value is outside of the unitarity ellipse. So A is a monotonic function. The largest value occurs on the boundary. The first term goes to zero and the second term reaches its maximum value:

$$A_{\max} = 2\cos\alpha \quad \text{or} \quad A_{\max} = 2[1-P^2(x_m)]^{1/2}.$$

In expressions (16) or (17) the symmetry of the integrand with respect to x and y can also be used in a different way. Instead of using this symmetry to find the extremum we can combine the second and third terms as twice of either one of them because the integration area also is symmetric in x and y (unitarity ellipse). Thus we obtain

$$A = \frac{1}{(1-z^2)^{1/2}}\left[\frac{K}{(1-x^2)^{1/2}(1-y^2)^{1/2}}\,|\cos\beta(x,y)| + 2\frac{|x-zy|}{(1-y^2)^{1/2}}|\cos(\beta(x,y)-\alpha(x))|\right] . \qquad (25)$$

Majorizing the cosines by 1 we obtain

$$A = \frac{1}{(1-z^2)^{1/2}}\left[\frac{K}{(1-x^2)^{1/2}(1-y^2)^{1/2}} + 2\,\frac{|x-zy|}{(1-y^2)^{1/2}}\right] . \qquad (26)$$

It is not *a priori* obvious whether (26) or (24) gives a better bound. This will depend on $P(x)$.

We now apply the considerations used for the non spin-flip amplitude also to $g(z)$: namely, that $|f|$ and $|g|$ cannot have their maximum simultaneously. Thus we write the integrand of (16) in the form

$$F = \frac{K}{(1-x^2)^{1/2}(1-y^2)^{1/2}}(D-x'^2)^{1/2}(E-y'^2)^{1/2}\cos(\phi_g(y)-\phi_g(x))$$
$$+\frac{(x-zy)}{(1-y^2)^{1/2}}x'(E-y'^2)^{1/2}\cos(\phi_g(y)-\phi_f(x)) + \frac{(y-zx)}{(1-x^2)^{1/2}}y'(D-x'^2)^{1/2}\cos(\phi_f(y)-\phi_g(x)) . \qquad (27)$$

We observe that this expression is symmetric under the exchange $x \longleftrightarrow y$ since

$$D(x) \longleftrightarrow D(y) \equiv E , \quad |f(x)| \equiv x' \longleftrightarrow y' \equiv |f(y)| .$$

The first term is symmetric by itself. The second and third terms go into each other under this exchange. We can now proceed in two manners:

(a) Since F is symmetric in x and y its extremum must be at $x = y$. This gives us

$$F = (1-z)\left[\frac{(1+z)-2x^2}{1-x^2}(D-x'^2) + 2xx'(D-x'^2)^{1/2}\cos\alpha(x)\right] . \qquad (28)$$

This is of the form

$$F = \beta(D-x'^2) + \gamma x'(D-x'^2)^{1/2} .$$

$\partial F/\partial x' = 0$ gives the location of the extremum:

$$x'^2 = \tfrac{1}{2}D\left[1 \pm \frac{\beta}{(\beta^2+\gamma^2)^{1/2}}\right] .$$

If this expression is used to majorize the integrand everything is known except $\cos\alpha(x)$ which can be majorized as

$$\cos\alpha(x) = [1-\sin^2\alpha(x)]^{1/2} \leq [1-P^2(x)]^{1/2} .$$

(b) As before, we can exploit the symmetry of the integration domain and write F as

$$F = \frac{K}{(1-x^2)^{1/2}(1-y^2)^{1/2}}(D-x'^2)^{1/2}(E-y'^2)^{1/2}\cos(\phi_g(y)-\phi_g(x)) + 2\frac{(x-zy)}{(1-y^2)^{1/2}}x'(E-y'^2)^{1/2}\cos(\phi_g(y)-\phi_f(x)) . \qquad (29)$$

The extremum of F occurs at

$$\frac{\partial F}{\partial x'}=0\,,\quad \frac{\partial F}{\partial y'}=0\,.$$

In the final expression the cosines are majorized by 1. Since those expressions are lengthy and complicated we shall not give them here. Instead we give the numerical results obtained by using (24) and (26) in the integral (23).

IV. COMPARISON WITH DATA

We give below some numerical calculations to test our inequalities.

(a) In relation (14) q is the center-of-mass projectile momentum in $\text{mb}^{-1/2}$. The conversion from MeV/c is achieved by multiplying the momentum with 0.0016,

$$q(\text{mb}^{-1/2})=0.0016q\ (\text{MeV}/c).$$

As an example we take π^+p data at $q_{\text{c.m.}}=91.4$ MeV (Refs. 21 and 22). With negligible polarization at this energy and $D_{\text{sup}}=1.58$ mb relation (14) gives

$$q_{\text{c.m.}}=0.0016\times 91.4=0.146\ \text{mb}^{-1/2},$$

$$\text{Im}f(z)\leqslant 0.146\times 1.58=0.231\ \text{mb}^{-1/2}.$$

This compares with $\text{Im}f(z)\leqslant[D(z)]^{1/2}$ as given by the following examples.

At $z=-0.9062$, $D=1.58$ mb, $\sqrt{D}=1.257\ \text{mb}^{-1/2}$.

At $z=-0.5446$, $D=1.13$ mb, $\sqrt{D}=1.063\ \text{mb}^{-1/2}$.

At $z=0.6018$, $D=0.130$ mb $\sqrt{D}=0.361\ \text{mb}^{-1/2}$.

We see that at this energy the bound (14) is better than $\sqrt{D}$ even for the smallest differential cross section. The goodness of inequality (14) depends on the test

$$qD_{\text{sup}}\leqslant[D(z)]^{1/2}$$

or (30)

$$q\leqslant\frac{[D(z)]^{1/2}}{D_{\text{sup}}},$$

which is satisfied at 91.4 MeV/c.

(b) For $\text{Im}f(z)$ we evaluated both (10) and (13). The maximum of the factor between those two expressions is $(1+a)$. As it turns out (13) is a better bound than (10). For $\text{Im}g(z)\sin\theta$ we used inequality (23) with (24) and (26).

Inequalities (13) and (23) with (24) and (26) are tested for the processes

$$\pi^+p\to\pi^+p,$$
$$\pi^-p\to\pi^-p,$$

and

$$K^+p\to K^+p$$

below the inelastic threshold. In the evaluation of these integrals the computer CDC Cyber 73 of the University of Western Ontario was used. For π^+p and π^-p the data were taken from the phase shifts of Almehed and Lovelace.[23] For K^+p the data were taken from the β phase shifts of Albrow *et al.*[18] The unitarity condition is given for the pure isospin states as discussed previously. For the π^-p case we form the proper combination of different isospin states. In general the lower the energy the better are our bounds as compared to $\sqrt{D}$ or $|f|_+$ and $|g\sin\theta|_+$. But when the energy increases the results are not as good as at lower energies. The bounds from some scattering angles are larger than $\sqrt{D}$. The typical features are given in Tables I–V. In the π^+p and π^-p cases our bounds for the $\text{Im}f$ and $\text{Im}g\sin\theta$ in the entire scattering region are better than $\sqrt{D}$ up to the $P_{\text{lab}}=0.14$ GeV/c (halfway to inelastic threshold). Near the inelastic threshold, the bounds of $\text{Im}g\sin\theta$ are better than $\sqrt{D}$ only in the forward direction. The results are shown in Tables I–IV. In the K^+p case all our bounds for the $\text{Im}f$ and $\text{Im}g\sin\theta$ are better than $\sqrt{D}$ in the entire angle region below the inelastic threshold. We present the highest-energy case below the inelastic threshold in Table V.

V. SUMMARY AND CONCLUSION

We have obtained bounds on the imaginary parts of the spin-nonflip and spin-flip amplitudes using the bounds on the moduli and the relative phase of the full amplitudes. Analyticity in energy has not been used. In principle experimental data can be used directly in numerical integrations. For the sake of convenience we used in numerical calculations the existing phase shifts. Comparison with experiment depends on the energy and values of D and P. Even a limited knowledge of the parameter R (like its sign)[24] would improve the results. The absolute phases are unknown, but a limited knowledge on the relative phase from the polarization can be exploited.

The price paid for using inequalities is recovered by the simpler form of the unitarity relations which make different majorizations feasible. We feel there is still room for improvements, some of which we did not follow because of their complexity for numerical calculations. These we hope to present in the future. For rigorous bounds on the phases from the positivity property of unitarity we refer the reader to Refs. 25 and 26.

TABLE I. $\pi^+ p$ scattering: $P_{lab}=0.1420$ GeV/c, $P_{c.m.}=0.1190$ GeV/c. z = cosine of the scattering angle, sup Imf = the right-hand side of Eq. (13), Imf = the imaginary part of spin-nonflip amplitude calculated from the phase shifts directly, $D^{1/2}$ = square root of the differential cross section, sup$|f|$ = the upper bound of the moduli of the spin-nonflip amplitude calculated from Eq. (1), $[\sup \mathrm{Im}g(1-z^2)^{1/2}]_1$ = the right-hand side of Eq. (23) with Eq. (26), $[\sup \mathrm{Im}g(1-z^2)^{1/2}]_2$ = the right-hand side of Eq. (23) with Eq. (24), $\mathrm{Im}g(1-z^2)^{1/2}$ = the left-hand side of Eq. (23) calculated from phase shifts directly, P = polarization.

z	sup Imf	Imf	$D^{1/2}$	sup$\|f\|$	$[\sup \mathrm{Im}g(1-z^2)^{1/2}]_1$	$[\sup \mathrm{Im}g(1-z^2)^{1/2}]_2$	$\mathrm{Im}g(1-z^2)^{1/2}$	P
0.90	0.216	0.207	0.720	0.717	0.316	0.150	0.037	0.182
0.80	0.213	0.189	0.695	0.689	0.329	0.196	0.051	0.261
0.70	0.211	0.171	0.682	0.674	0.336	0.227	0.061	0.312
0.60	0.208	0.154	0.683	0.673	0.341	0.248	0.068	0.338
0.50	0.206	0.136	0.697	0.687	0.342	0.265	0.073	0.340
0.40	0.204	0.118	0.724	0.714	0.343	0.278	0.078	0.322
0.30	0.202	0.101	0.761	0.753	0.344	0.289	0.081	0.293
0.20	0.200	0.083	0.809	0.802	0.342	0.298	0.083	0.257
0.10	0.198	0.066	0.864	0.859	0.340	0.306	0.084	0.219
−0.00	0.196	0.048	0.927	0.923	0.339	0.314	0.085	0.184
−0.10	0.194	0.030	0.994	0.991	0.334	0.320	0.084	0.153
−0.20	0.192	0.013	1.067	1.065	0.330	0.327	0.083	0.125
−0.30	0.191	−0.005	1.143	1.142	0.325	0.333	0.081	0.101
−0.40	0.189	−0.023	1.223	1.222	0.319	0.339	0.078	0.081
−0.50	0.187	−0.040	1.305	1.305	0.312	0.345	0.073	0.064
−0.60	0.186	−0.058	1.390	1.390	0.304	0.350	0.068	0.049
−0.70	0.184	−0.076	1.477	1.477	0.295	0.355	0.061	0.037
−0.80	0.183	−0.093	1.566	1.566	0.282	0.358	0.051	0.026
−0.90	0.181	−0.111	1.656	1.656	0.265	0.360	0.037	0.016

TABLE II. $\pi^+ p$ scattering: $P_{lab}=0.2370$ GeV/c, $P_{c.m.}=0.1866$ GeV/c. z = cosine of the scattering angle, sup Imf = the right-hand side of Eq. (13), Imf = the imaginary part of spin-nonflip amplitude calculated from the phase shifts directly, $D^{1/2}$ = square root of the differential cross section, sup$|f|$ = the upper bound of the moduli of the spin-nonflip amplitude calculated from Eq. (1), $[\sup \mathrm{Im}g(1-z^2)^{1/2}]_1$ = the right-hand side of Eq. (23) with Eq. (26), $[\sup \mathrm{Im}g(1-z^2)^{1/2}]_2$ = the right-hand side of Eq. (23) with Eq. (24), $\mathrm{Im}g(1-z^2)^{1/2}$ = the left-hand side of Eq.(23) calculated from phase shifts directly, P = polarization.

z	sup Imf	Imf	$D^{1/2}$	sup$\|f\|$	$[\sup \mathrm{Im}g(1-z^2)^{1/2}]_1$	$[\sup \mathrm{Im}g(1-z^2)^{1/2}]_2$	$\mathrm{Im}g(1-z^2)^{1/2}$	P
0.90	3.202	3.008	3.872	3.865	4.671	2.348	0.692	0.118
0.80	3.168	2.688	3.588	3.573	4.881	3.030	0.952	0.181
0.70	3.139	2.367	3.325	3.300	5.005	3.451	1.133	0.240
0.60	3.113	2.046	3.086	3.051	5.091	3.744	1.269	0.297
0.50	3.091	1.725	2.878	2.832	5.136	3.969	1.374	0.351
0.40	3.073	1.405	2.708	2.651	5.179	4.158	1.454	0.397
0.30	3.058	1.084	2.581	2.519	5.226	4.329	1.514	0.427
0.20	3.047	0.763	2.506	2.442	5.238	4.494	1.555	0.436
0.10	3.040	0.443	2.485	2.428	5.238	4.660	1.579	0.418
−0.00	3.036	0.122	2.522	2.475	5.262	4.829	1.587	0.377
−0.10	3.036	−0.199	2.612	2.578	5.232	4.999	1.579	0.320
−0.20	3.039	−0.520	2.752	2.729	5.226	5.183	1.555	0.257
−0.30	3.047	−0.840	2.935	2.921	5.207	5.367	1.514	0.196
−0.40	3.058	−1.161	3.153	3.145	5.155	5.561	1.454	0.143
−0.50	3.072	−1.482	3.399	3.395	5.105	5.755	1.374	0.100
−0.60	3.091	−1.802	3.670	3.668	5.053	5.950	1.269	0.066
−0.70	3.113	−2.123	3.959	3.958	4.962	6.136	1.133	0.040
−0.80	3.138	−2.444	4.264	4.264	4.832	6.298	0.952	0.021
−0.90	3.167	−2.765	4.583	4.583	4.617	6.421	0.692	0.008

TABLE III. $\pi^- p$ scattering: $P_{\text{lab}} = 0.1420$ GeV/c, $P_{\text{c.m.}} = 0.1190$ GeV/c. z = cosine of the scattering angle, sup Imf = the right-hand side of Eq. (13), Imf = the imaginary part of spin-nonflip amplitude calculated from the phase shifts directly, $D^{1/2}$ = square root of the differential cross section, sup$|f|$ = the upper bound of the moduli of the spin-nonflip amplitude calculated from Eq. (1), $[\sup \text{Im}g(1-z^2)^{1/2}]_1$ = the right-hand side of Eq. (23) with Eq. (26), $[\sup \text{Im}g(1-z^2)^{1/2}]_2$ = the right-hand side of Eq. (23) with Eq. (24), $\text{Im}g(1-z^2)^{1/2}$ = the left-hand side of Eq. (23) calculated from phase shifts directly, P = polarization.

z	sup Imf	Imf	$D^{1/2}$	sup$\|f\|$	$[\sup \text{Im}g(1-z^2)^{1/2}]_1$	$[\sup \text{Im}g(1-z^2)^{1/2}]_2$	$\text{Im}g(1-z^2)^{1/2}$	P
0.90	0.128	0.127	0.537	0.536	0.188	0.086	0.011	0.088
0.80	0.127	0.120	0.528	0.527	0.197	0.114	0.015	0.120
0.70	0.126	0.114	0.519	0.518	0.202	0.133	0.018	0.141
0.60	0.125	0.108	0.509	0.508	0.205	0.147	0.020	0.155
0.50	0.124	0.102	0.499	0.498	0.206	0.158	0.022	0.166
0.40	0.123	0.096	0.489	0.487	0.207	0.167	0.023	0.173
0.30	0.122	0.089	0.477	0.475	0.209	0.175	0.024	0.178
0.20	0.121	0.083	0.464	0.463	0.208	0.181	0.025	0.180
0.10	0.121	0.077	0.451	0.449	0.207	0.188	0.025	0.181
−0.00	0.120	0.071	0.436	0.434	0.207	0.193	0.025	0.180
−0.10	0.119	0.064	0.419	0.417	0.204	0.198	0.025	0.178
−0.20	0.118	0.058	0.400	0.399	0.203	0.202	0.025	0.174
−0.30	0.117	0.052	0.380	0.378	0.200	0.206	0.024	0.170
−0.40	0.116	0.046	0.356	0.355	0.196	0.210	0.023	0.164
−0.50	0.116	0.040	0.330	0.329	0.193	0.214	0.022	0.158
−0.60	0.115	0.033	0.299	0.298	0.188	0.217	0.020	0.151
−0.70	0.114	0.027	0.262	0.262	0.183	0.220	0.018	0.145
−0.80	0.114	0.021	0.217	0.217	0.176	0.222	0.015	0.140
−0.90	0.113	0.015	0.156	0.156	0.166	0.224	0.011	0.144

TABLE IV. $\pi^- p$ scattering: $P_{\text{lab}} = 0.2370$ GeV/c, $P_{\text{c.m.}} = 0.1866$ GeV/c. z = cosine of the scattering angle, sup Imf = the right-hand side of Eq. (13), Imf = the imaginary part of spin-nonflip amplitude calculated from the phase shifts directly, $D^{1/2}$ = square root of the differential cross section, sup$|f|$ = the upper bound of the moduli of the spin-nonflip amplitude calculated from Eq. (1), $[\sup \text{Im}g(1-z^2)^{1/2}]_1$ = the right-hand side of Eq. (23) with Eq. (26), $[\sup \text{Im}g(1-z^2)^{1/2}]_2$ = the right-hand side of Eq. (23) with Eq. (24), $\text{Im}g(1-z^2)^{1/2}$ = the left-hand side of Eq. (23) calculated from phase shifts directly, P = polarization.

z	sup Imf	Imf	$D^{1/2}$	sup$\|f\|$	$[\sup \text{Im}g(1-z^2)^{1/2}]_1$	$[\sup \text{Im}g(1-z^2)^{1/2}]_2$	$\text{Im}g(1-z^2)^{1/2}$	P
0.90	1.151	1.089	1.554	1.554	1.631	0.838	0.231	0.020
0.80	1.140	0.982	1.444	1.444	1.757	1.084	0.318	0.029
0.70	1.129	0.874	1.339	1.339	1.802	1.237	0.379	0.035
0.60	1.121	0.767	1.239	1.239	1.833	1.345	0.424	0.039
0.50	1.113	0.660	1.146	1.146	1.850	1.428	0.459	0.038
0.40	1.106	0.552	1.060	1.060	1.865	1.497	0.486	0.031
0.30	1.101	0.445	0.983	0.983	1.882	1.560	0.506	0.016
0.20	1.097	0.338	0.916	0.916	1.886	1.620	0.520	−0.008
0.10	1.094	0.230	0.850	0.860	1.886	1.681	0.528	−0.045
−0.00	1.093	0.123	0.818	0.817	1.894	1.741	0.530	−0.093
−0.10	1.092	0.016	0.791	0.789	1.883	1.802	0.528	−0.148
−0.20	1.093	−0.091	0.780	0.776	1.880	1.867	0.520	−0.204
−0.30	1.096	−0.199	0.785	0.779	1.873	1.931	0.506	−0.252
−0.40	1.099	−0.306	0.806	0.798	1.853	1.999	0.486	−0.284
−0.50	1.104	−0.413	0.841	0.831	1.834	2.067	0.459	−0.297
−0.60	1.109	−0.520	0.888	0.878	1.815	2.135	0.424	−0.291
−0.70	1.116	−0.628	0.944	0.936	1.781	2.199	0.379	−0.267
−0.80	1.125	−0.735	1.009	1.003	1.733	2.255	0.318	−0.225
−0.90	1.134	−0.842	1.080	1.077	1.655	2.297	0.231	−0.162

TABLE V. K^+p scattering: $P_{lab} = 0.5200$ GeV/c, $P_{c.m.} = 0.3100$ GeV/c. z = cosine of the scattering angle, sup Imf = the right-hand side of Eq. (13), Imf = the imaginary part of spin-nonflip amplitude calculated from the phase shifts directly, $D^{1/2}$ = square root of the differential cross section, sup$|f|$ = the upper bound of the moduli of the spin-nonflip amplitude calculated from Eq. (1), $[\sup \mathrm{Im} g(1-z^2)^{1/2}]_1$ = the right-hand side of Eq. (23) with Eq. (26), $[\sup \mathrm{Im} g(1-z^2)^{1/2}]_2$ = the right-hand side of Eq. (23) with Eq. (24), $\mathrm{Im} g(1-z^2)^{1/2}$ = the left-hand side of Eq. (23) calculated from phase shifts directly, P = polarization.

z	sup Imf	Imf	$D^{1/2}$	sup$\|f\|$	$[\sup \mathrm{Im} g(1-z^2)^{1/2}]_1$	$[\sup \mathrm{Im} g(1-z^2)^{1/2}]_2$	$\mathrm{Im} g(1-z^2)^{1/2}$	P
0.90	0.514	0.515	1.015	1.013	0.748	0.306	−0.010	0.138
0.80	0.514	0.511	1.024	1.019	0.787	0.419	−0.013	0.186
0.70	0.514	0.507	1.030	1.024	0.813	0.500	−0.016	0.216
0.60	0.514	0.503	1.036	1.029	0.831	0.565	−0.018	0.237
0.50	0.514	0.499	1.040	1.032	0.845	0.621	−0.019	0.253
0.40	0.514	0.495	1.044	1.034	0.856	0.669	−0.020	0.263
0.30	0.514	0.491	1.046	1.036	0.864	0.712	−0.021	0.270
0.20	0.514	0.487	1.046	1.036	0.869	0.750	−0.022	0.275
0.10	0.514	0.483	1.046	1.036	0.873	0.784	−0.022	0.277
−0.00	0.514	0.479	1.044	1.034	0.874	0.815	−0.022	0.276
−0.10	0.514	0.475	1.042	1.032	0.873	0.844	−0.022	0.274
−0.20	0.514	0.471	1.038	1.028	0.869	0.870	−0.022	0.269
−0.30	0.514	0.467	1.032	1.023	0.864	0.894	−0.021	0.262
−0.40	0.514	0.463	1.026	1.018	0.856	0.916	−0.020	0.253
−0.50	0.514	0.459	1.018	1.011	0.845	0.936	−0.019	0.240
−0.60	0.514	0.455	1.009	1.003	0.831	0.955	−0.018	0.223
−0.70	0.514	0.451	0.999	0.994	0.812	0.971	−0.016	0.202
−0.80	0.514	0.447	0.987	0.984	0.787	0.986	−0.013	0.172
−0.90	0.514	0.443	0.974	0.972	0.748	1.000	−0.010	0.127

APPENDIX

The maximum of

$$F(\beta) = |\cos(\Delta\alpha + \beta)| + a|\cos\beta|$$

occurs at

$$-\sin(\Delta\alpha + \beta) \pm a \sin\beta = 0.$$

This leads to

$$|\cos\beta| = \frac{|\cos\Delta\alpha \mp a|}{(1 + a^2 \mp 2a\cos\Delta\alpha)^{1/2}}.$$

$F(\beta)$ then becomes

$$F(\beta) = \frac{1 \mp a\cos\Delta\alpha + a|\cos\Delta\alpha \mp a|}{(1 + a^2 \mp 2a\cos\Delta\alpha)^{1/2}}.$$

The following cases arise:

(I) (a) Sign (−), $\cos\Delta\alpha > 0$, $a < \cos\Delta\alpha$: In this case

$$F(\beta) = \frac{1 - a^2}{(1 + a^2 - 2a\cos\Delta\alpha)^{1/2}}.$$

(b) Sign (−), $\cos\Delta\alpha > 0$, $a > \cos\Delta\alpha$: In this case

$$F(\beta) = (1 + a^2 - 2a\cos\Delta\alpha)^{1/2}.$$

(c) Sign (−), $\cos\Delta\alpha < 0$:

$$F(\beta) = (1 + a^2 + 2a|\cos\Delta\alpha|)^{1/2}.$$

(II) (a) Sign (+), $\cos\Delta\alpha > 0$:

$$F(\beta) = (1 + a^2 + 2a|\cos\Delta\alpha|)^{1/2}.$$

(b) Sign (+), $\cos\Delta\alpha < 0$, $a < |\cos\Delta\alpha|$:

$$F(\beta) = \frac{1 - a^2}{(1 + a^2 - 2a|\cos\Delta\alpha|)^{1/2}}.$$

(c) Sign (+), $\cos\Delta\alpha < 0$, $a > |\cos\Delta\alpha|$:

$$F(\beta) = (1 + a^2 - 2a|\cos\Delta\alpha|)^{1/2}.$$

It is readily shown that if $\cos\Delta\alpha > 0$ [II(a)] gives a larger $F(\beta)$ than [(I)(a)] or [(I)(b)]. Similarly if $\cos\Delta\alpha < 0$ [(I)(c)] gives a larger $F(\beta)$ than [(II)(b)] or [(II)(c)]. Thus when $\cos\Delta\alpha >$ we choose [(II)(a)]; when $\cos\Delta\alpha < 0$ we choose [(I)(c)]. In either case

$$F(\beta) = (1 + a^2 + 2a|\cos\Delta\alpha|)^{1/2}.$$

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