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Study of (3, 3*) ⊕ (3*,3) ⊕ (6, 6*) ⊕ (6*,6) and linear and bilinear (3, 3*) ⊕ (3*,3) chiral-symmetry breaking in a linear σ model*

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Two chiral-symmetry-breaking schemes are considered within the context of a linear SU(3) σ model. In the first the symmetry-breaking terms are constructed from the usual linear (in the fields) $(3,3^*)\oplus(3^*,3)$ contribution together with bilinear combinations belonging to the $(6,6^*)\oplus(6^*,6)$ representation of SU(3) × SU(3). In the second scheme a combination of linear and bilinear $(3,3^*)\oplus(3^*,3)$ terms is employed. The pseudoscalar mesons (π, K, η, η') and scalar mesons $(\epsilon, \kappa, \sigma, \sigma')$ are assigned to the $(3,3^*)\oplus(3^*,3)$ representation in this model which is used to describe their mass spectra, various scalar meson decays, $\eta'\to\eta\pi\pi$, and the $\pi\pi$ and πK scattering lengths. With the addition of isospin-violating terms to the Lagrangian additional, electromagnetic effects can be taken into account. In each of the symmetry-breaking schemes considered it is found that a two-parameter form in which the linear $(3,3^*)\oplus(3^*,3)$ contribution is SU(2) × SU(2)-invariant and the bilinear contribution is SU(3)-invariant is about as successful as the usual two-parameter linear $(3,3^*)\oplus(3^*,3)$ form.

I. INTRODUCTION

Chiral SU(3)×SU(3) symmetry[1] is an attractive framework in which to study the interactions of hadrons. As this symmetry is not exact, however, a great deal of effort has been devoted to the exploration of the manner in which it is broken. One popular proposal is that the symmetry is spontaneously broken[2-4]; i.e., in the limit where the Lagrangian is chiral-symmetric, the vacuum is not invariant under chiral transformations. The octet of pseudoscalar mesons, which would be massless Nambu-Goldstone bosons in the chiral-symmetric limit, acquire a mass through the explicit symmetry-breaking terms present in the interaction Lagrangian.

Of the many possible forms for the symmetry-breaking Lagrangian, a simple choice[1,5,6] is to assign it to the $(3,3^*)\oplus(3^*,3)$ representation of SU(3)×SU(3). The symmetry-breaking Hamiltonian density, $\mathcal{H}_{SB}$, then takes the form

$$\mathcal{H}_{SB} = -c_0 u_0 - c_8 u_8 , \tag{1.1}$$

where the u_i $(i=0,\ldots,8)$ are the nine scalar density operators belonging to the $(3,3^*)\oplus(3^*,3)$ representation of SU(3)×SU(3). While this choice is very appealing, especially the Gell-Mann, Oakes, Renner (GMOR)[5] scheme with SU(2)×SU(2) a better symmetry than SU(3) (thus accounting for the relatively small pion mass), it is not the only appealing scheme and, in addition, it may have problems. Indications of the latter could be the rather large πN σ term found by many authors,[7] the possibly large $\pi\pi$ scattering lengths,[8] and the calculations of the decay rate[9] for $\eta\to\pi^+\pi^-\pi^0$.

Consequently, other forms of chiral-symmetry breaking have been investigated. The most popular choices assign $\mathcal{H}_{SB}$ to the (8, 8) (see Ref. 10) and the $(6,6^*)\oplus(6^*,6)$ (see Ref. 11) representations of SU(3)×SU(3). However, if one associates the smallness of the pion mass with an approximate SU(2)×SU(2) symmetry of the strong-interaction Hamiltonian, these assignments must be rejected in favor of those that retain the GMOR type of symmetry breaking as the dominant type and add contributions from other possible types of symmetry breaking as small correction terms.[12] We would then have

$$\mathcal{H}_{SB} = -c_0(u_0 - \sqrt{2}\,u_8) + \mathcal{H}'_{SB} , \tag{1.2}$$

where $\mathcal{H}'_{SB}$ is a small SU(2)×SU(2)-breaking part of $\mathcal{H}_{SB}$, with its SU(3)×SU(3)-symmetry-breaking properties being *a priori* unspecified.

A model of this type was first introduced by Okubo.[13] He argued that, if the only SU(3)-breaking part of $\mathcal{H}_{SB}$ belonged to the $(3,3^*)\oplus(3^*,3)$ representation, then $\mathcal{H}_{SB}$ must have the structure of Eq. (1.2) with $\mathcal{H}'_{SB}$ restricted to be SU(3)-invariant. The GMOR scheme is a member of this class of models.

A Hamiltonian of the form in Eq. (1.2) has also been considered by Sirlin and Weinstein,[14] who assign $\mathcal{H}'_{SB}$ to the (8, 8) representation and, in addition, allow it to break SU(3).

In an earlier work[15] we investigated these proposals, and others, for $\mathcal{H}_{SB}$ within the context of a linear SU(3) σ model,[16] with $\mathcal{H}_{SB}$ assigned to the $(3,3^*)\oplus(3^*,3)\oplus(8,8)$ representation of SU(3)×SU(3). It was found that attractive alternatives to the GMOR model do exist. In particular, the Okubo form seems to describe the phenomena considered at least as well as the GMOR model.

In this paper we continue our investigation of various types of symmetry breaking within the framework of the σ model. We will consider here two cases. In the first, $\mathcal{H}_{SB}$ is assigned to the $(3,3^*)\oplus(3^*,3)\oplus(6,6^*)\oplus(6^*,6)$ representation, while in the second it is constructed out of two different $(3,3^*)\oplus(3^*,3)$ representations.

Our methods of investigation in the present study, as in I, will parallel those of Schechter and Ueda and their collaborators,[17-19] who employed the linear SU(3) σ model to study $(3,3^*)\oplus(3^*,3)$ symmetry breaking.[20] This σ model is based on a Lagrangian which is constructed out of SU(3) nonets of pseudoscalar (π, K, η, η') and scalar $(\epsilon, \kappa, \sigma, \sigma')$ fields assigned to the $(3,3^*)\oplus(3^*,3)$ representation.[21] The "potential energy" part of the Lagrangian can be separated into an SU(3)×SU(3)-invariant part, V_0, and one which breaks chiral symmetry, V_{SB}. The ground state of the system is assumed to occur at those (constant) values of the fields which minimize V_0+V_{SB}. The Lagrangian is expanded in terms of fields displaced from their ground-state values, and the constant coefficients in the expansion are identified with particle masses and couplings.

We are interested in two forms of V_0. In the general model V_0 is allowed to be any arbitrary nonderivative function of the basic fields. In the renormalizable model[19,22] V_0 is forbidden to contain terms of degree greater than 4 in the fields. In the general model most of our results can be obtained by specifying only the form of V_{SB}, the symmetry-breaking part of the Lagrangian. However, a complete description is not possible without further constraints. To obtain these restrictions one can consider, as a special case of the general model, the case where V_0 is required to be invariant under scale transformations.[23]

The GMOR symmetry-breaking scheme can be most simply accommodated in this model by taking V_{SB} to be a linear combination of the appropriate scalar meson fields. This has been the usual practice.[16-20,22] In I we studied the effect of adding to this linear $(3,3^*)\oplus(3^*,3)$ combination terms which are bilinear in the fields and which transform like (8, 8). For the isospin-conserving part of V_{SB} there are, in general, four parameters required to describe this model compared with the two-parameter GMOR scheme. Various combinations of terms belonging to $(3,3^*)\oplus(3^*,3)\oplus(8,8)$ were studied and it was found that, even with one or two more parameters, most of these were unable to fit the data as well as the simple GMOR model. However, the two-parameter Okubo scheme was found to be at least as successful as that of GMOR.

Here we will consider two further possibilities. One consists of the addition of bilinear $(6,6^*)\oplus(6^*,6)$ terms to the usual linear $(3,3^*)\oplus(3^*,3)$ combination, leading to a four-parameter model of the symmetry-breaking Lagrangian. It is found that the two-parameter Okubo form of this scheme is essentially as effective in fitting the data as is the GMOR. For the other case we will add bilinear $(3,3^*)\oplus(3^*,3)$ terms to the usual linear ones.[24] This is a four-parameter model of the Okubo form. In this model there is the *a priori* possibility of an alternative GMOR model in which V_{SB} contains, for example, only bilinear $(3,3^*)\oplus(3^*,3)$ terms. While this choice turns out not to work, we do find another two-parameter model, whose linear and bilinear parts are respectively SU(2)×SU(2)- and SU(3)-invariant, which works about as well as the linear GMOR model. The precise structure of the $(3,3^*)\oplus(3^*,3)\oplus(6,6^*)\oplus(6^*,6)$ and the linear and bilinear $(3,3^*)\oplus(3^*,3)$ symmetry-breaking contributions in terms of the fields will be derived in Sec. II.

In the following investigation we will consider many special cases of the two general symmetry-breaking forms mentioned above. All quantities are calculated in the tree approximation. A brief outline of the calculations will be given in Sec. III. For our strong-interaction calculations we consider the meson mass spectra, the scalar meson decays, the decay $\eta'\to\eta\pi\pi$, and the $\pi\pi$ and πK scattering lengths.

Isospin violations are introduced by adding an $I=1$ term to V_{SB}. Terms of this type may arise from higher-order electromagnetic tadpole contributions. The Lagrangian in this form is employed to investigate the electromagnetic mass shifts and the decay $\eta\to 3\pi$.

In I an effective nonleptonic part of the Lagrangian was included in a not-too-successful attempt to describe the $K\to 2\pi$ and $K\to 3\pi$ decays. We found that these decays could not be described any more successfully in the present schemes. As the K decays do not provide a reliable means of discriminating between the various symmetry-breaking schemes, we will not discuss them further here.

As inputs for our numerical computations we used the experimentally better determined quantities such as the pseudoscalar meson masses and the pion and kaon leptonic decay constants. In some cases scalar meson masses must be input as well. These are chosen, where possible, from the latest data compilation.[25] Details of the numerical analysis are given in Sec. IV.

A discussion of our results will be presented in Sec. V.

II. THE FORM OF THE SYMMETRY-BREAKING TERMS

As in I we assume that the nonweak symmetry-breaking part of the Lagrangian has the decomposition

$$V_{\mathrm{SB}} = V_{\mathrm{SB}}^{I=0} + V_{\mathrm{SB}}^{I=1} + V_{\mathrm{SB}}^{\gamma} , \tag{2.1}$$

where $I = 0$ and $I = 1$ denote the isospin content of the terms. The isospin-conserving term is necessary for a realistic description of strong-interaction phenomena. The isospin-violating terms, $V_{\mathrm{SB}}^{I=1}$ and V_{SB}^{γ}, are required for a treatment of electromagnetic phenomena.

We are interested in two types of chiral-SU(3) $\times$SU(3)-symmetry breaking. These involve expressions both linear and bilinear in the basic fields. As indicated above, the two forms of interest are

$$\mathcal{L}_{\mathrm{SB}} \in \text{linear } (3, 3^*)\oplus(3^*, 3)\oplus \text{bilinear } (6, 6^*)\oplus(6^*, 6)$$

and (2.2)

$$\mathcal{L}_{\mathrm{SB}} \in \text{linear } (3, 3^*)\oplus(3^*, 3)\oplus\text{bilinear } (3, 3^*)\oplus(3^*, 3) .$$

The linear $(3, 3^*)\oplus(3^*, 3)$ has been discussed in detail in I and, consequently, we now investigate the bilinear forms. In particular, we are interested in the even-parity nonets contained in these expressions.

The basic nonets of fields contained in the Lagrangian are

$$M_{\bar a}^{b} = (S + i\phi)_a^b$$

and (2.3)

$$M_a^{\bar b} = (S - i\phi)_a^b \quad (a, b = 1, 2, 3) ,$$

which transform as $(3, 3^*)$ and $(3^*, 3)$ respectively under SU(3)$\times$SU(3). S and ϕ denote the nonets of scalar $(\epsilon, \kappa, \sigma, \sigma')$ and pseudoscalar (π, K, η, η') fields, respectively. The upper (lower) index denotes the 3 (3*) SU(3) index. The barred (unbarred) index denotes the right- (left-) hand space of SU(3)$\times$SU(3). For further discussion of the properties of the basic fields see I.

Using the standard SU(3) Clebsch-Gordan series for the direct product of irreducible representations,

$$3\times 3 = 6\oplus 3^*$$

and (2.4)

$$3^* \times 3^* = 6^*\oplus 3,$$

we have

$$(3, 3^*)\times(3, 3^*) = (6, 6^*)\oplus(6, 3)\oplus(3^*, 6^*)\oplus(3^*, 3)$$

and (2.5)

$$(3^*, 3)\times(3^*, 3) = (6^*, 6)\oplus(6^*, 3^*)\oplus(3, 6)\oplus(3, 3^*).$$

Decomposing the direct product of the M's into forms which generate irreducible representations under SU(3)$\times$SU(3) we have

$$\begin{aligned} M_{\bar b}^{a} M_{\bar d}^{c} = {} & \tfrac14(M_{\bar b}^{a} M_{\bar d}^{c} + M_{\bar b}^{c} M_{\bar d}^{a} + M_{\bar d}^{a} M_{\bar b}^{c} + M_{\bar d}^{c} M_{\bar b}^{a}) \Rightarrow (6, 6^*) \\ & + \tfrac14(M_{\bar b}^{a} M_{\bar d}^{c} - M_{\bar b}^{c} M_{\bar d}^{a} + M_{\bar d}^{a} M_{\bar b}^{c} - M_{\bar d}^{c} M_{\bar b}^{a}) \Rightarrow (3^*, 6^*) \\ & + \tfrac14(M_{\bar b}^{a} M_{\bar d}^{c} + M_{\bar b}^{c} M_{\bar d}^{a} - M_{\bar d}^{a} M_{\bar b}^{c} - M_{\bar d}^{c} M_{\bar b}^{a}) \Rightarrow (6, 3) \\ & + \tfrac14(M_{\bar b}^{a} M_{\bar d}^{c} - M_{\bar b}^{c} M_{\bar d}^{a} - M_{\bar d}^{a} M_{\bar b}^{c} + M_{\bar d}^{c} M_{\bar b}^{a}) \Rightarrow (3^*, 3) , \end{aligned} \tag{2.6}$$

where the representations to be generated are indicated. A similar decomposition can be written for $M_b^{\bar a} M_d^{\bar c}$ with the representations generated as indicated in Eq. (2.5). Consequently, the tensors of interest are

$$T_{1\bar b \bar d}^{\ a\ c} = M_{\bar b}^{a} M_{\bar d}^{c} + M_{\bar b}^{c} M_{\bar d}^{a} + M_{\bar d}^{a} M_{\bar b}^{c} + M_{\bar d}^{c} M_{\bar b}^{a} \Rightarrow (6, 6^*) ,$$

(2.7)

$$T_{2 b\, d}^{\ \bar a\, \bar c} = M_b^{\bar a} M_d^{\bar c} + M_b^{\bar c} M_d^{\bar a} + M_d^{\bar a} M_b^{\bar c} + M_d^{\bar c} M_b^{\bar a} \Rightarrow (6^*, 6) ,$$

$$T_{3\bar b \bar d}^{\ a\ c} = M_{\bar b}^{a} M_{\bar d}^{c} - M_{\bar b}^{c} M_{\bar d}^{a} - M_{\bar d}^{a} M_{\bar b}^{c} + M_{\bar d}^{c} M_{\bar b}^{a} \Rightarrow (3^*, 3) ,$$

(2.8)

$$T_{4 b\, d}^{\ \bar a\, \bar c} = M_b^{\bar a} M_d^{\bar c} - M_b^{\bar c} M_d^{\bar a} - M_d^{\bar a} M_b^{\bar c} + M_d^{\bar c} M_b^{\bar a} \Rightarrow (3, 3^*) .$$

In Secs. II A and II B below we isolate the even-parity SU(3) nonet (octet + singlet) for bilinear $(6, 6^*)\oplus(6^*, 6)$ and $(3, 3^*)\oplus(3^*, 3)$, respectively.

A. Bilinear $(6,6^*)\oplus(6^*,6)$

First we note that for a tensor to transform as an irreducible representation it must be both symmetric and traceless. Consequently, to develop the SU(3) properties of $T_{1\bar b \bar d}^{\ a\ c} \in (6, 6^*)$ as given in Eq. (2.7) we write (with repeated indices summed over)

$$\begin{aligned} T_{1\bar b \bar d}^{\ a\ c} = {} & [M_{\bar b}^{a} M_{\bar d}^{c} + M_{\bar b}^{c} M_{\bar d}^{a} + M_{\bar d}^{a} M_{\bar b}^{c} + M_{\bar d}^{c} M_{\bar b}^{a} - \tfrac23 \delta_b^a (M_{\bar\alpha}^{\alpha} M_{\bar d}^{c} + M_{\bar\alpha}^{c} M_{\bar d}^{\alpha})] \Rightarrow \underline{27} \\ & + \tfrac23 \delta_b^a [M_{\bar\alpha}^{\alpha} M_{\bar d}^{c} + M_{\bar\alpha}^{c} M_{\bar d}^{\alpha} - \tfrac13 \delta_d^c (M_{\bar\alpha}^{\alpha} M_{\bar\beta}^{\beta} + M_{\bar\alpha}^{\beta} M_{\bar\beta}^{\alpha})] \Rightarrow \underline{8} \\ & + \tfrac29 \delta_b^a \delta_d^c (M_{\bar\alpha}^{\alpha} M_{\bar\beta}^{\beta} + M_{\bar\beta}^{\alpha} M_{\bar\alpha}^{\beta}) \Rightarrow \underline{1} , \end{aligned} \tag{2.9}$$

where the irreducible SU(3) representations generated are again indicated. Thus, the nonet in (6, 6*) can be written as

$$N_{1}{}^{\underline{c}}_{d}=M^{\underline{\alpha}}_{\alpha}M^{\underline{c}}_{d}+M^{\underline{c}}_{\alpha}M^{\underline{\alpha}}_{d}\,. \tag{2.10}$$

A similar decomposition can be written for $T_{2}{}^{\bar{a}}_{b}{}^{\bar{c}}_{d}$ with the nonet of interest being

$$N_{2}{}^{\bar{c}}_{d}=M^{\bar{\alpha}}_{\alpha}M^{\bar{c}}_{d}+M^{\bar{c}}_{\alpha}M^{\bar{\alpha}}_{d}\,. \tag{2.11}$$

Using Eq. (2.3) to express N_1 in terms of the scalar and pseudoscalar fields we find

$$\begin{aligned} N_{1}{}^{c}_{d} &= S^{\alpha}_{\alpha}S^{c}_{d}-\phi^{\alpha}_{\alpha}\phi^{c}_{d}+S^{c}_{\alpha}S^{\alpha}_{d}-\phi^{c}_{\alpha}\phi^{\alpha}_{d} \\ &\quad + i\,(S^{\alpha}_{\alpha}\phi^{c}_{d}+\phi^{\alpha}_{\alpha}S^{c}_{d}+S^{c}_{\alpha}\phi^{\alpha}_{d}+\phi^{c}_{\alpha}S^{\alpha}_{d}). \end{aligned} \tag{2.12}$$

After writing $N_{2}{}^{c}_{d}$ in a similar manner we find that the even-parity nonet has the form

$$(E^{6})^{t}_{s}=S^{\alpha}_{\alpha}S^{t}_{s}+S^{t}_{\alpha}S^{\alpha}_{s}-\phi^{\alpha}_{\alpha}\phi^{t}_{s}-\phi^{t}_{\alpha}\phi^{\alpha}_{s}\,. \tag{2.13}$$

This expression will be used to generate the symmetry-breaking terms. In anticipation of calculations to be made below we note

$$(E^{6})^{\alpha}_{\alpha}=S^{\alpha}_{\alpha}S^{\beta}_{\beta}+S^{\alpha}_{\beta}S^{\beta}_{\alpha}-\phi^{\alpha}_{\alpha}\phi^{\beta}_{\beta}-\phi^{\alpha}_{\beta}\phi^{\beta}_{\alpha} \tag{2.14}$$

and

$$(E^{6})^{3}_{3}=S^{\alpha}_{\alpha}S^{3}_{3}+S^{3}_{\alpha}S^{\alpha}_{3}-\phi^{\alpha}_{\alpha}\phi^{3}_{3}-\phi^{3}_{\alpha}\phi^{\alpha}_{3}\,. \tag{2.15}$$

B. Bilinear (3,3*)⊕(3*,3)

From the forms of T_3 and T_4 given in Eq. (2.8) we see that we must contract $T_{3}{}^{\underline{a}}_{b}{}^{\underline{c}}_{d}$ and $T_{4}{}^{\bar{a}}_{b}{}^{\bar{c}}_{d}$ with $\epsilon^{\bar{t}\,\bar{b}\,\bar{d}}\,\epsilon_{sac}$ and $\epsilon^{tbd}\,\epsilon_{\bar{s}\bar{a}\bar{c}}$, respectively, in order to generate a symmetric tensor. Consequently, the nonets of interest in (3*, 3) and (3, 3*) are

$$N_{3}{}^{\bar{t}}_{s}=\epsilon^{\bar{t}\,\bar{b}\,\bar{d}}\,\epsilon_{sac}M^{\underline{a}}_{b}M^{\underline{c}}_{d} \tag{2.16}$$

and

$$N_{4}{}^{t}_{\bar{s}}=\epsilon^{tbd}\,\epsilon_{\bar{s}\,\bar{a}\,\bar{c}}M^{\bar{a}}_{b}M^{\bar{c}}_{d}\,, \tag{2.17}$$

respectively. Again substituting for the M's, using Eq. (2.3), and keeping the even-parity terms we have the even-parity nonet

$$\begin{aligned} (E^{3})^{t}_{s} &= 2S^{t}_{s}S^{\alpha}_{\alpha}-2S^{t}_{\alpha}S^{\alpha}_{s}-2\phi^{t}_{s}\phi^{\alpha}_{\alpha}+2\phi^{t}_{\alpha}\phi^{\alpha}_{s} \\ &\quad -\delta^{t}_{s}\,(S^{\alpha}_{\alpha}S^{\beta}_{\beta}-S^{\alpha}_{\beta}S^{\beta}_{\alpha}-\phi^{\alpha}_{\alpha}\phi^{\beta}_{\beta}+\phi^{\alpha}_{\beta}\phi^{\beta}_{\alpha})\,. \end{aligned} \tag{2.18}$$

For calculations below we again note

$$(E^{3})^{\alpha}_{\alpha}=-(S^{\alpha}_{\alpha}S^{\beta}_{\beta}-S^{\alpha}_{\beta}S^{\beta}_{\alpha}-\phi^{\alpha}_{\alpha}\phi^{\beta}_{\beta}+\phi^{\alpha}_{\beta}\phi^{\beta}_{\alpha}) \tag{2.19}$$

and

$$\begin{aligned} (E^{3})^{3}_{3} &= 2S^{3}_{3}S^{\alpha}_{\alpha}-2S^{3}_{\alpha}S^{\alpha}_{3}-2\phi^{3}_{3}\phi^{\alpha}_{\alpha}+2\phi^{3}_{\alpha}\phi^{\alpha}_{3} \\ &\quad -(S^{\alpha}_{\alpha}S^{\beta}_{\beta}-S^{\alpha}_{\beta}S^{\beta}_{\alpha}-\phi^{\alpha}_{\alpha}\phi^{\beta}_{\beta}+\phi^{\alpha}_{\beta}\phi^{\beta}_{\alpha})\,. \end{aligned} \tag{2.20}$$

With the above expressions we can now give our symmetry-breaking terms explicitly. We consider the isospin-conserving part of $V_{\rm SB}$ in Sec. II C below and the isospin-violating component of $V_{\rm SB}$ in Sec. II D.

C. The isospin-conserving part of V

We assume that the isospin-conserving part of $V_{\rm SB}$ has the form

$$V^{I=0}_{\rm SB}=-c_0u_0-c_8u_8-\begin{pmatrix} e_0w_0 \\ c_0'u_0' \end{pmatrix}-\begin{pmatrix} e_8w_8 \\ c_8'u_8' \end{pmatrix}, \tag{2.21}$$

where u_0 and u_8 are the scalar SU(3) singlet and $I=0$ octet members of the linear (3, 3*)⊕(3*, 3) representation of SU(3)⊗SU(3). w_0 (u_0') and w_8 (u_8') are the same respective components of bilinear (6, 6*)⊕(6*, 6) (bilinear (3, 3*)⊕(3*, 3)). Following the analysis of I we rewrite Eq. (2.21) as

$$V^{I=0}_{\rm SB}=-g_0S^{\alpha}_{\alpha}-g_3S^{3}_{3}-\begin{pmatrix} k_0(E^{6})^{\alpha}_{\alpha} \\ g_0'(E^{3})^{\alpha}_{\alpha} \end{pmatrix}-\begin{pmatrix} k_3(E^{6})^{3}_{3} \\ g_3'(E^{3})^{3}_{3} \end{pmatrix}, \tag{2.22}$$

where

$$\begin{aligned} g_0 &= \frac{1}{\sqrt{6}}\,(\sqrt{2}\,c_0+c_8)\,, \\ g_3 &= -(\tfrac{3}{2})^{1/2}c_8\,, \end{aligned} \tag{2.23}$$

$$\begin{aligned} k_0 &= \frac{1}{\sqrt{6}}\,(\sqrt{2}\,e_0+e_8)\,, \\ k_3 &= -(\tfrac{3}{2})^{1/2}e_8\,, \end{aligned} \tag{2.24}$$

$$g_0'=\frac{1}{\sqrt{6}}\,(\sqrt{2}\,c_0'+c_8')\,,$$

and

$$g_3'=-(\tfrac{3}{2})^{1/2}c_8'\,. \tag{2.25}$$

Finally, in terms of the scalar and pseudoscalar fields, we have for the (6, 6*)⊕(6*, 6) case, using Eqs. (2.14) and (2.15),

$$\begin{aligned} V^{I=0}_{\rm SB} &= -g_0S^{\alpha}_{\alpha}-g_3S^{3}_{3} \\ &\quad -k_0(S^{\alpha}_{\alpha}S^{\beta}_{\beta}+S^{\alpha}_{\beta}S^{\beta}_{\alpha}-\phi^{\alpha}_{\alpha}\phi^{\beta}_{\beta}-\phi^{\alpha}_{\beta}\phi^{\beta}_{\alpha}) \\ &\quad -k_3(S^{\alpha}_{\alpha}S^{3}_{3}+S^{3}_{\alpha}S^{\alpha}_{3}-\phi^{\alpha}_{\alpha}\phi^{3}_{3}-\phi^{3}_{\alpha}\phi^{\alpha}_{3})\,. \end{aligned} \tag{2.26}$$

For the bilinear (3, 3*)⊕(3*, 3) case we have, using Eqs. (2.19) and (2.20),

$$\begin{aligned} V^{I=0}_{\rm SB} &= -g_0S^{\alpha}_{\alpha}-g_3S^{3}_{3} \\ &\quad +g_0'(S^{\alpha}_{\alpha}S^{\beta}_{\beta}-S^{\alpha}_{\beta}S^{\beta}_{\alpha}-\phi^{\alpha}_{\alpha}\phi^{\beta}_{\beta}+\phi^{\alpha}_{\beta}\phi^{\beta}_{\alpha}) \\ &\quad -g_3'[\,2S^{3}_{3}S^{\alpha}_{\alpha}-2S^{3}_{\alpha}S^{\alpha}_{3}-2\phi^{3}_{3}\phi^{\alpha}_{\alpha}+2\phi^{3}_{\alpha}\phi^{\alpha}_{3} \\ &\qquad -(S^{\alpha}_{\alpha}S^{\beta}_{\beta}-S^{\alpha}_{\beta}S^{\beta}_{\alpha}-\phi^{\alpha}_{\alpha}\phi^{\beta}_{\beta}+\phi^{\alpha}_{\beta}\phi^{\beta}_{\alpha})]\,. \end{aligned} \tag{2.27}$$

D. The isospin-violating part of V

This component of $V_{\rm SB}$ receives contributions from two sources, $V^{I=1}_{\rm SB}$ and $V^{\gamma}_{\rm SB}$. $V^{I=1}_{\rm SB}$ may be considered to arise from an electromagnetic tadpole mechanism. The term $V^{\gamma}_{\rm SB}$ is introduced to account for the effect of ordinary single-photon ex-

TABLE I. Expressions for the derivatives of V_{SB} with respect to the scalar and pseudoscalar fields, evaluated at the equilibrium point, where V_{SB} is contained in the linear $(3,3^*)\oplus(3^*,3)$ and bilinear $(6,6^*)\oplus(6^*,6)$ representations of $\mathrm{SU}(3)\times\mathrm{SU}(3)$. We write the expressions as $\langle\ \rangle_0 = a_1 g_0 + a_2 g_3 + a_3 k_0 + a_4 k_3 + a_5 g_1 + a_6 k_1$. All derivatives not listed vanish at the equilibrium point.

Equilibrium point derivative	a_1	a_2	a_3	a_4	a_5	a_6
$\left\langle \frac{\partial V_{\mathrm{SB}}}{\partial S_1^1} \right\rangle_0$	-1	0	$-2(2\alpha_1+\alpha_2+\alpha_3)$	$-\alpha_3$	-1	$-(4\alpha_1+\alpha_3)$
$\left\langle \frac{\partial V_{\mathrm{SB}}}{\partial S_2^2} \right\rangle_0$	-1	0	$-2(\alpha_1+2\alpha_2+\alpha_3)$	$-\alpha_3$	1	$4\alpha_2+\alpha_3$
$\left\langle \frac{\partial V_{\mathrm{SB}}}{\partial S_3^3} \right\rangle_0$	-1	-1	$-2(\alpha_1+\alpha_2+2\alpha_3)$	$-(\alpha_1+\alpha_2+4\alpha_3)$	0	$-(\alpha_1-\alpha_2)$
$\left\langle \frac{\partial^2 V_{\mathrm{SB}}}{\partial S_1^1 \partial S_1^1} \right\rangle_0, -\left\langle \frac{\partial^2 V_{\mathrm{SB}}}{\partial \phi_1^1 \partial \phi_1^1} \right\rangle_0$	0	0	-4	0	0	-4
$\left\langle \frac{\partial^2 V_{\mathrm{SB}}}{\partial S_1^1 \partial S_2^2} \right\rangle_0, -\left\langle \frac{\partial^2 V_{\mathrm{SB}}}{\partial \phi_1^1 \partial \phi_2^2} \right\rangle_0$	0	0	-2	0	0	0
$\left\langle \frac{\partial^2 V_{\mathrm{SB}}}{\partial S_1^1 \partial S_3^3} \right\rangle_0, -\left\langle \frac{\partial^2 V_{\mathrm{SB}}}{\partial \phi_1^1 \partial \phi_3^3} \right\rangle_0$	0	0	-2	-1	0	-1
$\left\langle \frac{\partial^2 V_{\mathrm{SB}}}{\partial S_2^2 \partial S_2^2} \right\rangle_0, -\left\langle \frac{\partial^2 V_{\mathrm{SB}}}{\partial \phi_2^2 \partial \phi_2^2} \right\rangle_0$	0	0	-4	0	0	4
$\left\langle \frac{\partial^2 V_{\mathrm{SB}}}{\partial S_2^2 \partial S_3^3} \right\rangle_0, -\left\langle \frac{\partial^2 V_{\mathrm{SB}}}{\partial \phi_2^2 \partial \phi_3^3} \right\rangle_0$	0	0	-2	-1	0	1
$\left\langle \frac{\partial^2 V_{\mathrm{SB}}}{\partial S_3^3 \partial S_3^3} \right\rangle_0, -\left\langle \frac{\partial^2 V_{\mathrm{SB}}}{\partial \phi_3^3 \partial \phi_3^3} \right\rangle_0$	0	0	-4	-4	0	0
$\left\langle \frac{\partial^2 V_{\mathrm{SB}}}{\partial S_2^1 \partial S_1^2} \right\rangle_0, -\left(\left\langle \frac{\partial^2 V_{\mathrm{SB}}}{\partial \phi_1^2 \partial \phi_2^1} \right\rangle_0 - d_\pi\right)$	0	0	-2	0	0	0
$\left\langle \frac{\partial^2 V_{\mathrm{SB}}}{\partial S_1^3 \partial S_3^1} \right\rangle_0, -\left(\left\langle \frac{\partial^2 V_{\mathrm{SB}}}{\partial \phi_1^3 \partial \phi_3^1} \right\rangle_0 - d_K\right)$	0	0	-2	-1	0	-1
$\left\langle \frac{\partial^2 V_{\mathrm{SB}}}{\partial S_3^2 \partial S_2^3} \right\rangle_0, -\left\langle \frac{\partial^2 V_{\mathrm{SB}}}{\partial \phi_3^2 \partial \phi_2^3} \right\rangle_0$	0	0	-2	-1	0	1

change on the pion and kaon masses. This component has the form

$$V_{\mathrm{SB}}^{\gamma} = d_\pi \phi_1^2 \phi_2^1 + d_K \phi_1^3 \phi_3^1 . \tag{2.28}$$

The term $V_{\mathrm{SB}}^{I=1}$ is assumed to have the form

$$V_{\mathrm{SB}}^{I=1} = -c_3 u_3 - \begin{pmatrix} e_3 w_3 \\ c_3' u_3' \end{pmatrix} . \tag{2.29}$$

Again we rewrite this as

$$V_{\mathrm{SB}}^{I=1} = -g_1(S_1^1 - S_2^2) - \begin{pmatrix} k_1[(E^6)_1^1 - (E^6)_2^2] \\ g_1'[(E^3)_1^1 - (E^3)_2^2] \end{pmatrix}, \tag{2.30}$$

where

$$g_1 = c_3/\sqrt{2} \;,$$

$$k_1 = e_3/\sqrt{2} \;,$$

and (2.31)

$$g_1' = c_3'/\sqrt{2} \;.$$

Finally, for the $(6,6^*)\oplus(6^*,6)$ case we have, using Eq. (2.13),

$$\begin{aligned} V_{\mathrm{SB}}^{I=1} = &-g_1(S_1^1 - S_2^2) \\ &- k_1[2(S_1^1 S_1^1 - S_2^2 S_2^2) + (S_1^1 - S_2^2)S_3^3 + S_3^1 S_1^3 - S_3^2 S_2^3 \\ &\quad - 2(\phi_1^1\phi_1^1 - \phi_2^2\phi_2^2) - (\phi_1^1 - \phi_2^2)\phi_3^3 - \phi_3^1\phi_1^3 + \phi_3^2\phi_2^3] \end{aligned} \tag{2.32}$$

For the bilinear $(3,3^*)\oplus(3^*,3)$ case we have, using Eq. (2.18),

$$\begin{aligned} V_{\mathrm{SB}}^{I=1} = &-g_1(S_1^1 - S_2^2) - 2g_1'[(S_1^1 - S_2^2)S_3^3 - S_3^1 S_1^3 + S_3^2 S_2^3 \\ &- (\phi_1^1 - \phi_2^2)\phi_3^3 + \phi_3^1\phi_1^3 - \phi_3^2\phi_2^3] . \end{aligned} \tag{2.33}$$

In summary we note that the complete Lagrangian density to be employed in the following calculations is

TABLE II. Expressions for the derivative of V_{SB} with respect to the scalar and pseudoscalar fields, evaluated at the equilibrium point, where V_{SB} is contained in the linear and bilinear $(3,3^*)\oplus(3^*,3)$ representation of $SU(3)\times SU(3)$. We write the expression as $\langle\,\rangle_0 = a_1 g_0 + a_2 g_3 + a_3 g_0' + a_4 g_3' + a_5 g_1 + a_6 g_1'$. All derivatives not listed vanish at the equilibrium point.

Equilibrium point derivative	a_1	a_2	a_3	a_4	a_5	a_6
$\left\langle \frac{\partial V_{SB}}{\partial S_1^1} \right\rangle_0$	-1	0	$2(\alpha_2+\alpha_3)$	$2\alpha_2$	-1	$-2\alpha_3$
$\left\langle \frac{\partial V_{SB}}{\partial S_2^2} \right\rangle_0$	-1	0	$2(\alpha_1+\alpha_3)$	$2\alpha_1$	1	$2\alpha_3$
$\left\langle \frac{\partial V_{SB}}{\partial S_3^3} \right\rangle_0$	-1	-1	$2(\alpha_1+\alpha_2)$	0	0	$-2(\alpha_1-\alpha_2)$
$\left\langle \frac{\partial^2 V_{SB}}{\partial S_1^1 \partial S_2^2} \right\rangle_0, -\left\langle \frac{\partial^2 V_{SB}}{\partial \phi_1^1 \partial \phi_2^2} \right\rangle_0$	0	0	2	2	0	0
$\left\langle \frac{\partial^2 V_{SB}}{\partial S_1^1 \partial S_3^3} \right\rangle_0, -\left\langle \frac{\partial^2 V_{SB}}{\partial \phi_1^1 \partial \phi_3^3} \right\rangle_0$	0	0	2	0	0	-2
$\left\langle \frac{\partial^2 V_{SB}}{\partial S_2^2 \partial S_3^3} \right\rangle_0, -\left\langle \frac{\partial^2 V_{SB}}{\partial \phi_2^2 \partial \phi_3^3} \right\rangle_0$	0	0	2	0	0	2
$\left\langle \frac{\partial^2 V_{SB}}{\partial S_1^2 \partial S_2^1} \right\rangle_0, -\left(\left\langle \frac{\partial^2 V_{SB}}{\partial \phi_1^2 \partial \phi_2^1} \right\rangle_0 - d_\pi\right)$	0	0	-2	-2	0	0
$\left\langle \frac{\partial^2 V_{SB}}{\partial S_3^1 \partial S_1^3} \right\rangle_0, -\left(\left\langle \frac{\partial^2 V_{SB}}{\partial \phi_1^3 \partial \phi_3^1} \right\rangle_0 - d_K\right)$	0	0	-2	0	0	2
$\left\langle \frac{\partial^2 V_{SB}}{\partial S_3^2 \partial S_2^3} \right\rangle_0, -\left\langle \frac{\partial^2 V_{SB}}{\partial \phi_3^2 \partial \phi_2^3} \right\rangle_0$	0	0	-2	0	0	-2

$$
\begin{aligned}
\mathcal{L} = {} & \tfrac{1}{2}\mathrm{Tr}[\partial_\mu \phi \partial^\mu \phi] + \tfrac{1}{2}\mathrm{Tr}[\partial_\mu S \partial^\mu S] \\
& - V_0 - V_{SB}^{I=0} - V_{SB}^{I=1} \\
& - d_\pi \phi_1^2 \phi_2^1 - d_K \phi_1^3 \phi_3^1 \, . \qquad (2.34)
\end{aligned}
$$

For calculational purposes the derivatives of the symmetry-breaking Lagrangian with respect to the scalar and pseudoscalar fields are given in Tables I and II for bilinear $(6,6^*)\oplus(6^*,6)$ and bilinear $(3,3^*)\oplus(3^*,3)$, respectively.

At this stage two significant differences between these cases and the bilinear (8, 8) case considered in I are apparent from the tables. First we note that, in the present cases, derivatives of the form $\langle \partial^2 V_{SB}/\partial S \partial S\rangle_0$ have signs opposite to $\langle \partial^2 V_{SB}/\partial\phi\partial\phi\rangle_0$ (when they are nonvanishing). Second, we notice that, in Table II, the g_3' term vanishes identically for $\langle \partial V_{SB}/\partial S_3^3\rangle_0$. This will be quite important in any calculation involving the $I=0$ fields. For instance, it is the primary reason that the form of symmetry breaking employing only the bilinear $(3,3^*)\oplus(3^*,3)$ term fails. In particular, after inputting the π and K masses and their decay constants, this type of symmetry breaking cannot accommodate an acceptable value for both the η and η' masses.

III. AN OUTLINE OF THE σ-MODEL CALCULATIONS

In this section we summarize the calculations used as a basis by which to assess the various types of symmetry breaking. As in I the analysis was carried out as follows. The ground state of the system described by the Lagrangian of Eq. (2.34) is assumed to be characterized by those values, $\langle S_a^b\rangle_0$ and $\langle \phi_a^b\rangle_0$, of the fields for which

$$
\frac{\partial}{\partial\phi}(V_0+V_{SB})\bigg|_{\langle S\rangle_0,\,\langle\phi\rangle_0} = 0 = \frac{\partial}{\partial S}(V_0+V_{SB})\bigg|_{\langle S\rangle_0,\,\langle\phi\rangle_0} . \qquad (3.1)
$$

It is assumed that $\langle\phi\rangle_0 = 0$ to avoid spontaneous parity violation. We set $\langle S_a^a\rangle = \alpha_a$ $(a=1,2,3)$. SU(2) invariance of the ground state requires that $\alpha_1=\alpha_2$.

The Lagrangian (2.34) is expanded about $\langle S\rangle_0$ with second, third, and higher derivatives of $V=V_0+V_{SB}$ with respect to the fields being identified with masses, three-point couplings, and higher-order couplings, respectively. For example, the pion mass is given by $\langle \partial^2 V/\partial\phi_2^1\partial\phi_1^2\rangle_0$ and the $\kappa^+ \to K^0\pi^+$ coupling by $\langle \partial^3 V/\partial S_3^1 \partial\phi_2^3 \partial\phi_1^2\rangle_0$. Most of the masses and couplings depend[15,17] only on the α's and the symmetry-breaking parameters. The latter may thus be determined from the well-known pseudoscalar meson (i.e., π, K, and η) masses.

For further details of the computational techniques the reader should refer to I. With the general formulas given in I all results can be calculated using Tables I and II. We consider the strong- and electromagnetic-interaction calculations in subsections A and B below, respectively. All computations are done in the tree approximation.

A. Strong-interaction calculations

For the strong-interaction calculations we neglect all the symmetry-breaking terms in the Lagrangian except $V_{SB}^{I=0}$ and set $\alpha_1=\alpha_2$, $\omega\equiv\alpha_3/\alpha$ [corresponding to an SU(2)-invariant ground state]. We then consider the scalar and pseudoscalar meson mass spectra, the strong decays, and the $I=0$ and 2 $\pi\pi$ and $I=\frac{1}{2}$ and $\frac{3}{2}$ πK S-wave scattering lengths a_0, a_2, $a_{1/2}$, and $a_{3/2}$, respectively. Two types of models are considered: the general model and the renormalizable model. The general model can be further constrained by imposing scale invariance on V_0, if desired.

In the general model we can derive expressions for the π, K, η, η', and κ masses, and θ_P (the η-η' mixing angle), in terms of the symmetry-breaking parameters. The σ, σ', and ϵ masses, and θ_S (the σ-σ' mixing angle), remain unconstrained. If one imposes scale invariance on V_0, two further conditions can be obtained relating σ, σ', and θ_S. However, the ϵ mass remains unrelated to the symmetry-breaking parameters. In our computation the quantities not related to the symmetry-breaking parameters are input. We prefer the values $\sigma\approx 660$, $\sigma'\approx 997$, and $\epsilon\approx 970$ MeV; $\theta_S\approx 120°$.

Once the mass spectra are determined we investigate the strong decays. In the general model we consider the decays $\kappa\to K\pi$, $\epsilon\to\eta\pi$, $\sigma\to\pi\pi$, $\sigma'\to\pi\pi$, and $\sigma'\to K\bar{K}$. As the chiral invariance of V_0 does not give enough information to calculate coupling constants involving three isoscalar fields, we must again impose scale invariance on V_0 to get information about coupling constants such as $g_{\sigma\eta\eta'}$. With this information, and the value of a (the Dalitz-plot slope parameter), we can estimate the width for the decay $\eta'\to\eta\pi\pi$. In the general model we also compute the $\pi\pi$ and πK S-wave scattering lengths in the usual manner.

In the renormalizable model the condition of renormalizability determines the structure of V_0. Thus, all quantities can be calculated directly in terms of V_0 and V_{SB}, if desired. This gives us more information than was available, even using scale invariance, in the general model. Consequently, we can reproduce all the calculations of the general model, occasionally with fewer inputs.

Our results for the general and renormalizable models are given in Tables III and IV, respective-

TABLE III. Tree-approximation calculations in the general model for various combinations of $(3,3^*)\oplus(3^*,3)\oplus(6,6^*)\oplus(6^*,6)$ chiral-symmetry breaking. $V_{SB}=-c_0u_0-c_8u_8-e_0w_0-e_8w_8$, $g_0=(\sqrt{2}c_0+c_8)/\sqrt{6}$. For these calculations we have $f_\pi=135$, $\pi=135$, $K=495.8$, $\eta=548$, $\sigma=660$, $\sigma'=997$, and $\epsilon=970$ (MeV). Other quantities used as input are underlined.

	$e_0=e_8=0$	$e_8=0$	$e_0=0$	$g_0=e_8=0$	$g_0=e_0=0$	$g_0=0$	$g_0=-0.3$
ω	$\underline{1.56}$	$\underline{1.56}$	$\underline{1.66}$	$\underline{1.56}$	$\underline{1.56}$	$\underline{1.66}$	$\underline{1.66}$
c_0 (π^3)	10.26	21.60	10.02	9.33	10.53	8.89	8.02
c_8 (π^3)	−13.28	−10.29	−10.63	−13.20	−14.90	−12.57	−12.08
e_0 (π^2)	0.00	−1.91	0.00	0.16	0.00	0.25	0.38
e_8 (π^2)	0.00	0.00	−0.73	0.00	0.38	−0.25	−0.35
c_8/c_0	−1.30	−0.66	−1.06	$-\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$
e_8/e_0	···	0.00	∞	0.00	∞	−0.99	−0.90
η' (MeV)	1019	$\underline{958}$	$\underline{958}$	1024	1054	$\underline{980}$	$\underline{980}$
θ_P (deg)	−0.3	$\underline{-0.4}$	$\underline{-2.9}$	−0.3	1.7	$\underline{-0.6}$	$\underline{-1.1}$
κ (MeV)	1029	1067	936	1026	1046	951	944
θ_S (deg)	$\underline{120}$	$\underline{120}$	$\underline{120}$	$\underline{120}$	$\underline{120}$	$\underline{120}$	$\underline{120}$
$\Gamma(\kappa\to K\pi)$ (MeV)	778	760	489	778	833	540	529
$\Gamma(\epsilon\to\eta\pi)$ (MeV)	201	153	190	205	198	199	203
$\Gamma(\sigma\to\pi\pi)$ (MeV)	781	283	630	834	870	808	831
$\Gamma(\sigma'\to\pi\pi)$ (MeV)	25	54	20	24	29	20	18
$\Gamma(\sigma'\to K\bar{K})$ (MeV)	68	42	72	70	62	70	72
a_0 (π^{-1})	0.168	−0.447	−0.078	0.253	0.316	0.209	0.244
a_2 (π^{-1})	−0.038	−0.219	−0.116	−0.011	0.010	−0.025	−0.014
$a_{1/2}$ (π^{-1})	0.152	0.088	0.169	0.162	0.148	0.178	0.188
$a_{3/2}$ (π^{-1})	−0.050	−0.078	−0.047	−0.044	−0.050	−0.039	−0.034

TABLE IV. Tree-approximation calculations in the renormalizable model for several combinations of $(3,3^*)\oplus(3^*,3)\oplus(6,6^*)\oplus(6^*,6)$ chiral-symmetry breaking. $V_{SB}=-c_0u_0-c_8u_8-e_0w_0-e_8w_8$, $g_0=(\sqrt{2}\,c_0+c_8)/\sqrt{6}$. We set $f_\pi=135$, $\pi=135$, $K=495.8$, and $\eta=548$ (MeV). Other input quantities are underlined.

	$e_0=e_8=0$	$g_0=e_8=0$	$g_0=0$	$g_0=-0.3$
ω	$\underline{1.66}$	$\underline{1.66}$	$\underline{1.66}$	$\underline{1.66}$
c_0 (π^3)	10.65	9.71	8.89	8.02
c_8 (π^3)	−13.83	−13.74	−12.57	−12.08
e_0 (π^2)	0.00	0.15	0.25	0.38
e_8 (π^2)	0.00	0.00	−0.25	−0.35
c_8/c_0	−1.30	$-\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$
e_8/e_0	···	0.00	−1.00	−0.92
η' (MeV)	988	993	$\underline{980}$	$\underline{980}$
θ_P (deg)	0.7	0.7	−0.6	−1.1
κ (MeV)	967	964	951	944
ϵ (MeV)	952	948	935	929
σ (MeV)	$\underline{640}$	$\underline{640}$	$\underline{660}$	$\underline{660}$
σ' (MeV)	1123	1126	1124	1119
θ_S (deg)	123.4	124.1	126.1	126.5
$\Gamma(\eta'\to\eta\pi\pi)$ (MeV)	5.4	5.6	4.8	4.7
a	−0.03	−0.03	0.00	0.01
$\Gamma(\kappa\to K\pi)$ (MeV)	565	565	540	529
$\Gamma(\epsilon\to\eta\pi)$ (MeV)	169	169	157	154
$\Gamma(\sigma\to\pi\pi)$ (MeV)	709	757	811	830
$\Gamma(\sigma'\to\pi\pi)$ (MeV)	5	2	2	5
$\Gamma(\sigma'\to K\bar{K})$ (MeV)	506	540	584	596
a_0 (π^{-1})	0.170	0.254	0.208	0.243
a_2 (π^{-1})	−0.038	−0.012	−0.025	−0.014
$a_{1/2}$ (π^{-1})	0.157	0.167	0.178	0.189
$a_{3/2}$ (π^{-1})	−0.049	−0.043	−0.039	−0.034

ly, for $(3,3^*)\oplus(3^*,3)\oplus(6.6^*)\oplus(6^*,6)$ symmetry breaking and in Tables V and VI, respectively, for linear and bilinear $(3,3^*)\oplus(3^*,3)$ symmetry breaking.

B. Electromagnetic-interaction calculations

For the electromagnetic calculations we add to the Lagrangian of part A the isospin-violating terms $V_{SB}^{I=1}$ and V_{SB}^{γ}. We can then consider the mass splitting in the π, K, and κ isospin multiplets as well as the electromagnetic mixing between $I=0$ and $I=1$ members of both scalar and pseudoscalar nonets. The value of the ϵ mass can now be obtained in the general model, since $\alpha_1\neq\alpha_2$. In the renormalizable model we can investigate the width and Dalitz-plot slope parameter (β) of the decay $\eta\to\pi^+\pi^-\pi^0$. The most interesting information obtained in these computations is given in Tables VII and VIII for the $(6,6^*)\oplus(6^*,6)$ and $(3,3^*)\oplus(3^*,3)$ models, respectively.

IV. THE NUMERICAL ANALYSIS AND DISCUSSION OF RESULTS

During the course of this work many types of symmetry breaking were investigated. Tables III–IX summarize the most interesting results for the general and renormalizable models. We do not present the results for the general model using scale invariance as we feel they are not acceptable. Before we discuss our results in detail we will outline the inputs and assumptions that went into the numerical analysis.

We begin by considering the value of the pseudoscalar masses employed to determine the basic symmetry-breaking parameters in the strong-interaction calculations. As we do not expect a shift in the π^0 mass to lowest order in the electromagnetic interactions, we identify the pion mass, π, with the π^0 mass.[26] We use this as our basic mass unit.

Both the K^+ and K^0 masses will shift to lowest order in the electromagnetic interaction. As an estimate of the kaon mass due to the strong interactions we prefer to average the K^+ and K^0 masses. Naturally, the symmetry-breaking parameters are sensitive to the value of K chosen, but the effect is not severe for small changes in K. Chan and Haymaker[22] have pointed out that the value of α_3 is sensitive to that of K. We avoid this problem by employing f_K/f_π to determine α_3, as will be

TABLE V. Tree-approximation calculations in the general model for various combinations of linear and bilinear $(3,3^*)\oplus(3^*,3)$ chiral-symmetry breaking, $V_{SB}=-c_0u_0-c_8u_8-c_0'u_0'-c_8'u_8'$, $g_0=(\sqrt{2}\,c_0+c_8)/\sqrt{6}$, and similarly for g_0'. For these calculations we have $f_\pi=135$, $\pi=135$, $K=495.8$, $\eta=548$, $\sigma=660$, $\sigma'=997$, $\epsilon=970$ (MeV), and $f_K/f_\pi=1.28$. Other input quantities are underlined.

	$c_0'=c_8'=0$	$c_8'=0$	$c_0'=0$	$g_0=c_8'=0$	$g_0=c_0'=0$	$g_0=0$	$g_0=0.3$	$c_0=0$	$g_0'=0$	$g_0'=0.3$	$g_0=0$
c_0 (π^3)	10.26	12.53	10.31	9.60	10.15	9.14	9.60	0.0	9.90	10.61	12.64
c_8 (π^3)	−13.28	−12.27	−12.70	−13.57	−14.36	−12.92	−12.83	−14.68	−12.78	−12.64	−17.88
c_0' (π^2)	0.0	1.91	0.0	−0.56	0.0	−1.02	−0.62	−8.91	−0.36	0.26	2.49
c_8' (π^2)	0.0	0.0	0.43	0.0	−0.79	0.65	0.56	2.41	0.50	0.37	−4.30
c_8/c_0	−1.30	−0.98	−1.23	$-\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$	−1.34	∞	−1.29	−1.19	$-\sqrt{2}$
c_8'/c_0'	...	0.0	∞	0.0	∞	−0.64	−0.91	−0.27	$-\sqrt{2}$	1.42	−1.73
c'	−1.30	−1.29	−1.30	−1.30	−1.29	−1.30	−1.30	−1.33	−1.30	−1.29	−1.27
η' (MeV)	1019	$\underline{958}$	$\underline{958}$	1036	1123	$\underline{958}$	$\underline{958}$	$\underline{958}$	$\underline{958}$	$\underline{958}$	$\underline{1450}$
θ_P (deg)	−0.3	−0.4	−0.4	−0.3	−0.2	−0.4	−0.4	−0.4	−0.4	−0.4	−0.1
κ (MeV)	1029	989	989	1040	1099	989	989	989	989	989	1330
θ_S (deg)	$\underline{120}$	$\underline{120}$	$\underline{120}$	$\underline{120}$	$\underline{120}$	$\underline{120}$	$\underline{120}$	$\underline{110}$	$\underline{120}$	$\underline{120}$	$\underline{130}$
$\Gamma(\kappa\to K\pi)$ (MeV)	778	785	658	776	1015	698	620	219	637	675	2350
$\Gamma(\epsilon\to\eta\pi)$ (MeV)	201	254	209	186	186	187	196	57	201	215	185
$\Gamma(\sigma\to\pi\pi)$ (MeV)	781	763	778	788	788	788	785	955	781	777	778
$\Gamma(\sigma'\to\pi\pi)$ (MeV)	25	67	30	16	16	16	22	7	25	35	30
$\Gamma(\sigma'\to K\bar{K})$ (MeV)	68	68	68	66	66	66	66	34	66	68	84
a_0 (π^{-1})	0.168	0.184	0.168	0.170	0.170	0.170	0.168	0.485	0.168	0.168	0.169
a_2 (π^{-1})	−0.038	−0.030	−0.038	−0.038	−0.038	−0.038	−0.038	0.066	−0.038	−0.038	−0.037
$a_{1/2}$ (π^{-1})	0.152	0.213	0.163	0.138	0.137	0.140	0.149	0.048	0.154	0.170	0.134
$a_{3/2}$ (π^{-1})	−0.050	−0.038	−0.048	−0.052	−0.053	−0.051	−0.050	−0.022	−0.050	−0.047	−0.053

TABLE VI. Tree-diagram calculations in the renormalizable model for several combinations of linear and bilinear (3, 3*) ⊕ (3*, 3) chiral-symmetry breaking. $V_{SB} = -c_0 u_0 - c_8 u_8 - c_0' u_0' - c_8' u_8'$, $g_0 = (\sqrt{2}\, c_0 + c_8)/\sqrt{6}$, and similarly for g_0'. We set $f_\pi = 135$, $\pi = 135$, $K = 495.8$, and $\eta = 548$ (MeV). Other input quantities are underlined.

	$c_0' = c_8' = 0$	$c_8' = 0$	$c_0' = 0$	$g_0 = c_8' = 0$	$g_0 = c_0' = 0$	$g_0 = 0$	$g_0 = 0.3$	$c_0 = 0$	$g_0' = 0$	$g_0' = 0.3$
ω	$\underline{1.66}$	$\underline{1.66}$	$\underline{1.66}$	$\underline{1.66}$	$\underline{1.66}$	$\underline{1.56}$	$\underline{1.56}$	$\underline{1.56}$	$\underline{1.56}$	$\underline{1.56}$
c_0 (π^3)	10.65	11.78	10.68	10.01	10.53	9.14	9.60	0.0	9.90	10.61
c_8 (π^3)	−13.83	−13.25	−13.49	−14.16	−14.89	−12.92	−12.83	−14.68	−12.78	−12.64
c_0' (π^2)	0.0	0.93	0.0	0.52	0.0	−1.02	−0.62	−8.91	−0.36	0.26
c_8' (π^2)	0.0	0.0	0.24	0.0	−0.74	0.65	0.56	2.41	0.50	0.38
c_8/c_0	−1.30	−1.13	−1.26	$-\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$	−1.34	∞	−1.29	−1.19
c_8'/c_0'	⋯	0.0	∞	0.0	∞	−0.64	−0.91	−0.27	$-\sqrt{2}$	1.42
c'	−1.30	−1.30	−1.30	−1.30	−1.30	−1.30	−1.30	−1.33	−1.30	−1.29
η' (MeV)	988	$\underline{958}$	$\underline{958}$	1005	1076	$\underline{958}$	$\underline{958}$	$\underline{958}$	$\underline{958}$	$\underline{958}$
θ_P (deg)	0.7	0.8	0.8	0.7	0.6	−0.4	−0.4	−0.4	−0.4	−0.4
κ (MeV)	967	947	947	979	1028	989	989	989	989	989
ϵ (MeV)	952	931	931	963	1014	955	955	955	955	955
σ (MeV)	$\underline{640}$	$\underline{650}$	$\underline{650}$	$\underline{640}$	$\underline{560}$	$\underline{660}$	$\underline{660}$	$\underline{660}$	$\underline{660}$	$\underline{660}$
σ' (MeV)	1123	1121	1121	1129	1129	1129	1129	1129	1129	1129
θ_S (deg)	123.4	126.3	126.3	122.5	114.0	121.6	121.6	121.6	121.6	121.6
$\Gamma(\eta' \to \eta\pi\pi)$ (MeV)	5.4	5.6	4.4	5.4	27.0	2.6	2.9	0.3	3.0	3.6
a	−0.03	−0.01	−0.01	−0.04	3.06	0.00	0.00	0.08	0.00	0.00
$\Gamma(\kappa \to K\pi)$ (MeV)	565	565	515	564	734	597	620	219	638	675
$\Gamma(\epsilon \to \eta\pi)$ (MeV)	169	168	150	170	234	170	178	48	183	197
$\Gamma(\sigma \to \pi\pi)$ (MeV)	709	750	749	710	439	787	786	838	786	781
$\Gamma(\sigma' \to \pi\pi)$ (MeV)	5	~0	2	6	148	12	15	51	18	25
$\Gamma(\sigma' \to K\bar{K})$ (MeV)	506	538	538	512	394	538	540	520	540	542
a_0 (π^{-1})	0.170	0.178	0.169	0.172	0.185	0.170	0.168	0.382	0.168	0.168
a_2 (π^{-1})	−0.038	−0.037	−0.038	−0.038	−0.038	−0.038	−0.038	0.040	−0.038	−0.037
$a_{1/2}$ (π^{-1})	0.157	0.188	0.164	0.143	0.142	0.141	0.149	0.052	0.155	0.169
$a_{3/2}$ (π^{-1})	−0.049	−0.045	−0.048	−0.050	−0.050	−0.051	−0.050	−0.016	−0.050	−0.048

discussed below.

The η mass is used in the determination of the η-η' system as discussed in Appendix A of I. Depending on the type of symmetry breaking being considered, η' is either calculated or fitted. If not fitted, the value of η' can be sensitive to the value of η. For example, with linear (3, 3*)⊕(3*, 3) symmetry breaking alone we have a value of 1019 MeV for η' when $\eta = 548$ MeV. If we set $\eta = 543$ MeV then the η' mass shifts to 960 MeV. Nevertheless, we use $\eta = 548$ throughout our calculations. In general, we expect the sensitivity to be characteristic of the linear (3, 3*)⊕(3*, 3) form of symmetry breaking for all types of symmetry breaking, as small admixtures of the bilinear form are preferred. We also include some interesting calculations with the η' mass set at 1450 MeV.[27] This is near the $E(1420)$ (see Ref. 25) which has been suggested as a possible candidate for the η'. In summary we employ $\pi = 135$, $K = 495.8$, $\eta = 548$, and, when possible, $\eta' = 958$ (or occasionally 1450) MeV.

With a value of 135 MeV for f_π we have[15] $\alpha = 0.5\ \pi$, which is used throughout all our calculations. We determine α_3 using[15] f_K/f_π. We prefer[28] $f_K/f_\pi = 1.28$ which sets $\alpha_3 = 0.78\ \pi$. As there is some uncertainty in this quantity we allow f_K/f_π to be as large as 1.33.

Since the variable α_3 appears throughout all our calculations, we must consider the sensitivity of our results to the value of f_K/f_π. As an example,

TABLE VII. Electromagnetic-interaction calculations for two of the solutions in the renormalizable model presented in Table IV. The additional quantities used as input are underlined.

	$e_0 = e_8 = 0$		$g_0 = e_8 = 0$	
$\alpha_1 - \alpha_2$ (π)	$\underline{-0.0148}$	$\underline{-0.0139}$	$\underline{-0.0147}$	$\underline{-0.0137}$
d_K (π)	$\underline{0.21}$	$\underline{0.21}$	$\underline{0.21}$	$\underline{0.21}$
c_3 (π^3)	−0.527	−0.483	−0.522	−0.472
e_3 (π^2)	⋯	−0.009	⋯	−0.010
ϵ_+ (MeV)	958	$\underline{970}$	957	$\underline{970}$
$\Gamma(\eta \to \pi^+\pi^-\pi^0)$ (eV)	114	202	121	229
β	−0.477	−0.477	−0.479	−0.478

TABLE VIII. Electromagnetic-interaction calculations for three of the solutions in the renormalizable model presented in Table VI. The additional quantities used as input are underlined.

	$c_0' = c_8' = 0$		$c_8' = 0$		$g_0 = c_8' = 0$	
$\alpha_1 - \alpha_2$ (π)	$\underline{-0.0148}$	$\underline{-0.0150}$	$\underline{-0.0116}$	$\underline{-0.0115}$	$\underline{-0.0151}$	$\underline{-0.0153}$
d_K (π^2)	$\underline{0.21}$	$\underline{0.21}$	$\underline{0.21}$	$\underline{0.21}$	$\underline{0.21}$	$\underline{0.21}$
c_3 (π^3)	−0.527	−0.536	−0.445	−0.435	−0.543	−0.551
c_3' (π^2)	⋯	−0.007	⋯	0.010	⋯	−0.005
ϵ_+ (MeV)	958	$\underline{970}$	995	$\underline{970}$	963	$\underline{970}$
$\Gamma(\eta \to \pi^+\pi^-\pi^0)$ (eV)	114	123	329	307	94	98
β	−0.477	−0.479	−0.479	−0.478	−0.480	−0.478

consider again the linear $(3, 3^*)\oplus(3^*, 3)$ case. With $f_K/f_\pi = 1.28$ (and $\eta = 548$) we have $\eta' = 1019$ MeV. If we set $f_K/f_\pi = 1.4$ we then have $\eta' = 958$ MeV. (Naturally this result is dependent on the value of K employed.) This value of f_K/f_π is probably too large. At this point, we might also note that the scalar masses increase as α_3 approaches α [see, for example, the expression for the κ mass in Eq. (4.13) of I]. Thus, a larger value of f_K/f_π favors smaller scalar masses. In summary, we prefer to set $f_K/f_\pi = 1.28$ and deviate from this choice only when a change improves the calculations as a whole.

While the masses of the pseudoscalar nonet are accurately known (except for a possible uncertainty with regard to the choice of η'), the scalar masses are subject to a great deal of uncertainty. First, we identify the isovector $\delta(970)$ (see Ref. 25) with the ϵ. Second, we identify the broad $K\pi$ signal[25] in the 1200–1400 MeV region with the κ. Finally, we use the results of Protopopescu *et al.*[29] to complete the nonet with the σ around 660 MeV and the σ' at approximately 997 MeV. In our calculations the above masses are chosen, if possible. If the model being considered applies constraints to these masses, we try to fit them, while maintaining reasonable results for the other quantities of interest. Such a compromise is not always possible. In the general model we input σ, σ', θ_S, and ϵ. In the renormalizable model we input only σ.

It may be of some value for the reader to study parts A, B, and C of Sec. VII of I for two reasons. First, more information is given in I. This information should be compared with the calculations presented in this paper in order to assess the merits of any given type of symmetry breaking. Second, the discussion in I concerns linear $(3, 3^*)\oplus(3^*, 3)\oplus$bilinear $(8, 8)$ symmetry breaking which can be used to judge the relative success of the symmetry-breaking types we consider below.

We will now present a brief discussion of the $(3, 3^*)\oplus(3^*, 3)\oplus(6, 6^*)\oplus(6^*, 6)$ and linear and bilinear $(3, 3^*)\oplus(3^*, 3)$ results in parts A and B, respectively.

A. $(3,3^*)\oplus(3^*,3)\oplus(6,6^*)\oplus(6^*,6)$

These calculations did not work particularly well. From the results presented in Table III for the general model we consider only the cases $e_0 = e_8 = 0$ (the GMOR[5,6] scheme), $g_0 = e_8 = 0$ (the Okubo[13] form), $g_0 = 0.0$, and $g_0 = -0.3$ to be acceptable. Even the last two cases may be rejected as they do not allow the η' mass to be at 958 MeV, but prefer a larger value. This mass is required to be larger for a smaller value of $\omega = \alpha_3/\alpha$. The computations with $\eta' = 1450$ MeV are, however,

TABLE IX. Contributions to the kaon mass squared (in π^2) from the c_0, c_8, c_0', and c_8' terms in the kaon mass formula $K^2 = [2g_0 + g_3 - 2\alpha(2g_0' + g_3')]/(\alpha + \alpha_3)$ for various types of $(3,3^*)\oplus(3^*,3)$ symmetry breaking. We have $K = 3.63\pi$, $K^2 = 13.49\pi^2$, and $\omega = 1.56$. In all but the last case $\eta' = 958$ MeV, whenever possible.

	$c_0' = c_8' = 0$	$c_8' = 0$	$c_0' = 0$	$g_0 = c_8' = 0$	$g_0 = c_0' = 0$	$g_0 = 0$	$g_0 = 0.3$	$c_0 = 0$	$g_0' = 0$	$g_0' = 0.3$	$g_0 = 0$ $\eta' = 1450$ (MeV)
c_0	9.25	11.30	9.30	8.66	9.16	8.24	8.66	0.0	8.93	9.58	11.40
c_8	4.24	3.91	4.05	4.33	4.58	4.12	4.09	4.68	4.08	4.03	5.70
c_0'	0.0	−1.72	0.0	0.50	0.0	0.92	0.56	8.04	0.32	−0.23	−2.25
c_8'	0.0	0.0	0.14	0.0	−0.25	0.20	0.18	0.77	0.16	0.12	−1.37

also not satisfactory. The results for the renormalizable model follow the pattern of the general model. The more interesting cases are presented in Table IV. We also rejected the general model with scale invariance in all cases for reasons similar to those given in I.

Consequently, from these results, we feel that only the GMOR and Okubo cases merit serious consideration. The model seems to reject the cases that allow e_8 to be nonvanishing. The $g_0 = e_8 = 0$ case involves a small (6, 6*)⊕(6*, 6) SU(3) scalar symmetry-breaking term. From Tables III and IV it is difficult to see where this case gives more attractive results than the GMOR model. Note that both give acceptable results[30] for the slope parameter a of the decay $\eta' \to \eta\pi\pi$ [assuming that the $X^0(957)$ is the η'].

Although we feel that our electromagnetic-interaction calculations should not be taken too seriously, there is one interesting result given by the e_3 term. From Table VII we note that the addition of the e_3 term generates a larger width for the $\eta \to \pi^+\pi^-\pi^0$ decay, a width roughly twice as large as that given by the c_3 term alone. In fact, the e_3 term alone was not included in the tables as it gives a width that is many times too large. We feel this is worth pointing out as we have not had much success in obtaining a width for $\eta \to \pi^+\pi^-\pi^0$ that is near the experimental one,[31] while maintaining a reasonable value for ϵ and β.[32]

B. Linear and bilinear (3,3*)⊕(3*,3)

These results are more interesting than those of the previous case. In the first place, there are more combinations which work. In the second place, we can rule out a possible GMOR scheme in which the symmetry breaking is purely bilinear in the fields (the case $c_0 = c_8 = 0$).

Several other forms of symmetry breaking can also be rejected immediately. These are the cases $c_8 = 0$, $g_0' = c_0 = 0$, and $g_0' = c_8 = 0$. From Table V (general model) and Table VI (renormalizable model) we feel that the $c_0 = 0$ and $g_0 = c_0' = 0$ cases must also be rejected. From these tables it is also clear that there are many acceptable alternatives to the linear (3, 3*)⊕(3*, 3) GMOR scheme.

In the tables the variable c' appears. This is defined as

$$c' = \frac{c_8 - 2\alpha_3 c_8'}{c_0 - 2\alpha_3 c_0'} \ . \tag{4.1}$$

This expression reduces to the usual definition of c if either the linear or bilinear terms are omitted. We also have $c' = -\sqrt{2}$ when SU(2)×SU(2) symmetry becomes exact and the pion mass vanishes.

Before going further, perhaps we should comment on the size of the bilinear (3, 3*)⊕(3*, 3) contribution. As a measure of this contribution we use the magnitude of each term in the kaon mass. From Table IX we see that all bilinear contributions are small, except for the $c_0 = 0$ case, which gives unacceptable results.

At this point we also note that the solutions with $\eta' = 1450$ MeV are successful in the general model. An example is given in the last column of Table V. The results, for all forms of symmetry breaking that allow η' to be 1450 MeV, generally are similar numerically to those with $\eta' = 958$ MeV, except that the κ mass increases to about 1330 MeV (and the width to 2.3 GeV). However, in the renormalizable model we cannot accept the $\eta' = 1450$ MeV solutions, since they do not allow the σ and σ' masses to be near our required values of 660 and 1000 MeV, respectively.

From the numerical results in Tables V and VI we feel that the most interesting cases are $c_0' = c_8' = 0$ (GMOR), $c_8' = 0$, $c_0' = 0$, $g_0 = c_8' = 0$, $g_0 = 0$, and $g_0' = 0$. However, most of these cases involve more parameters than the GMOR model, but do not give significantly better results. In the other two-parameter model ($g_0 = c_8' = 0$) the η' mass is predicted to be further from the $X^0(957)$ than in the GMOR model. For all of the above cases the $\eta' \to \eta\pi\pi$ slope parameter is predicted to be in accord with the recent data.[30] Another interesting observation is that the model requires both the c_0 and c_8 terms. Omitting either or both generates unacceptable results.

If we consider the electromagnetic-interaction results we can reduce the number of successful cases to three (see Table VIII). These cases are the GMOR, and those with $c_8' = 0$, and $g_0 = c_8' = 0$, respectively. The results with $c_8' = 0$ are the best as they allow a width for the decay $\eta \to \pi^+\pi^-\pi^0$ that is large enough to be consistent with the latest data.[31] Most types of symmetry breaking give a width that is much too small.

V. DISCUSSION

The main results of the preceding analysis can be summarized as follows. We first considered the case in which the symmetry-breaking part of the Lagrangian belongs to the (3, 3*)⊕(3*, 3) ⊕(6, 6*)⊕(6 *, 6) representations of SU(3)×SU(3). The (3, 3*)⊕(3 *, 3) terms are linear in the basic fields while the (6, 6*)⊕(6*, 6) terms are bilinear. While this model involves four parameters in general, it was found that few of the possible combinations of terms could give as good a description of the data as does the two-parameter GMOR[5,6] model of linear $(3, 3^*)\oplus(3^*, 3)$ breaking. In fact, only the Okubo form[13] gives results which are competitive with the GMOR scheme. In the electromagnetic-interaction calculations the addition of the $I=1$ scalar octet part of $(6, 6^*)\oplus(6^*, 6)$ to that of the $(3, 3^*)\oplus(3^*, 3)$ representation gives an improved value for the $\eta\to\pi^+\pi^-\pi^0$ width.

When the symmetry-breaking terms were chosen from linear and bilinear $(3, 3^*)\oplus(3^*, 3)$ terms, we found more combinations of terms to be acceptable than was the case for $(3, 3^*)\oplus(3^*, 3)\oplus(6, 6^*)\oplus(6^*, 6)$ breaking. Interestingly enough, the possible GMOR scheme consisting of bilinear $(3, 3^*)\oplus(3^*, 3)$ symmetry breaking alone does not work. It may also be of some interest that an η' close to the $E(1420)$ (see Ref. 25) may be accommodated in the general model of this scheme. This choice for η' is about as successful as the usual assignment of the $X^0(957)$ as the η'. However, a high-mass η' does not work well in the renormalizable model and, hence, may not be a truly viable possibility. There is another two-parameter combination which works almost as well as the linear $(3, 3^*)\oplus(3^*, 3)$ GMOR model. In it the linear $(3, 3^*)\oplus(3^*, 3)$ contribution is $SU(2)\times SU(2)$ symmetric and the bilinear terms are SU(3)-invariant. However, neither of the two-parameter schemes predicts the width of $\eta\to\pi^+\pi^-\pi^0$ as well as the three-parameter combination which has no bilinear $I=0$ octet component. The latter scheme predicts a decay width consistent with the latest data.[31]

The analysis presented here, together with that of our earlier investigation,[15] supports the notion that chiral $SU(3)\times SU(3)$ symmetry is broken in a very simple way, consistent with having nearly exact $SU(2)\times SU(2)$ invariance. Although it is not possible from these studies alone to select a "best" symmetry-breaking scheme, the most successful of these share several common features. All have dominant, linear $(3, 3^*)\oplus(3^*, 3)$ components with (except in the GMOR case) small admixtures of bilinear terms transforming as $(8, 8)$, $(6, 6^*)\oplus(6^*, 6)$, or $(3, 3^*)\oplus(3^*, 3)$. In fact, despite the additional parameters available in these schemes, it is the two-parameter versions characterized by an $SU(2)\times SU(2)$-invariant linear $(3, 3^*)\oplus(3^*, 3)$ combination combined with an SU(3)-invariant bilinear contribution which are essentially as successful as the GMOR model. These two-parameter models are of the Okubo form for the cases in which the bilinear symmetry-breaking combination belongs to $(8, 8)$ or $(6, 6^*)\oplus(6^*, 6)$. If we restrict ourselves to strong-interaction results alone the Okubo form of $(3, 3^*)\oplus(3^*, 3)\oplus(8, 8)$ breaking is perhaps the most successful of all the two-parameter schemes, including the GMOR.

Finally, we found no viable symmetry-breaking scheme in which the chiral-invariant part of the Lagrangian could also be scale-invariant.[23] Thus, these models appear to demand an operator in the Hamiltonian which is chiral-invariant, but scale-noninvariant.[33]

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