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Strings, monopoles, and meson states*

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Dirac's formulation of the monopole theory modified by an additional mass term for the gauge field has been considered as a possible simplified model for quark binding. We discuss methods for the consistent regularization of the infinities present in the resulting action and equations of motion. In this way we are led to an action which is the same as that suggested by previous authors. We show that the expression for the energy of the modified action still has infinities unless the mass of the gauge field is infinite. Thus the regularization procedure is incomplete when the gauge field has finite mass. Applications of the regularized model to charmonium and other meson states are discussed.

I. INTRODUCTION

In the preceding paper,[1] we considered the canonical formulation of Dirac's action[2] for electrodynamics with magnetic charges, modified by the addition of an arbitrary mass term for the gauge field. For zero-mass gauge fields, the equivalence of Dirac's treatment and other treatments which do not use strings was established. In the static limit, the string was seen not to appear in the expression for the energy of the system. In the same static limit, in the massive case, the string contributed to the energy of the system a term which could be interpreted as a confinement potential for the monopoles. However, this energy expression contained certain infinities which had to be suitably reinterpreted.

Here we investigate the regularization of these infinities more carefully. It turns out that the method we use is very similar to that of Barut and Bornzin.[3] In contrast to our preceding paper,[1] we use the Lagrangian equations of motion rather than a Hamiltonian approach for this purpose. The modified action which leads to the correct regularized equations of motion is the same as that suggested by Nambu.[4] In the limit when the mass μ of the gauge field becomes infinite, the modified action reduces exactly to the Nambu-Goto[5] string action plus kinetic energy terms for the monopoles. The corresponding static energy is finite and leads to a confinement potential. For finite nonzero μ, there is also a Yukawa force between the monopoles. However, the static energy now contains infinities showing the incompleteness of the regularization procedure.

In this approach the strength of the confinement potential can be readily related to the over-all constant $-1/2\pi\alpha'$ appearing in the Nambu-Goto action. This identification leads, as we have pointed out elsewhere,[6] to a suggestive experimental consistency if the monopoles are considered to be charmed quarks and the new narrow resonances as their bound states. Some further consequences of this approach are discussed.

II. REGULARIZATION OF THE MODIFIED DIRAC ACTION

For simplicity we consider the following action containing two particles with equal and opposite magnetic charges and no electric charges:

$$A = \int d^4x\,(-\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \tfrac{1}{2}\mu^2 A_\mu A^\mu) - \sum_N m_{MN} \int d\tau \left[\left(\frac{dz^N_{M\mu}}{d\tau}\right)\left(\frac{dz^{N\mu}_M}{d\tau}\right)\right]^{1/2} . \quad (2.1)$$

Here A_μ is the vector field of mass μ, $z^N_{M\mu}(\tau)$ ($N=1,2$) are the monopole coordinates, m_{MN} are the corresponding masses, and

$$F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) + {}^\star G^{\mu\nu} , \quad (2.2)$$

where

$$\begin{aligned} {}^\star G^{\mu\nu} &= \tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta} G_{\alpha\beta} , \\ G_{\alpha\beta}(x) &= g \int d\tau\, d\sigma\, \delta^4(x - y(\tau,\sigma))\sigma_{\alpha\beta}(\tau,\sigma) , \\ \sigma_{\alpha\beta}(\tau,\sigma) &= \dot{y}_\alpha(\tau,\sigma) y_\beta'(\tau,\sigma) - \dot{y}_\beta(\tau,\sigma) y_\alpha'(\tau,\sigma) , \\ \dot{y}_\alpha &= \frac{\partial y_\alpha}{\partial\tau} , \quad y_\alpha' = \frac{\partial y_\alpha}{\partial\sigma} . \end{aligned} \quad (2.3)$$

In (2.3), g is the magnetic charge. Furthermore, the monopoles are located at the ends of the string $y_\mu(\tau,\sigma)$ so that the σ integration runs between the monopole positions. (In Ref. 1, we used an "infinite" string. For our present purposes, however, it is adequate to assume that the string is of finite extension.)

The field equations which follow from (2.1) are

$$\partial_\mu F^{\mu\nu} + \mu^2 A^\nu = 0\,, \tag{2.4a}$$

$$m_{MN}\frac{\partial}{\partial\tau}\left[\frac{\dot{z}^N_{M\mu}}{(\dot{z}^N_{M\lambda}\dot{z}^{N\lambda}_M)^{1/2}}\right] = g_N\, {}^\star F_{\mu\nu}(z^N_M)\dot{z}^{N\nu}_M \quad \text{(no } N \text{ sum)}, \tag{2.4b}$$

$$\sigma^{\mu\nu}\left[\frac{\partial}{\partial y^\rho}\,{}^\star F_{\mu\nu}(y) + \frac{\partial}{\partial y^\nu}\,{}^\star F_{\rho\mu}(y) + \frac{\partial}{\partial y^\mu}\,{}^\star F_{\nu\rho}(y)\right] = 0. \tag{2.4c}$$

Here $g_1 = +g$ and $g_2 = -g$ corresponding to monopole 1 at the beginning of the string and monopole 2 at the end of the string, while ${}^\star F_{\mu\nu}$ is defined analogously to ${}^\star G_{\mu\nu}$. Note also that by construction of $G_{\mu\nu}$,

$$\frac{\partial}{\partial x_\mu}\,{}^\star F_{\mu\nu} = -\frac{\partial}{\partial x_\mu} G_{\mu\nu}(x) = \sum_N g_N \int d\tau\, \delta^4(x - z^N_M(\tau))\dot{z}^N_{M\nu}\,. \tag{2.5}$$

Equation (2.5) indicates the presence of monopole sources for the vector field.

Note that both (2.4b) and (2.4c) contain infinities since they involve $G_{\mu\nu}(y)$ which is seen from (2.3) to be infinite. We will now describe a regularization procedure for these infinities. This procedure will also be consistent with (2.5) when it is evaluated on the string. Unlike other authors[3,4] who have attempted to regularize the action directly, we will regularize the equations of motion and then construct an action which leads to the regularized equations of motion. It turns out that the two procedures are not exactly equivalent as we shall discuss later.

The basic idea behind this regularization is to interpret an infinity $\delta^2(0)$ which occurs in the expressions involving $G_{\mu\nu}(y)$ as a finite constant. This is similar to giving the string a finite lateral dimension as other authors[3,4] have pointed out. The technical problem involved is the unambiguous covariant separation of the transverse directions from the others. This will be achieved by requiring the regularized equations to retain covariance under reparametrizations in the (τ,σ) space.

First, consider the part J_ρ of (2.4c) which contains the singularities:

$$J_\rho \equiv -\sigma^{\mu\nu}\frac{\partial}{\partial y^\rho} G_{\mu\nu}(y) - 2\left[\dot{y}^\mu \frac{\partial}{\partial\sigma} G_{\rho\mu}(y) - y'^\mu \frac{\partial}{\partial\tau} G_{\rho\mu}(y)\right]. \tag{2.6}$$

The singularity in the first term of (2.6) is of a different nature from those in the last two terms. The latter can be regularized by giving a suitable interpretation of $G_{\rho\mu}(y)$ while the first term, in addition, requires an interpretation of the derivative $\partial G_{\mu\nu}(y)/\partial y_\rho$. Thus we will treat the two cases separately.

The expression for $G_{\rho\mu}(y(\tau,\sigma))$,

$$G_{\rho\mu}(y(\tau,\sigma)) = g\int d\tau'\, d\sigma'\, \delta^4(y(\tau,\sigma) - y(\tau',\sigma')) \times \sigma_{\rho\mu}(\tau',\sigma')\,,$$

is clearly peaked around $\sigma = \sigma'$, $\tau = \tau'$. As a first approximation we therefore set

$$y(\tau',\sigma') = y(\tau,\sigma) + (\tau' - \tau)\dot{y}(\tau,\sigma) + (\sigma' - \sigma)y'(\tau,\sigma). \tag{2.7}$$

Resolving $\dot{y}$ into components parallel and perpendicular to y',

$$\dot{y}_\mu = \frac{\dot{y}\cdot y'}{y'^2}\, y'_\mu + \dot{y}^\perp_\mu\,, \qquad \dot{y}^\perp\cdot y' = 0\,, \tag{2.8}$$

and using (2.7), we find

$$\delta^4(y(\tau,\sigma) - y(\tau',\sigma')) = \delta^4\left(\left[(\sigma' - \sigma) + (\tau' - \tau)\frac{\dot{y}\cdot y'}{y'^2}\right] y' + (\tau' - \tau)\dot{y}^\perp\right).$$

Resolving this four-dimensional δ function into directions perpendicular to the world plane of the string and along y' and $\dot{y}^\perp$ gives

$$\delta^4(y(\tau,\sigma) - y(\tau',\sigma')) = \delta^2(0)\delta\left(\left[(\sigma' - \sigma) + (\tau' - \tau)\frac{\dot{y}\cdot y'}{y'^2}\right](-y'^2)^{1/2}\right) \times \delta((\tau' - \tau)\,(\dot{y}^{\perp 2})^{1/2})\,,$$
$$= \frac{\delta^2(0)\delta(\sigma' - \sigma)\delta(\tau' - \tau)}{[(\dot{y}\cdot y')^2 - \dot{y}^2 y'^2]^{1/2}}\,. \tag{2.9}$$

Thus

$$G_{\rho\mu}(y) = g\delta^2(0)\,\frac{\sigma_{\rho\mu}(\tau,\sigma)}{[(\dot{y}\cdot y')^2 - \dot{y}^2 y'^2]^{1/2}}\,. \tag{2.10}$$

Next, consider the first term of (2.6). $\partial G_{\mu\nu}(y)/\partial y_\rho$ can be written as

$$\frac{\partial}{\partial y^\rho} G_{\mu\nu}(y) = \left[\frac{y'_\rho y'_\lambda}{y'^2} + \frac{\dot{y}^\perp_\rho \dot{y}^\perp_\lambda}{\dot{y}^{\perp 2}}\right]\frac{\partial}{\partial y_\lambda} G_{\mu\nu}(y) + (n^{(1)}_\mu n^{(1)}_\lambda + n^{(2)}_\mu n^{(2)}_\lambda)\frac{\partial}{\partial y_\lambda} G_{\mu\nu}(y)\,, \tag{2.11}$$

where $y'/(-y'^2)^{1/2}$, $\dot{y}^{\perp}/(\dot{y}^{\perp 2})^{1/2}$, $n^{(1)}, n^{(2)}$ form an orthonormal coordinate system. The last two terms in (2.11) can be consistently defined to be zero by evaluating the derivative as a symmetrical limit

$$\lim_{\Delta\to 0}\frac{1}{\Delta}[G_{\mu\nu}(y+\tfrac{1}{2}\Delta)-G_{\mu\nu}(y-\tfrac{1}{2}\Delta)],$$

where Δ is normal to the world plane of the string. This limit vanishes because $G_{\mu\nu}(x)$ has support only on the world plane of the string. On substituting $\dot{y}^{\perp}_{\lambda}=\dot{y}_{\lambda}-(\dot{y}\cdot y'/y'^2)y'_{\lambda}$ and using (2.10), the first term of (2.6) becomes

$$\sigma^{\mu\nu}\frac{\partial}{\partial y^{\rho}}G_{\mu\nu}(y)$$
$$=g\delta^2(0)\sigma^{\mu\nu}\left\{\frac{1}{y'^2}\left[y'_{\rho}-\frac{(\dot{y}\cdot y')\dot{y}^{\perp}_{\rho}}{(\dot{y}^{\perp})^2}\right]\frac{\partial}{\partial\sigma}+\frac{\dot{y}^{\perp}_{\rho}}{(\dot{y}^{\perp})^2}\frac{\partial}{\partial\tau}\right\}$$
$$\times\frac{\sigma_{\mu\nu}}{[(\dot{y}\cdot y')^2-\dot{y}^2y'^2]^{1/2}}\,. \qquad (2.12)$$

Since

$$\left\{\frac{\sigma_{\mu\nu}}{[(\dot{y}\cdot y')^2-\dot{y}^2y'^2]^{1/2}}\right\}^2=-2\,,$$

we have

$$\sigma^{\mu\nu}\frac{\partial}{\partial\sigma}\frac{\sigma_{\mu\nu}}{[(\dot{y}\cdot y')^2-\dot{y}^2y'^2]^{1/2}}=\sigma^{\mu\nu}\frac{\partial}{\partial\tau}\frac{\sigma_{\mu\nu}}{[(\dot{y}\cdot y')^2-\dot{y}^2y'^2]^{1/2}}=0\,.$$

Thus (2.12) vanishes and (2.6) simplifies to

$$J_{\rho}=-2g\delta^2(0)\left(\dot{y}^{\mu}\frac{\partial}{\partial\sigma}-y'^{\mu}\frac{\partial}{\partial\tau}\right)\frac{\sigma_{\rho\mu}}{[(\dot{y}\cdot y')^2-\dot{y}^2y'^2]^{1/2}}\,. \qquad (2.13)$$

If we identify $\delta^2(0)$ with a finite constant, Eq. (2.10) defines a finite expression for $G_{\rho\mu}(y)$ and its derivatives. Then the equations of motion (2.4b) and (2.4c) become

$$m_{MN}\frac{\partial}{\partial\tau}\left[\frac{\dot{z}^N_{M\mu}}{(\dot{z}^N_M\cdot\dot{z}^N_M)^{1/2}}\right]=g_N(\epsilon_{\mu\nu\lambda\rho}\partial^{\lambda}A^{\rho}(z^N_M)G_{\mu\nu}(z^N_M))\dot{z}^{N\nu}_M \quad (\text{no } N \text{ sum}), \qquad (2.14)$$

$$\epsilon_{\rho\mu\nu\sigma}\dot{y}^{\mu}y'^{\nu}\frac{\partial}{\partial y^{\lambda}}\left[\frac{\partial}{\partial y_{\lambda}}A^{\sigma}(y)-\frac{\partial}{\partial y_{\sigma}}A^{\lambda}(y)\right]-g\delta^2(0)\left(\dot{y}^{\mu}\frac{\partial}{\partial\sigma}-y'^{\mu}\frac{\partial}{\partial\tau}\right)\frac{\sigma_{\rho\mu}}{[(\dot{y}\cdot y')^2-\dot{y}^2y'^2]^{1/2}}=0\,. \qquad (2.15)$$

Note that for $A_{\lambda}=0$, (2.15) reduces to the equation of motion of the Nambu-Goto string.

Next consider Eq. (2.5). When x is on the string but away from the ends, the right-hand side vanishes. The left-hand side, using arguments similar to the above, may be shown after some calculation to vanish. Also when $x=y(\tau,\sigma)$ is near the position $y(\tau,\sigma_N)=z^N_M(\tau)$ of the monopole N, the right-hand side of (2.5) may be written as

$$g_N\int d\sigma'\,d\tau'\,\delta(\sigma'-\sigma_N)\delta^4(y(\tau,\sigma)-y(\tau',\sigma'))\dot{y}_{\nu}(\tau',\sigma') \quad (\text{no } N \text{ sum}).$$

With our approximation (2.9) for the δ function this becomes

$$\frac{g_N\delta^2(0)\delta(\sigma-\sigma_N)\dot{y}_{\nu}(\tau,\sigma_N)}{\{[\dot{y}(\tau,\sigma_N)\cdot y'(\tau,\sigma_N)]^2-\dot{y}^2(\tau,\sigma_N)y'^2(\tau,\sigma_N)\}^{1/2}}\,.$$

This is easily seen to be the same as the left-hand side of (2.5) when $\partial G_{\mu\nu}(y)/\partial y_{\mu}$ is evaluated according to the prescription used in arriving at (2.12) and taking the surface terms into account.

Now the modified action which reproduces the equations of motion (2.4a), (2.14), and (2.15) is seen to be

$$A'=\int d^4x[-\tfrac{1}{4}(\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu})(\partial^{\mu}A^{\nu}-\partial^{\nu}A^{\mu})+\tfrac{1}{2}\mu^2A_{\mu}A^{\mu}]$$
$$-g\int d\tau\,d\sigma\,\partial^{\mu}A^{\nu}(y)^{\star}\sigma_{\mu\nu}-g^2\delta^2(0)\int d\tau\,d\sigma[(\dot{y}\cdot y')^2-\dot{y}^2y'^2]^{1/2}-m_{MN}\int d\tau(\dot{z}^N_M\cdot\dot{z}^N_M)^{1/2}. \qquad (2.16)$$

We see that the third term in (2.16) is exactly the Nambu-Goto action if we set

$$g^2\delta^2(0)=\frac{1}{2\pi\alpha'}\,, \qquad (2.17)$$

where α' is the universal slope of the Regge trajectories. This same action has been suggested by others.[3,4]

Note that the choice (2.17) regularizes the equations of motion (2.14) and (2.15) by making $\delta^2(0)$ finite. However, we shall see later that there are still infinities present in the expression for the energy of the system.

The comparison of A' and A shows that we have simply made the replacement

$$\tfrac{1}{4}\int d^4x\, G_{\mu\nu}G^{\mu\nu} = \frac{g^2}{4}\int d\tau\; d\sigma\, d\tau'\, d\sigma'\,\delta^4(y(\tau,\sigma)-y(\tau',\sigma'))\sigma_{\mu\nu}(\tau,\sigma)\sigma^{\mu\nu}(\tau',\sigma')$$

$$\to -g^2\delta^2(0)\int d\tau\; d\sigma\,[(\dot{y}\cdot y')^2 - \dot{y}^2 y'^2]^{1/2} \tag{2.18}$$

in the term involving $F_{\mu\nu}F^{\mu\nu}$ in A to get A'. It is amusing that if we had applied the approximation (2.9) directly to the δ function appearing in (2.18), we would have obtained $\frac{1}{2}g^2\delta^2(0)$ rather than $g^2\delta^2(0)$ in the last expression of (2.18). The expression in (2.18) without this factor $\frac{1}{2}$ is the one which with a consistent interpretation of infinities leads to the same canonical momenta and Hamiltonian given in the preceding paper (cf. footnote 7). Furthermore we note that regularization is not required to interpret the zero-mass theory since the Hamiltonian turns out to be finite except for the well-known self-energy infinities associated with point particles.[1]

III. THE ENERGY OF THE SYSTEM

To understand the physical situation represented by the action (2.16) it is helpful to compute the energy of the system. Making the convenient choices

$$z^N_{M0} = y_0 = t\,, \quad \tau = t\,, \tag{3.1}$$

we write (2.16) as

$$A' = \int dt\; L \tag{3.2}$$

and compute the energy using the equation

$$E = -L + \int d^3x\,\dot{A}_\mu\frac{\delta L}{\delta\dot{A}_\mu} + \int d\sigma\,\dot{y}_i\frac{\delta L}{\delta\dot{y}_i} + \dot{z}^N_{Mi}\frac{\partial L}{\partial\dot{z}^N_{Mi}}\;. \tag{3.3}$$

Here the latin indices are to be summed from 1 to 3 and

$$\frac{\partial L}{\partial\dot{z}^N_{Mi}} = \frac{m_{MN}\dot{z}^N_{Mi}}{(1-\dot{\vec{z}}^N_M\cdot\dot{\vec{z}}^N_M)^{1/2}} \quad (\text{no } N \text{ sum}), \tag{3.4a}$$

$$\frac{\delta L}{\delta\dot{A}_\mu(x)} = F^{\mu 0}(x)\,, \tag{3.4b}$$

$$\frac{\delta L}{\delta\dot{y}_i(\sigma,\tau)} = -g\left\{F^{ik} + G^{ik} - g\,\delta^2(0)\frac{\sigma^{ik}}{[(\dot{\vec{y}}\cdot\vec{y}')^2 + \vec{y}'^2 - \dot{\vec{y}}^2\vec{y}'^2]^{1/2}}\right\}y_k'\,. \tag{3.4c}$$

In (3.4c) σ is in the interior of the string.[7] The expression for the energy becomes

$$E = \sum_N \frac{m_{MN}}{(1-\dot{\vec{z}}^N_M\cdot\dot{\vec{z}}^N_M)^{1/2}} + \int d^3x\left(-\tfrac{1}{2}F^{i0}F_{i0} + \tfrac{1}{4}F^{ij}F_{ij} - \tfrac{1}{2}\mu^2A^\mu A_\mu - \frac{\partial}{\partial x^k}F^{k0}A_0\right)$$

$$+\tfrac{1}{4}\int d^3x\,G^{ij}G_{ij} + g^2\delta^2(0)\int d\sigma\,\frac{\vec{y}'\cdot\vec{y}'}{[(\dot{\vec{y}}\cdot\vec{y}')^2 + \vec{y}'^2 - \dot{\vec{y}}^2\vec{y}'^2]^{1/2}} - \tfrac{1}{2}\int d^3x\,G^{i0}G_{i0}\,. \tag{3.5}$$

Equation (3.5) will be used in the next two sections to study various limits of the theory.

IV. THE $\mu\to\infty$ CASE AND ITS STATIC LIMIT

In the preceding paper[1] it was shown that for $\mu=0$, we recover from Dirac's action the electrodynamics of electrically and magnetically charged particles and that the string played, in a certain sense, no dynamical role. In the $\mu\to\infty$ limit we shall see that the opposite occurs, namely that the field A_μ disappears and we get a theory of a Nambu-Goto string with monopoles at its ends.

The field equation (2.4a) for A_α has the solution[8]

$$A_\alpha(x)=-(\mu^2+\Box)^{-1}\frac{\partial}{\partial x_\lambda}{}^\star G_{\lambda\alpha}(x), \tag{4.1}$$

where we have used $\partial^\mu A_\mu=0$, which also is a consequence of (2.4a). The precise definition of the Green's function $(\mu^2+\Box)^{-1}$ is not necessary for our purposes. Note that $G_{\lambda\alpha}$ has no dependence on the mass μ. Thus as $\mu\to\infty$, $A_\alpha\to 0$. The remaining field equations (2.14) and (2.15) reduce to

$$g\,\delta^2(0)\left(\dot{y}^\mu\frac{\partial}{\partial\sigma}-y'^\mu\frac{\partial}{\partial\tau}\right)\frac{\sigma_{\rho\mu}(\sigma,\tau)}{[(\dot{y}\cdot y')^2-\dot{y}^2y'^2]^{1/2}}=0\,, \tag{4.2}$$

$$m_{MN}\frac{\partial}{\partial\tau}\left(\frac{\dot{z}^N_{M\mu}}{(\dot{z}^N_M\cdot\dot{z}^N_M)^{1/2}}\right)=-gg_N\delta^2(0)\frac{\sigma_{\mu\nu}(\sigma_N,\tau)\dot{z}^{N\nu}_M}{[(\dot{y}\cdot y')^2-\dot{y}^2y'^2]^{1/2}}$$

$$(\text{no } N \text{ sum}), \tag{4.3}$$

where in (4.3) $\dot{y}$ and y' are evaluated at the monopole position $\sigma=\sigma_N$. These field equations may be derived from the action A'' obtained by setting $A_\mu=0$ in (2.16), namely

$$A''=-g^2\delta^2(0)\int d\tau\, d\sigma\,[(\dot{y}\cdot y')^2-\dot{y}^2y'^2]^{1/2}$$
$$-m_{MN}\int d\tau\,(\dot{z}^N_M\cdot\dot{z}^N_M)^{1/2}\,. \tag{4.4}$$

The energy expression corresponding to A'' is obtained by setting $A_\mu=0$ in (3.5):

$$E\underset{\mu\to\infty}{\to}\sum_N\frac{m_{MN}}{(1-\dot{\vec{z}}^N_M\cdot\dot{\vec{z}}^N_M)^{1/2}}$$
$$+g^2\delta^2(0)\int d\sigma\,\frac{\vec{y}'\cdot\vec{y}'}{[(\dot{\vec{y}}\cdot\vec{y}')^2+\vec{y}'^2-\vec{y}^2\dot{\vec{y}}'^2]^{1/2}}\,. \tag{4.5}$$

It is interesting to consider the limit of this theory when the string is static (when y_i does not depend on time[9]). The equation of motion (4.2) then becomes, on dropping dotted quantities,

$$\frac{\partial}{\partial\sigma}\frac{\vec{y}'}{|\vec{y}'|}=0\,. \tag{4.6}$$

The fourth component of (4.2) gives an identity. This equation just says that the direction in space of the string does not change along the string. Hence the string must be a straight line. The potential energy V between the monopoles owing to the last term of (4.5) becomes

$$V=g^2\delta^2(0)\int d\sigma\,|\vec{y}'|$$
$$=g^2\delta^2(0)r \tag{4.7}$$

where r is the distance between the monopoles. Some consequences of this potential will be discussed later.

V. STATIC LIMIT FOR FINITE μ

In this more general case, we consider A_μ as well as y_i to be time independent.[9] Then

$$\begin{aligned}&G_{ij}(x)=0\,,\\&G_{i0}(x)=-g\int d\sigma\,\delta^3(x-y)y_i'\,,\\&A_i(x)=-\frac{1}{\mu^2-\nabla^2}\frac{\partial}{\partial x_j}{}^\star G_{ji}(x)\,,\\&A_0(x)=0\,.\end{aligned} \tag{5.1}$$

We also have from (2.5) the identity

$$\frac{\partial}{\partial x_i}G_{i0}(x)=-\sum_N g_N\delta^3(x-z^N_M) \tag{5.2}$$

Substituting (5.1) and (5.2) in the energy expression (3.5) gives after a straightforward calculation the static potential

$$V=\tfrac{1}{2}\int d^3x\left[\frac{\nabla^2}{\mu^2-\nabla^2}G_{i0}(x)\right]G_{i0}(x)$$
$$+\tfrac{1}{2}\sum_{N,N'}g_Ng_{N'}\frac{\exp(-\mu|\vec{z}^N_M-\vec{z}^{N'}_M|)}{|\vec{z}^N_M-\vec{z}^{N'}_M|}$$
$$+g^2\delta^2(0)\int d\sigma\,|\vec{y}'|\,. \tag{5.3}$$

Note that for $\mu\to\infty$, only the third term, which we found in the last section, survives. There is now also a Yukawa interaction given by the second term for $N\neq N'$ (for $N=N'$ we have the usual infinite interaction self-energy, which may be dropped). The first term in (5.3) is seen to contain an infinity when we substitute for G_{i0} from (5.1).

Thus the regularized action A' with the interpretation $g^2\delta^2(0)=1/2\pi\alpha'$ still leads to an infinity in the energy when μ is finite.[10]

Finally, we show that the string equation of motion (2.15) in this case is identically satisfied for a straight string so that the last term in (5.3) still gives a linear potential between the monopoles. In the last section, we showed that the last term of (2.15) vanishes for a straight static string. The first term of (2.15) also can be seen to vanish. To show this we use (2.4a) to rewrite this term as

$$-\epsilon_{\rho\mu\nu\sigma}\dot{y}^{\mu}y'^{\nu}\left(\mu^2A^{\sigma}(y)-\frac{\partial}{\partial y^{\lambda}}{}^{\star}G^{\lambda\sigma}(y)\right).$$

In this expression, by (5.1), $A_0(y)$ is zero while

$$\vec{A}(y(\sigma))=\frac{g}{4\pi}\int d\sigma'\,F(\sigma,\sigma')\,\frac{\vec{y}(\sigma)-\vec{y}(\sigma')}{|\vec{y}(\sigma)-\vec{y}(\sigma')|}\times\vec{y}\,'(\sigma'),\tag{5.4a}$$

where $\vec{y}(\sigma)\equiv\vec{y}(\tau,\sigma)$ and

$$F(\sigma,\sigma')=\frac{d}{dr}\left(\frac{e^{-\mu r}}{r}\right)\quad\text{for } r=|\vec{y}(\sigma)-\vec{y}(\sigma')|.\tag{5.4b}$$

Thus $A_\sigma(y)$ vanishes for straight strings owing to the cross product in (5.4a). We have still to show that $\partial{}^{\star}G^{\lambda\sigma}(y)/\partial y^{\lambda}$ is zero. By (5.1), this vanishes for $\sigma=0$, while for $\sigma=i$ it becomes

$$\frac{\partial}{\partial y_l}{}^{\star}G_{li}(y(\sigma))=-g\,\epsilon_{lij}\frac{\partial}{\partial y_l}\times\int d\sigma'\,\delta^3(y(\sigma)-y(\sigma'))y_j'(\sigma').\tag{5.4c}$$

Here $\partial/\partial y_l$ can be resolved into components along and perpendicular to y_l'. The latter gives no contribution based on an argument similar to that following (2.11), while the former also gives zero because of the ϵ symbol. Thus a straight string is consistent with the static approximation.

VI. CHARMONIUM

Even though the above model is certainly far too simple for it to be a fully realistic description of hadron dynamics, it contains the interesting feature that a term in the potential between the monopoles increases linearly with their separation. Such a term would of course prevent the monopoles from escaping each other and hence suggests their possible identification with quarks. Several authors[11-13] have recently tried successfully to explore the recently discovered narrow resonances as S-wave states of a charmed quark-antiquark system bound by such a potential. Eichten *et al.*[11] actually fit the energy levels as well as the electronic decay widths with the potential

$$V(r)=\frac{-\alpha_1}{r}+\alpha_2 r,\qquad \alpha_1=0.2,\quad \alpha_2=0.19\ \text{GeV}^2.\tag{6.1}$$

They mention that the linear term is the important one for getting the energy levels right. We notice that our model gives a prediction for α_2,

$$\alpha_2=\frac{1}{2\pi\alpha'}=0.18\ \text{GeV}^2,\tag{6.2}$$

where we have used $\alpha'=0.895\ \text{GeV}^{-2}$ as the experimental Regge slope. The close agreement between the two values for α_2 seems to indicate that the notion of the dual string as providing a quark binding force may have some validity.

It may be of some interest to check further the consistency of this simple model. The S-wave energy levels[14] of a quark-antiquark system nonrelativistically bound by the potential $\alpha_2 r$ are given by

$$E_n=-z_n\left(\frac{\alpha_2^{\,2}}{m}\right)^{1/3}+2m,\tag{6.3}$$

where m is the quark mass (we have dropped the subscripts M and N from the monopole mass m_{MN}) and z_n are the zeros of the Airy function. The first few of them are[15]

$$\begin{aligned}z_1&=-2.338,\\ z_2&=-4.088,\\ z_3&=-5.521,\\ z_4&=-6.786.\end{aligned}\tag{6.4}$$

It has been pointed out[12] that the prediction of (6.3) and (6.4),

$$m(\psi_3)-m(\psi_2)=\frac{z_2-z_3}{z_1-z_2}[m(\psi_2)-m(\psi_1)],\tag{6.5}$$

where ψ_1, ψ_2, and ψ_3 are the meson states $\psi(3105)$, $\psi(3695)$, and $\psi(4170)$, leads to good agreement with experiment. Of course, one has equations similar to (6.5) for the higher radially excited S states. Beside this, one can look for rough consistencies involving bound states of noncharmed quark-antiquark pairs. Denoting by m_u the mass of the uncharmed nonstrange quark and by m_c the mass of the charmed quark, and interpreting ρ' as the first radial excitation of the ρ, we get from (6.3)

$$\frac{m_u}{m_c}=\left[\frac{m(\psi_2)-m(\psi_1)}{m(\rho')-m(\rho)}\right]^3\simeq 0.4.\tag{6.6}$$

To derive (6.6) we have also assumed that a universal $r/2\pi\alpha'$ potential holds for both the charmed and uncharmed cases. The prediction of (6.6) that m_u is appreciably less than m_c seems reasonable and encourages us to believe that this approach has some merit.

Now our static potential (5.3) contains, in addition to the $r/2\pi\alpha'$ term, a Yukawa term and an infinite-string-like term. Neglecting the latter, we might try to improve the semiphenomenological binding potential by including the Yukawa term. Thus we arrive at the potential

$$V = c_1 \frac{e^{-\mu r}}{r} + \frac{r}{2\pi\alpha'}, \tag{6.7}$$

where c_1 is a negative constant. For large but not infinite μ one can get an analytic expression for the energy shift by treating the first term of (6.7) as a perturbation. This energy shift is, by first-order perturbation theory,

$$\Delta E_n = \frac{c_1(m/2\pi\alpha')^{1/3} \int_{z_n}^{\infty} dz\{[\mathrm{Ai}(z)]^2/(z-z_n)\} \exp[-\mu(z-z_n)(m/2\pi\alpha')^{1/3}]}{\int_{z_n}^{\infty} dz[\mathrm{Ai}(z)]^2}, \tag{6.8}$$

where Ai(z) is the Airy function. The denominator can be evaluated as

$$\int_{z_n}^{\infty} [\mathrm{Ai}(z)]^2 dz = [\mathrm{Ai}'(z_n)]^2, \qquad \mathrm{Ai}'(z) \equiv \frac{d}{dz}\mathrm{Ai}(z) \tag{6.9}$$

by using the differential equation for Ai(z).[16] Also for $\mu \gg (m/2\pi\alpha')^{1/3}$,[17]

$$\int_{z_n}^{\infty} dz \frac{[\mathrm{Ai}(z)]^2}{z-z_n} \exp\left[-\mu(z-z_n)\left(\frac{m}{2\pi\alpha'}\right)^{1/3}\right] \simeq \frac{1}{\mu^2}\left(\frac{m}{2\pi\alpha'}\right)^{2/3}[\mathrm{Ai}'(z_n)]. \tag{6.10}$$

From (6.8), (6.9), and (6.10), we get the energy shift as

$$\Delta E_n = \frac{m c_1}{2\pi\alpha'\mu^2}. \tag{6.11}$$

Note that ΔE_n is independent of n. Thus the predictions of (6.5) and (6.6) which depend only on energy differences still hold.

Combining (6.2), (6.3), and (6.11), we find the following formula for the energy levels in this large-μ case:

$$E_n + \Delta E_n = 2m - \frac{z_n}{(2\pi\alpha' m^{1/2})^{2/3}} + \frac{mc_1}{2\pi\alpha'\mu^2}. \tag{6.12}$$

If we apply (6.12) to the [$\psi(3105) - \psi(3695)$] mass difference, we get an estimate for the mass of the charmed quark,

$$m_c \simeq 0.84 \text{ GeV}. \tag{6.13}$$

Putting (6.13) back into (6.12) gives an absolute prediction for the mass of ψ_1 (for example):

$$m(\psi_1) \leq 2.47 \text{ GeV}. \tag{6.14}$$

The inequality in (6.14) comes about because the last term in (6.12) must be negative. To improve this result while keeping the present over-all framework, we may of course consider changing the value of $1/2\pi\alpha'$ given in (6.2). Actually a relatively small change in this quantity from 0.18 to 0.21 is sufficient to shift m_c in (6.13) from 0.84 GeV to 1.16 GeV and change (6.14) to $m(\psi_1) \leq 3.1$ GeV.[12]

It is natural to speculate that the parameters $1/2\pi\alpha'$ and c_1/μ^2 are the same for charmed and uncharmed vector mesons. This would mean that the formula (6.12) should give the ω, ϕ, or ψ masses (for example) depending on the value of the quark mass m. However, the minimum of (6.12) as a function of m is uniquely determined once we specify $m(\psi_1)$ and $m(\psi_2)$. It is independent of m_c and is given by (noting $z_n < 0$)

$$\min E_n = \frac{4}{3^{3/4}}(-z_n)^{3/4}\Lambda^{1/4},$$

where

$$\Lambda = \frac{[m(\psi_1) - m(\psi_2)]^3}{(z_1 - z_2)^4}[z_1 m(\psi_2) - z_2 m(\psi_1)]. \tag{6.15}$$

Inserting the values for $m(\psi_n)$ and z_n, we find

$$E_n \geq 1.8 \text{ GeV}, \tag{6.16}$$

which is clearly in disagreement with the observed vector-meson masses. The simplest way to remedy the situation is to add different constants to the potential for different cases, as several authors[11,12] have done. Of course, in a presumably more realistic non-Abelian version of the theory, modifications of the parameters may occur naturally, giving rise to the correct mass spectrum.

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[7]Note that if $G_{ij}(y)$ in (3.4c) is replaced by the approximation (2.10), the second and third terms will cancel and we will be left with the first term which is the same as the string canonical momentum obtained directly from the action A of Ref. 1. See also the remarks after Eq. (2.18).

[8]The solutions of the homogeneous equation $(\Box+\mu^2)A_\alpha=0$ vanish as $\mu\to\infty$ (in the sense of distributions) and hence need not be added to (4.1).

[9]Note that we are allowing the monopoles themselves to move. If we also set $\dot{\vec{z}}_M^N=0$, (4.3) implies the formula $0=\hat{y}_\mu'/m_N$, where $\hat{y}_\mu'=y_\mu'/(y_\lambda' y'^\lambda)^{1/2}$. Since $\hat{y}_\mu'\hat{y}'^\mu=-1$, this is consistent only for $m_{MN}=\infty$ corresponding to the fact that infinitely massive particles will not move when acted on by a potential.

[10]In the $\mu=0$ case the first and third terms of (5.3) must cancel to yield the pure Coulomb interaction

$$-\frac{1}{2}\sum_{N,N'}\frac{g_N g_{N'}}{|\vec{z}_M^N-\vec{z}_M^{N'}|}.$$

This cancellation requires that the infinite first term be evaluated in such a way that

$$\int d^3x\, G_{i0}(x)G_{i0}(x)\equiv -g^2\int d\sigma\, d\sigma'\,\delta^3(y(\sigma)-y(\sigma'))\wedge y_i'(\sigma)y_i'(\sigma')$$
$$=-2g^2\delta^2(0)\int d\sigma(y_i' y_i')^{1/2}.$$

However, if we approximate

$$\delta^3(y(\sigma)-y(\sigma'))\simeq\delta^3((\sigma-\sigma')y')$$
$$=[1/(y_i' y_i')^{1/2}]\delta^2(0)\delta(\sigma-\sigma').$$

we would get only $\frac{1}{2}$ the required value. The situation is similar to that described in the discussion after (2.18).

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[15]*Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun, National Bureau of Standards, Applied Mathematics Series, No. 55 (U.S.G.P.O., Washington, D. C., 1964).

[16]To derive (6.9), we multiply the equation $d^2\mathrm{Ai}(z)/dz^2-z\mathrm{Ai}(z)=0$ by $d\mathrm{Ai}(z)/dz$ and integrate from z_n to ∞.

[17]Since the integrand on the left-hand side of (6.10) is peaked around $z=z_n$ for large μ, we approximate $\mathrm{Ai}(z)\simeq(z-z_n)\mathrm{Ai}'(z_n)$. Then the integral is easily evaluated.