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Gauge invariance in the effective action and potential

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The gauge dependence of the effective action Γ and potential V are studied in general gauge theories. Explicit expressions which manifest all the gauge dependences of Γ and V are obtained. From these equations, it is concluded that Γ (or V) has gauge-invariant values for any solution of the Euler-Lagrange equation $\delta\Gamma/\delta\phi = 0$ (or $\partial V/\partial\phi = 0$). Introduction of a certain concept about the categories of gauge conditions resolves the appearance of a gauge-dependent unphysical "vacuum." Any gauge can be used to calculate Γ. A wide class of gauges are allowed for the effective potential V; for instance, in scalar QED the allowed gauges are $-(1/2\alpha)(\partial A)^2$, $-(1/2\alpha)(\partial A - v\Phi_2)^2$ (where the direction of condensation is restricted to Φ_1), the Coulomb gauge, and the axial gauge. In particular the R_ξ gauge is also an allowed gauge.

I. INTRODUCTION AND SUMMARY

The effective potential V has recently received much attention and has been widely discussed by many authors.[1] It is the ground-state average energy density given as a function of an order parameter, such as the expectation value ϕ of a scalar field Φ. The corresponding quantity in statistical physics is the Gibbs free energy. For discussions of spontaneous breakdown of symmetries, the effective potential plays a fundamental role; that is, the true ground state of the theory should realize an absolute minimum of $V(\phi)$.

When gauge fields are present in the theory, however, V depends on the choice of the gauge. Although theories with gauge fields are much more interesting and important theoretically, quantities derived from V seem to depend on the gauge.[2-5] An interesting possibility of symmetry breaking due to radiative corrections, discussed by Coleman and Weinberg[6] in the model of scalar quantum electrodynamics, may have gauge dependence, as pointed out by Jackiw.[3] Weinberg[2] calculated in the R_ξ gauge[7] the tadpole graphs T at the one-loop level and was led to the following conclusions: V itself cannot be defined in general in such a way that T is given as $\partial V/\partial\phi$, and only in the Landau gauge $\xi\to\infty$ can we define V. Dolan and Jackiw[4] chose another gauge and calculated V, but they found a peculiar unphysical minimum of V. They claimed that calculation of V should be done in the unitary gauge.

One usually says that the effective potential V (or action Γ) is an *off-shell* quantity and is not directly related to a measurable quantity; therefore, it may depend on the gauge. But one can also argue the other way: *The effective potential V at any stationary point gives average energy densities of the (quasi-) stationary state of "vacuum" and is a physical quantity.* So *the values of V at any stationary point should not depend on the gauge.* Our main conclusion of this article is that these latter statements are true, and we see that this supplies a sufficient *raison d'être* for V in gauge theories.

Consider $W[J]$, the generating functional of the connected Green's functions. Let us expand it around $J=0$ (with all the indices and the symbols of summation and integration omitted):

$$W[J]=W[0]+J\left.\frac{\delta W}{\delta J}\right|_{J=0}+\frac{1}{2}G^{(2)}J^2+\frac{1}{3!}G^{(3)}J^3+\cdots. \quad (1.1)$$

We know that the position of the poles of $\tilde{G}^{(2)}(p)$, $\tilde{G}^{(3)}(p,q),\ldots$ (the Fourier transforms of $G^{(2)}$, $G^{(3)},\ldots$) in any channel is gauge invariant and that the gauge dependence of residues of these poles can be absorbed into wave-function renormalizations, i.e., the S matrix is gauge invariant. Further, the value $W[0]$ is gauge invariant, although there is no concept of "on-shell" for $W[0]$. This is proved in Sec. II and is directly related to the above arguments about the physical meaning of V. Translate (1.1) into the language of effective action Γ, $\Gamma\equiv W-J(\delta W/\delta J)$:

$$\Gamma[\phi]=\Gamma[\tilde\phi]+\frac{1}{2}(\phi-\tilde\phi)^2\Gamma^{(2)}+\frac{1}{3!}(\phi-\tilde\phi)^3\Gamma^{(3)}+\cdots, \quad (1.2)$$

where $\tilde\phi$ corresponds to $J=\delta\Gamma/\delta\phi|_{\phi=\tilde\phi}=0$. Since $\Gamma[\tilde\phi]=W[0]$, gauge invariance of $W[0]$ leads to the invariance of $\Gamma[\tilde\phi]$ and also to gauge invariance of the effective potential $V(\tilde\phi)\equiv-\Gamma[\tilde\phi]/\int d^4x$. Thus we have the following: In the expansion (1.1), each term except for the second term $J(\delta W/\delta J)|_{J=0}$

has the gauge-invariant content. And in the expansion (1.2), *the value* $\Gamma[\tilde{\phi}]$ and the *zeros* of $\tilde{\Gamma}^{(2)}(p)$ are gauge invariant. [For $\tilde{\Gamma}^{(3)}(p,q)$, $\tilde{\Gamma}^{(4)},\ldots$ gauge-invariant contents are somewhat complicated.] We can say from this statement that *symmetry breaking is a gauge-invariant concept*. We remark here that if we expand $W[J]$ around some $J=J_0\neq 0$, then everything depends on the gauge, for example the value of $W[J_0]$ and the position of the poles of the Green's functions.

Precisely speaking, the gauge invariance of $W[0]$ is not sufficient for the invariance of $\Gamma[\tilde{\phi}]$ because $W[J]$ is, in general, multivalued around $J=0$. So the proof must be given directly for the single-valued functional Γ. This is done affirmatively also in Sec. II together with the consideration of renormalization complexities. There we will get the expressions which manifest all the gauge dependences of the renormalized effective action $\Gamma[\phi]$ [(2.26) and (2.27)], and which directly shows gauge invariance of Γ at its any stationary point.

We stress that *any gauge can be used to calculate the effective action* Γ. Some care is needed for the gauge choice to calculate the effective potential $V(\phi)$. Consider the definition of the function $V(\phi)$ from the functional $\Gamma[\phi(x), a_\mu(x)]$. (Up to this point, the argument ϕ of $\Gamma[\phi]$ represents all the fields. But in this paragraph we denote scalar fields as $\phi(x)$ and vector fields as $a_\mu(x)$ for definiteness.) $V(\phi)$ is defined as $-\Gamma[\phi(x)=\phi, a_\mu(x)=0]/\int d^4x$ and its stationary point $\phi=\tilde{\phi}$ (x-independent) is supposed to correspond to the true ground state of the theory. But we must note that the true ground state does not necessarily realize $\phi(x)=\tilde{\phi}$ and $a_\mu(x)=0$ because of the existence of gauge freedom. In a certain class of gauges, which we call "good gauges" in Sec. III, the true ground state realizes $\phi(x)=\tilde{\phi}$, $a_\mu(x)=0$. But in another class of gauges, "bad gauges," $\phi(x)=\tilde{\phi}$ and $a_\mu(x)=0$ do not satisfy the Euler-Lagrange equation $\delta\Gamma/\delta\phi=\delta\Gamma/\delta a_\mu=0$. In the latter case, the true ground state has the expectation values of x-dependent ϕ and nonvanishing a_μ. So in bad gauges the stationary point of $V(\phi)$ is deceptive. In order to search for the true ground state of the theory by the use of the effective potential $V(\phi)$ we must use good gauges. We have to work directly with the effective action $\Gamma[\phi(x), a_\mu(x)]$ in the case of bad gauges; we *define* in any gauge the average energy density V at stationary points of Γ, $\phi(x)=\tilde{\phi}(x)$, $a_\mu(x)=\tilde{a}_\mu(x)$:

$$V\equiv -\left.\frac{\Gamma[\phi(x),a_\mu(x)]}{\int d^4x}\right|_{\substack{\phi(x)=\tilde{\phi}(x)\\ a_\mu(x)=\tilde{a}_\mu(x)}} . \tag{1.3}$$

Gauge invariance of this quantity for *any* gauge is a direct consequence of the invariance of Γ. The definition (1.3) of the effective potential V may be objected to because it is not a true local quantity and has only the meaning of *average*. But we should note that even the "genuine" effective potential $V(\phi)$ in good gauges is not a local quantity and it becomes inevitably nonlocal by quantum corrections. Thus, the defect of bad gauges lies only in the inconvenience in practical calculations.

Wide varieties of gauges are included in good gauges: for instance, the Coulomb gauge, the axial gauge, the gauge $-(1/2\alpha)(\partial A)^2$ for any α, and especially the R_ξ gauge. All these can be used to calculate the usual effective potential $V(\phi)$ and give gauge-independent stationary values, in contrast to the conclusion of Dolan and Jackiw[4] and Weinberg.[2] These are discussed in detail in Sec. III.

The remaining contents of the present paper are as follows. Section IV includes discussions. The explicit calculation up to $O(\hbar)$, which gives all the gauge dependences and gauge-independent stationary values of $V(\phi)$, is contained in Appendix A. Appendix B includes a combinatorial direct proof of the gauge invariance of $V(\phi)$ and a stationary point up to $O(\hbar^2)$. Examples of good gauges in some non-Abelian gauge theories are given in Appendix C. Finally, in Appendix D the equivalence of two approaches using the effective potential and tadpoles is proved.

II. GAUGE INVARIANCE OF THE EFFECTIVE ACTION AND POTENTIAL AT STATIONARY POINTS

A. Ward-Takahashi identities—A review

In order to study the gauge dependence of the effective action and potential, we need the Ward-Takahashi (WT) identities.[8] So we briefly review the WT identities for the generating functional of the connected Green's functions. Let us start with the infinitesimal local gauge transformation

$$\Phi_i \to \Phi_i' = \Phi_i + (\Lambda_i^\alpha + t_{ij}^\alpha \Phi_j)\,u_\alpha + O(u^2)\ , \tag{2.1}$$

where we have used the "condensed notation."[9] Φ_i stands for all the fields and the index i for all attributes of them. For the gauge field $A_\mu^\alpha(x)$, i stands for the group index α, the Lorentz index μ, and the space-time variable x; for the scalar field $\Phi_a(x)$ (we use the same letter Φ for the scalar field and for the condensed notation of a general field, but it will cause no confusion), i stands for the representation index a and x. Summation and integration over repeated indices will always be understood unless noted otherwise. The inhomogeneous term Λ_i^α in (2.1) is nonzero only for the gauge field $\Phi_i=A_\mu^\alpha(x)$ and is given as

$$\Lambda_i^\alpha = \frac{1}{g}\,\partial_\mu \delta^4(x - x_\alpha)\ , \qquad (2.2)$$

and t_{ij}^α is a representation of the generator labeled by α on the fields Φ_i.

The generating functional $W_F[J]$ of the connected Green's functions for the given Lagrangian L in the F gauge, is expressed as

$$\exp\left(\frac{i}{\hbar} W_F[J]\right) = \int [d\Phi]\,\Delta_F[\Phi]\exp\left(\frac{i}{\hbar}\{S[\Phi] - \tfrac{1}{2}F_\alpha^2(\Phi) + J_i\Phi_i\}\right), \qquad (2.3)$$

where

$$S[\Phi] = \int d^4x\, L(\Phi) \qquad (2.4)$$

and $\Delta_F[\Phi]$ is the Faddeev-Popov determinant,[10]

$$\Delta_F[\Phi] = \det M_F\ . \qquad (2.5)$$

M_F is determined by the character of the gauge-fixing term $F_\alpha(\Phi)$, which transforms under the gauge transformation (2.1) as

$$F_\alpha(\Phi) \to F_\alpha(\Phi') = F_\alpha(\Phi) + (M_F)_{\alpha\beta}u_\beta + O(u^2)\ . \qquad (2.6)$$

If we change, in (2.3), the integration variables Φ to Φ' as in (2.1), we can obtain the WT identity for $W_F[J]$:

$$\left\{-F_\beta\left(\frac{\hbar}{i}\frac{\delta}{\delta J}\right)\left[M_F\left(\frac{\hbar}{i}\frac{\delta}{\delta J}\right)\right]_{\beta\alpha} + J_i\left(\Lambda_i^\alpha + t_{ij}^\alpha \frac{\hbar}{i}\frac{\delta}{\delta J_j}\right)\right\}\exp\left(\frac{i}{\hbar}W_F[J]\right) = 0\ . \qquad (2.7)$$

A slightly different (nonlinear) change of integration variable,[11] such as

$$\Phi_i \to \Phi_i' = \Phi_i + (\Lambda_i^\alpha + t_{ij}^\alpha\Phi_j)(M_F^{-1})_{\alpha\beta}u^\beta + O(u^2)\ ,$$

gives another type of WT identity[9,11]:

$$\left\{-F_\alpha\left(\frac{\hbar}{i}\frac{\delta}{\delta J}\right) + J_i\left(\Lambda_i^\beta + t_{ij}^\beta\frac{\hbar}{i}\frac{\delta}{\delta J_j}\right)\left[M_F^{-1}\left(\frac{\hbar}{i}\frac{\delta}{\delta J}\right)\right]_{\beta\alpha}\right\}\exp\left(\frac{i}{\hbar}W_F[J]\right) = 0\ . \qquad (2.8)$$

With the help of this WT identity, we can estimate the change of $W_F[J]$ under the change of the gauge-fixing term F to $F + \Delta F$:

$$\exp\left(\frac{i}{\hbar}W_{F+\Delta F}[J]\right) - \exp\left(\frac{i}{\hbar}W_F[J]\right) = \int [d\Phi]\,\Delta_F[\Phi]\exp\left(\frac{i}{\hbar}\{S[\Phi] - \tfrac{1}{2}F_\alpha^2(\Phi) + J_i\Phi_i\}\right) \times \frac{i}{\hbar}J_i(\Lambda_i^\alpha + t_{ij}^\alpha\Phi_j)[M_F^{-1}(\Phi)]_{\alpha\beta}\Delta F_\beta(\Phi)\ . \qquad (2.9)$$

This gives the simple relation

$$W_{F+\Delta F}[J] - W_F[J] = J_i\langle f_i\rangle\ , \qquad (2.10)$$

with the following notations

$$f_i \equiv (\Lambda_i^\alpha + t_{ij}^\alpha\Phi_j)[M_F^{-1}(\Phi)]_{\alpha\beta}\Delta F_\beta(\Phi)\ , \qquad (2.11)$$

$$\langle \mathfrak{M}\rangle \equiv \frac{\int [d\Phi]\,\Delta_F[\Phi]\,\mathfrak{M}\exp((i/\hbar)\{S[\Phi] - \frac{1}{2}F_\alpha^2(\Phi) + J_i\Phi_i\})}{\int [d\Phi]\,\Delta_F[\Phi]\exp((i/\hbar)\{S[\Phi] - \frac{1}{2}F_\alpha^2(\Phi) + J_i\Phi_i\})}\ . \qquad (2.12)$$

B. Gauge dependence of the effective action

The generating functional of one-particle irreducible (1PI) vertices $\Gamma[\phi]$, or the effective action, is given by the Legendre transform of $W[J]$:

$$\Gamma_F[\phi^F] = W_F[J] - J_i\phi_i^F\ , \qquad (2.13)$$

where

$$\phi_i^F = \frac{\delta W_F[J]}{\delta J_i}\ , \qquad (2.14)$$

$$J_i = -\frac{\delta\Gamma_F[\phi^F]}{\delta\phi_i^F}\ . \qquad (2.15)$$

By the use of these relations, (2.10) is transformed into the equation for Γ:

$$\Gamma_{F+\Delta F}[\phi^{F+\Delta F}] - \Gamma_F[\phi^F] = -\frac{i}{\hbar}J_i(\langle\Phi_i f_j\rangle - \langle\Phi_i\rangle\langle f_j\rangle)J_j \equiv -\frac{i}{\hbar}J_i\langle\Phi_i f_j\rangle_c J_j\ . \qquad (2.16)$$

Here $\langle\cdots\rangle_c$ means the connected part of $\langle\cdots\rangle$ and J should be expressed in terms of ϕ, (2.15). Equation (2.16) gives the difference of two effective actions with the *same source J*.

We can also easily evaluate the difference with the *same argument* ϕ:

$$\Gamma_{F+\Delta F}[\phi^F]-\Gamma_F[\phi^F]=J_i\langle f_i\rangle \ . \tag{2.17}$$

Equations (2.16) and (2.17) give the response of the effective action to the infinitesimal change of the gauge. We see that *the values of* Γ *at its stationary points* ($J_i=0$) *are gauge independent*. This is obvious from (2.16). Some care is needed in order to derive the same conclusion from (2.17). We should compare the values of Γ at different ϕ's which realize the stationarity of Γ with different gauges. But the change of stationary points due to the gauge change causes no change to the values of Γ at stationary points:

$$\begin{aligned}\Gamma_{F+\Delta F}[\phi^{F+\Delta F}]-\Gamma_F[\phi^F]&=\Gamma_{F+\Delta F}[\phi^F]-\Gamma_F[\phi^F]\\&\quad+\frac{\delta\Gamma_F[\phi^F]}{\delta\phi^F}\,\Delta\phi^F+O(\Delta F^2)\\&=J_i\langle f_i\rangle+O(\Delta F^2)\ .\end{aligned}\tag{2.18}$$

Therefore (2.17) is sufficient to lead to the same conclusion. Up to this point all the quantities are unrenormalized. The problem is whether this conclusion about the gauge invariance of Γ at stationary points is also true for the renormalized Γ, and this is the subject of Sec. II C. Since the (unrenormalized) ϕ-fixed relation (2.17) is easier to discuss than (2.16), only the renormalization of (2.17) will be discussed. This is sufficient for our purpose.

C. Gauge dependence of the renormalized action and potential

Although the formulas of Secs. II A and II B are completely general, it will be more transparent to base the discussion on specific examples. Let us take the following unrenormalized Lagrangian density L of scalar quantum electrodynamics (scalar QED) and gauge-fixing term F:

$$\begin{aligned}L=&-\tfrac{1}{4}F^0_{\mu\nu}F^{0\mu\nu}+\tfrac{1}{2}(\partial_\mu\Phi_1^0-e_0A_\mu^0\Phi_2^0)^2\\&+\tfrac{1}{2}(\partial_\mu\Phi_2^0+e_0A_\mu^0\Phi_1^0)^2-\tfrac{1}{2}m_0^{\,2}(\Phi_1^{02}+\Phi_2^{02})\\&-\frac{1}{4!}\lambda_0(\Phi_1^{02}+\Phi_2^{02})^2\ ,\end{aligned}\tag{2.19}$$

$$F=\frac{1}{(\alpha_0)^{1/2}}\,(\partial\cdot A^0-v_0\Phi_2^0)\ ,\tag{2.20}$$

where the index zero indicates unrenormalized quantities and the direction of would-be symmetry breaking has been chosen to be Φ_1. The reason for this special choice of F will be discussed in the next section. The change of gauge ΔF in this case is due to the changes of the parameters α_0 and v_0. Equation (2.17) can be written in differential form:

$$\frac{\partial\Gamma}{\partial\alpha_0}\bigg|_{\phi^0\,\text{fixed}}=-\frac{1}{2\alpha_0}\left\langle\left(J_1^0\Phi_2^0-J_2^0\Phi_1^0-\frac{1}{e_0}\partial\cdot J^0\right)_x\left(\frac{e_0}{-\Box-e_0v_0\Phi_1^0}\right)_{xy}(\partial\cdot A^0-v_0\Phi_2^0)_y\right\rangle,\tag{2.21}$$

$$\frac{\partial\Gamma}{\partial v_0}\bigg|_{\phi^0\,\text{fixed}}=-\left\langle\left(J_1^0\Phi_2^0-J_2^0\Phi_1^0-\frac{1}{e_0}\partial\cdot J^0\right)_x\left(\frac{e_0}{-\Box-e_0v_0\Phi_1^0}\right)_{xy}\Phi_{2y}^0\right\rangle.\tag{2.22}$$

The renormalized version of these equations is obtained as follows. We introduce the renormalized quantities[9]

$$\begin{aligned}\Gamma&=\Gamma^{\text{unren}}([\phi_1^0,\phi_2^0,a_\mu^0],\alpha_0,v_0,\lambda_0,m_0^{\,2},e_0)\\&=\Gamma^{\text{ren}}([\phi_1,\phi_2,a_\mu],\alpha,v,\lambda,m^2,e),\end{aligned}\tag{2.23}$$

$$\lambda_0=Z_\lambda\lambda,\quad e_0=Z_3^{-1/2}e,\quad m_0^{\,2}=Z_mm^2,\quad \alpha_0=Z_3\alpha,\quad v_0=Z_2^{-1/2}Z_3^{1/2}v.\tag{2.24}$$

$$\phi_{1,2}^0=Z_2^{1/2}\phi_{1,2},\quad a_\mu^0=Z_3^{1/2}a_\mu,\quad J_{1,2}^0=Z_2^{-1/2}J_{1,2},\quad J_\mu^0=Z_3^{-1/2}J_\mu.\tag{2.25}$$

From these relations we have

$$\begin{aligned}&\left[(1-2\gamma_3^\alpha\alpha)\frac{\partial}{\partial\alpha}+(\gamma_2^\alpha-\gamma_3^\alpha)v\frac{\partial}{\partial v}-\gamma_\lambda^\alpha\lambda\frac{\partial}{\partial\lambda}-\gamma_m^\alpha m^2\frac{\partial}{\partial m^2}-\gamma_3^\alpha e\frac{\partial}{\partial e}-\gamma_2^\alpha\left(\phi_1\frac{\delta}{\delta\phi_1}+\phi_2\frac{\delta}{\delta\phi_2}\right)-\gamma_3^\alpha a_\mu\frac{\delta}{\delta a_\mu}\right]\Gamma\\&\qquad=-\frac{1}{2\alpha}\left\langle\left(J_1\Phi_2-J_2\Phi_1-\frac{1}{e}\partial\cdot J\right)_x\left(\frac{e}{-\Box-ev\Phi_1}\right)_{xy}(\partial\cdot A-v\Phi_2)_y\right\rangle,\end{aligned}\tag{2.26}$$

$$\left[[1+(\gamma_2^v-\gamma_3^v)v]\frac{\partial}{\partial v}-2\gamma_3^v\alpha\frac{\partial}{\partial\alpha}-\gamma_\lambda^v\lambda\frac{\partial}{\partial\lambda}-\gamma_m^v m^2\frac{\partial}{\partial m^2}-\gamma_3^v e\frac{\partial}{\partial e}-\gamma_2^v\left(\phi_1\frac{\delta}{\delta\phi_1}+\phi_2\frac{\delta}{\delta\phi_2}\right)-\gamma_3^v a_\mu\frac{\delta}{\delta a_\mu}\right]\Gamma$$
$$=-\left\langle\left(J_1\Phi_2-J_2\Phi_1-\frac{1}{e}\partial\cdot J\right)_x\left(\frac{e}{-\Box-ev\Phi_1}\right)_{xy}\Phi_{2y}\right\rangle\ . \quad (2.27)$$

The definitions of γ's are

$$\gamma_{2,3}^\alpha=\tfrac{1}{2}Z_3\frac{\partial}{\partial\alpha_0}\ln Z_{2,3},\ \gamma_{2,3}^v=\tfrac{1}{2}Z_2^{-1/2}Z_3^{1/2}\frac{\partial}{\partial v_0}\ln Z_{2,3}\ , \quad (2.28)$$

$$\gamma_{\lambda,m}^\alpha=Z_3\frac{\partial}{\partial\alpha_0}\ln Z_{\lambda,m},\quad \gamma_{\lambda,m}^v=\tfrac{1}{2}Z_2^{-1/2}Z_3^{1/2}\frac{\partial}{\partial v_0}\ln Z_{\lambda,m}, \quad (2.29)$$

where these derivatives are evaluated with the other bare quantities fixed.[12] Since Z_3 is independent of α_0 and v_0 in our Abelian case, $\gamma_3^\alpha=\gamma_3^v=0$. Some comments are required about (2.26) and (2.27). Though they are "renormalized" version of (2.21) and (2.22), they are not free from infinities. First γ's include divergences. Second, since the stationary point of Γ is infinitely displaced in the gauge with nonvanishing v_0 (this is the case[9] even in the symmetric theory, i.e., $m^2>0$), ϕ_1 should be replaced by ϕ_1+w, where w absorbs this infinity. Correspondingly, the term $\gamma_2^{\alpha,v}\phi_1(\delta\Gamma/\delta\phi_1)$ in (2.26) and (2.27) is replaced by $\delta_2^{\alpha,v}(\delta\Gamma/\delta\phi_1)$, where

$$\delta_2^\alpha=\gamma_2^\alpha(\phi_1+w)+Z_3\frac{\partial w}{\partial\alpha_0}\ ,$$
$$\delta_2^v=\gamma_2^v(\phi_1+w)+Z_2^{-1/2}Z_3^{1/2}\frac{\partial w}{\partial v_0}\ . \quad (2.30)$$

Because of these infinities, (2.26) and (2.27) are meaningful only when suitable regularizations are performed. We adopt the n-dimensional regularization, which is perhaps the most convenient.

For the effective potential

$$V(\phi)\equiv-\Gamma[\phi_1(x)=\phi,\ \phi_2(x)=a_\mu(x)=0]\Big/\int d^4x\ ,$$

we have, corresponding to (2.26) and (2.27),

$$\left(\frac{\partial}{\partial\alpha}+\gamma_2^\alpha v\frac{\partial}{\partial v}-\gamma_\lambda^\alpha\lambda\frac{\partial}{\partial\lambda}-\gamma_m^\alpha m^2\frac{\partial}{\partial m^2}\right)V$$
$$=\delta_2^\alpha\frac{\partial V}{\partial\phi}+\frac{1}{2\alpha}\frac{\partial V}{\partial\phi}\left\langle\Phi_{2x}\left(\frac{e}{-\Box-ev\Phi_1}\right)_{xy}(\partial\cdot A-v\Phi_2)_y\right\rangle\ , \quad (2.31)$$

$$\left[(1+\gamma_2^v v)\frac{\partial}{\partial v}-\gamma_\lambda^v\lambda\frac{\partial}{\partial\lambda}-\gamma_m^v m^2\frac{\partial}{\partial m^2}\right]V$$
$$=\delta_2^v\frac{\partial V}{\partial\phi}+\frac{\partial V}{\partial\phi}\left\langle\Phi_{2x}\left(\frac{e}{-\Box-ev\Phi_1}\right)_{xy}\Phi_{2y}\right\rangle\ . \quad (2.32)$$

[The matrix element $\langle\cdots\rangle$ here is evaluated with the theory (2.19) in the presence of an external source $J_1=\partial V/\partial\phi$.] The equations (2.31) and (2.32) are the final expressions, which manifest all the gauge dependences of the effective potential $V(\phi)$. These expressions clearly show, we remark here, that gauge dependences of the effective potential V are *not* solely due to wave-function renormalization in contrast to the conclusion of Ref. 13. Wave-function renormalization causes only a part of the first terms of the right-hand sides of (2.31) and (2.32) [the $\gamma_2^{\alpha,v}$ terms in (2.30)].

A direct consequence of (2.31) and (2.32) is *gauge independence of the values of the renormalized effective potential at its stationary points*; that is,

$$\left(\frac{\partial}{\partial\alpha}+\gamma_2^\alpha v\frac{\partial}{\partial v}-\gamma_\lambda^\alpha\lambda\frac{\partial}{\partial\lambda}-\gamma_m^\alpha m^2\frac{\partial}{\partial m^2}\right)V\Big|_{\phi=\bar\phi}=0\ , \quad (2.33)$$

$$\left[(1+\gamma_2^v v)\frac{\partial}{\partial v}-\gamma_\lambda^v\lambda\frac{\partial}{\partial\lambda}-\gamma_m^v m^2\frac{\partial}{\partial m^2}\right]V\Big|_{\phi=\bar\phi}=0\ , \quad (2.34)$$

where $\phi=\bar\phi$ may be any stationary point of V. A more explicit statement of gauge independence is obtained if on-shell renormalizations of λ and m^2 are performed, in which case Z_λ and Z_m become independent of α_0 and v_0, i.e., $\gamma_{\lambda,m}^\alpha=\gamma_{\lambda,m}^v=0$. So we get, at the stationary points,

$$\frac{\partial}{\partial\alpha}V=\frac{\partial}{\partial v}V=0\ , \quad (2.35)$$

Equations (2.33) and (2.34) indicate that the explicit α and v dependence of V is compensated for by the implicit dependence through λ and m^2.

We have checked (2.31) and (2.32) up to $O(\hbar^2)$. We explicitly reproduce this calculation up to $O(\hbar)$ in Appendix A, which is necessary for the discussions below, also. Further, there is an interesting proof of (2.33) and (2.34) which is shown in Appendix B up to $O(\hbar^2)$.

We want to emphasize the importance of the direct check. From a simple-minded point of view, it might be thought that the following arguments are sufficient to prove the gauge invariance of the value of V at a stationary point.

We have

$$
\begin{aligned}
-\frac{i}{\hbar} V(\phi) \int d^4x \,\Big|_{\phi=\tilde{\phi}} &= \frac{i}{\hbar} W[J=0] \\
&= \ln \int [d\Phi_1^0 \, d\Phi_2^0 \, dA_\mu^0] \exp\left(\frac{i}{\hbar} S[\Phi_1^0, \Phi_2^0, A_\mu^0]\right) \\
&= \ln \int [d\Phi_1^0 \, d\Phi_2^0 \, dA_\mu^0] \int [dg] \, \Delta_F^0[\Phi^0] \, \delta(F^0(\Phi^{0g}, A_\mu^{0g}) - C) \exp\left(\frac{i}{\hbar} S[\Phi_1^0, \Phi_2^0, A_\mu^0]\right) \\
&= \ln \int [dg] + \ln \int [dC] \exp\left(-\frac{i}{\hbar} \frac{C^2}{2}\right) \int [d\Phi_1^0 \, d\Phi_2^0 \, dA_\mu^0] \, \Delta_F^0[\Phi^0] \, \delta(F^0(\Phi^0, A_\mu^0) - C) \\
&\qquad \times \exp\left(\frac{i}{\hbar} S[\Phi_1^0, \Phi_2^0, A_\mu^0]\right) \\
&= \ln \int [dg] + \ln \int [d\Phi_1^0 \, d\Phi_2^0 \, dA_\mu^0] \, \Delta_F^0[\Phi^0] \exp\left(\frac{i}{\hbar} \{S[\Phi_1^0, \Phi_2^0, A_\mu^0] - \tfrac{1}{2} F^{02}(\Phi^0, A_\mu^0)\}\right) \\
&= \ln \int [dg] + \ln \int [d\Phi_1 \, d\Phi_2 \, dA_\mu] \, \Delta_F[\Phi] \exp\left(\frac{i}{\hbar} \{S[Z_2^{1/2}\Phi_1, Z_2^{1/2}\Phi_2, Z_3^{1/2}A_\mu] - \tfrac{1}{2} F^2(\Phi, A)\}\right) \\
&\quad - \int d^4x (\ln Z_2 + \tfrac{1}{2} \ln Z_3) \delta^4(0) ,
\end{aligned} \tag{2.36}
$$

where dg is the invariant Hurwitz measure[9] over the gauge group and the equations

$$
\Delta_F^0[\Phi^0] \int [dg] \, \delta(F^0(\Phi^{0g}, A^{0g}) - C) = 1 , \tag{2.37}
$$

$$
\Delta_F^0[\Phi^0] = \det\left(\frac{e_0(\alpha_0)^{1/2}}{-\Box - e_0 v_0 \Phi_1^0}\right) = \det\left(\frac{e\sqrt{\alpha}}{-\Box - e v \Phi_1}\right) = \Delta_F[\Phi] , \tag{2.38}
$$

$$
F^0(\Phi^0, A^0) = \frac{1}{(\alpha_0)^{1/2}} (\partial A^0 - v_0 \Phi_2^0) = \frac{1}{\sqrt{\alpha}} (\partial \cdot A - v \Phi_2) = F(\Phi, A) \tag{2.39}
$$

are used. Since all the procedures of the transformation of equations in (2.36) apply for any gauge F, the value of the effective potential at a stationary point given by

$$
\begin{aligned}
V(\phi) &= \frac{i\hbar}{\int d^4x} \ln \int [d\Phi_1 \, d\Phi_2 \, dA_\mu] \, \Delta_F[\Phi] \exp\left(\frac{i}{\hbar} \{S[Z_2^{1/2}\Phi_1, Z_2^{1/2}\Phi_2, Z_3^{1/2}A_\mu] - \tfrac{1}{2} F^2(\Phi, A)\}\right) \\
&\quad - i\hbar \delta^4(0) \ln Z_2 + (\text{gauge-independent infinite const})
\end{aligned} \tag{2.40}
$$

is manifestly gauge invariant. [Only the first term of the right-hand side is included in the ordinary definition of the effective potential. Then a gauge dependence appears due to the neglected second term $-i\hbar\delta^4(0)\ln Z_2$. However, this term gives no gauge dependence to the differences of the values of the effective potential between its stationary points.]

The above naive proof, although it is simple and is found in the end to be essentially correct, may create objections because of two ambiguities: One is due to the fact that $W[J]$ is a multivalued function(al) of J. The point is that we do not know whether the above argument applies to any branch of $W[J]$ about the point $J=0$. The other ambiguity is the question of whether the renormalizations may work destructively against the formal proof. In order to reject the former objection it is necessary to check (2.31) and (2.32) directly without reference to the multivalued $W[J]$, and direct calculation can answer the latter question also (see Appendixes A and B).

With the results in Appendix A, we cite here the difference of average energy densities between two stationary points of the theory (2.19) with $m^2 < 0$, a symmetric one ϕ^s and a broken one ϕ^b:

$$
\begin{aligned}
V(\phi^s) - V(\phi^b) &= \frac{3}{2} \frac{m^4}{\lambda} + \frac{\hbar}{64\pi^2} m^4 \left\{ \left[2 \ln\left(\frac{m^2}{\mu^2}\right) - 3\right] - 4 \left[\ln\left(\frac{-2m^2}{\mu^2}\right) - \frac{3}{2}\right] - 108 \frac{e^4}{\lambda^2} \left[\ln \frac{6e^2}{\lambda}\left(\frac{-m^2}{\mu^2}\right) - \frac{3}{2}\right] \right\} \\
&\quad - \hbar \frac{3}{2} \frac{m^4}{\lambda} (f_\lambda - 2 f_m) .
\end{aligned} \tag{2.41}
$$

This is gauge independent in the sense of (2.33) and (2.34); that is, the explicit gauge dependence of the last term (f_λ, f_m) cancels the implicit gauge dependence of the first term. (f_λ and f_m are finite parts of counterterms which are fixed by renormalization prescription.) A manifestly gauge-invariant, i.e., α and v independent, result is obtained if we renormalize on the mass shell (m^2, λ, f_m, and f_λ are α and v independent in this case) or if we take f_λ and f_m simply to be zero. Since our proof applies irrespectively whether the value of V is real or complex, gauge independence holds for the real and imaginary parts of V separately. The imaginary part of V may be interpreted as the decay probability[14] per unit space and time of an unstable "vacuum":

$$\mathrm{Im}\,V(\phi^s)=\mathrm{Im}\left[\frac{\hbar}{32\pi^2}m^4\ln\left(\frac{m^2-i\epsilon}{\mu^2}\right)\right] =-\frac{\hbar m^4}{32\pi}, \quad (2.42)$$

where we note that $m^2<0$.

III. GOOD GAUGES AND BAD GAUGES

In the theories of spontaneously broken gauge symmetry, the gauge fields A_μ do not necessarily have vanishing vacuum expectation values $a_\mu \equiv \langle A_\mu\rangle$. This is because the gauge freedoms are there. It may happen to be true in some gauge that $a_\mu=0$. But in some other gauge, it must be that the expectation values a_μ are not zero but are the gauge transforms of the above ones at least in the tree level[15]: $a_\mu=(1/g)[\partial_\mu U(\theta)]U^{-1}(\theta)+O(\hbar)$. $[a_\mu=(1/e)\partial_\mu\Lambda+O(\hbar)$ in Abelian gauge theory.$]$ Thus we are led to the concept of two categories of gauge conditions, namely, good gauges and bad gauges. The former ones are defined as the gauge conditions which realize the vanishing vacuum expectation values of the vector gauge fields A_μ. The others are called as bad gauges.

Let us further examine the meaning of this concept. In the search for the true vacuum, a_μ can be set to zero in the effective action in the case of good gauges; hence the effective action contains only scalar fields ϕ (fermion fields, if they are there, are of course set to zero). Then, setting ϕ to be a space-time independent constant, we can define the effective potential: $V(\phi) \equiv -\Gamma[\phi(x)=\phi,\ a_\mu=0]/\int d^4x$. The minimum point of $V(\phi)$ corresponds to a translationally invariant vacuum. That is, the vacuum realizes

$$\phi(x)=\tilde{\phi}=\text{const},\qquad a_\mu(x)=0 \quad (3.1)$$

in good gauges. It is of course true that the theories with bad gauges have the same physical content. We know that in classical theory ($\hbar\to 0$) ϕ and a_μ, which realize the minimum of Γ in different gauges, are connected with each other by gauge transformations[16]; thus,

$$\phi(x)=u(\theta(x))\,\tilde{\phi}\,u^{-1}(\theta(x))+O(\hbar),\qquad a_\mu(x)=\frac{1}{g}[\partial_\mu U(\theta(x))]U^{-1}(\theta(x))+O(\hbar) \quad (3.2)$$

in bad gauges, where $u(U)$ is a representation of gauge transformation on the fields $\Phi(A_\mu)$. Since $a_\mu\neq 0$ in bad gauges by definition, the gauge-transformation angle θ is x dependent; therefore, we conclude that *the scalar fields' vacuum expectation value $\phi(x)$ is x dependent in bad gauges*. This leads to the consequence that the true ground state must be searched for with the help not of the effective potential but of the effective action in the case of bad-gauge conditions. This fact makes bad gauges inconvenient from a practical point of view. The calculation of the effective action is much more tedious than that of the effective potential.

Let us see more specifically in the scalar QED model with Lagrangian (2.19) what types of gauges are good (or bad). First, let us start with the tree-level approximation. The effective action Γ^{tree} is a classical action with a gauge-fixing term:

$$\Gamma^{\text{tree}}=\int d^4x\,L',\qquad L'=[L\text{ of }(2.19)]-\tfrac{1}{2}F^2. \quad (3.3)$$

Choose $F=(1/\sqrt{\alpha})\partial\cdot A$. Then the Euler-Lagrange equations,

$$\frac{\delta\Gamma^{\text{tree}}}{\delta\phi_1}=\frac{\delta\Gamma^{\text{tree}}}{\delta\phi_2}=\frac{\delta\Gamma^{\text{tree}}}{\delta a_\mu}=0,$$

have a well-known solution which corresponds to the ground state in the case of $m^2<0$:

$$\phi_1(x)=\left(\frac{-6m^2}{\lambda}\right)^{1/2}\equiv\phi_0,\qquad \phi_2(x)=a_\mu(x)=0. \quad (3.4)$$

Next, consider the linear gauge $F=(1/\sqrt{\alpha})(\partial\cdot A-v\Phi_2)$. Since any solution in this gauge should be a gauge transformation of solution (3.4), we try the following form:

$$\phi_1(x)=\phi_0\sin\Lambda(x),\qquad \phi_2(x)=\phi_0\cos\Lambda(x),\qquad a_\mu(x)=\frac{1}{e}\partial_\mu\Lambda(x). \quad (3.5)$$

Equations of motion with (3.5) lead to

$$\sqrt{\alpha}\,F=\frac{1}{e}\Box\Lambda-\phi_0 v\cos\Lambda=0. \quad (3.6)$$

This has a constant solution $\Lambda(x)=\frac{1}{2}\pi$, which is nothing but (3.4), and a space-time–dependent solution $\Lambda(x)\sim\frac{1}{8}ev\phi_0 x_\mu{}^2$ (near $\Lambda\sim 0$). It should be noted that there is no solution such that ϕ_2 is nonzero *and* $a_\mu=0$ which gives a nonvanishing gauge-fixing term F. (It is easily proved in general that any gauge-fixing term F must vanish for any solutions of the Euler-Lagrange equation.[17]) Conversely speaking, *if we presuppose as usual that symmetry breakdown occurs, such as* $\langle\Phi_1\rangle=\text{const}$ *and* $\langle\Phi_2\rangle=\langle A_\mu\rangle=0$, *we cannot use the gauge* $\frac{1}{2}F^2=(1/2\alpha)(\partial\cdot A-v_1\Phi_1-v_2\Phi_2)^2(v_1\neq 0)$, *i.e., this is a bad gauge*. With the above presuppositions, a gauge of the type, $F=(1/\sqrt{\alpha})(\partial\cdot A-v\phi_2)$, is a good one.

The last statement can be proved in full-order arguments. Let us assign even and odd quantum numbers to fields as follows:

$$\begin{aligned}&\Phi_1:\ \text{even},\\&\Phi_2, A_\mu:\ \text{odd}.\end{aligned}\tag{3.7}$$

Even after the condensation, $\langle\Phi_1\rangle=\text{const}\neq 0$, the Lagrangian (2.19), the gauge-fixing term $-\frac{1}{2}F^2=(1/2\alpha)(\partial\cdot A-v\Phi_2)^2$, and the corresponding ghost interaction conserve this quantum number. So the effective action Γ must not include linear terms of ϕ_2 and a_μ such as $f(\phi_1)\phi_2$ or $g(\phi_1)a_\mu$. Therefore, the point $\phi_2=a_\mu=0$ remains a stationary point of Γ in any order; i.e., the gauge $F=(1/\sqrt{\alpha})(\partial\cdot A-v\Phi_2)$ is a good gauge in full theory. We stress here a relative nature of the concept of good and bad gauges. Under the presupposition of breakdown in the direction of ϕ_2, such as $\langle\Phi_2\rangle=\text{const}\neq 0$, $\langle\Phi_1\rangle=\langle A_\mu\rangle=0$, the above classifications are interchanged. Goodness or badness is nothing but a relation between the gauges and the solutions we choose.

Similarly, we can classify gauges as good and bad ones in any gauge theory, although classification in general is difficult. In case we can find simple even-odd rules such as (3.7), this classification becomes easy. Some examples in non-Abelian cases are given in Appendix C.

We comment here on the gauge-dependent unphysical solution of Dolan and Jackiw.[4] With the gauge-fixing term $\frac{1}{2}F^2=(1/2\alpha)(\partial\cdot A-v_1\Phi_1-v_2\Phi_2)^2$, the effective potential of the theory (2.19) is, in the tree approximation,

$$V(\phi_1,\phi_2)=\tfrac{1}{2}m^2(\phi_1{}^2+\phi_2{}^2)^2+\frac{1}{4!}\lambda(\phi_1{}^2+\phi_2{}^2)^2+\frac{1}{2\alpha}(v_1\phi_1+v_2\phi_2)^2\,.\tag{3.8}$$

They found the unphysical solution of $\partial V/\partial\phi_a=0$ $(a=1,2)$,

$$\phi_a=\frac{v_a}{(v^2)^{1/2}}\left[-\frac{6}{\lambda}\left(m^2-\frac{v}{\alpha}\right)\right]^{1/2},\tag{3.9}$$

besides the usual physical solution,

$$\phi_a=\epsilon_{ab}\frac{v_b}{(v^2)^{1/2}}\left(-\frac{6m^2}{\lambda}\right)^{1/2}.\tag{3.10}$$

The former solution (3.9) is manifestly unphysical because the value of the effective potential (3.8) at that point depends on the gauge parameters α and v. For any solution $\phi_a\| v_a$ like (3.9), the gauge condition $F=(1/\sqrt{\alpha})(\partial\cdot A-v_a\Phi_a)$ is easily seen to be a bad gauge. (Note that it does not make F vanish.) Therefore the constant "solution" (3.9) is not a solution,[18] because the true solution in a bad gauge is inevitably x dependent. In the case of the latter solution (3.10), $\phi_a\perp v_a$. So $F=(1/\sqrt{\alpha})(\partial\cdot A-v_a\Phi_a)$ is a good gauge for this solution [and therefore (3.10) is the physical solution, of course]. Note that this solution (3.10) makes F vanish and respect the conservation of even-odd quantum number similar to (3.7).

Next, consider the R_ξ gauge[7],

$$\tfrac{1}{2}F^2=\frac{1}{2\alpha}(\partial\cdot A-\alpha e\phi\Phi_2)^2,\tag{3.11}$$

where we have used the usual presupposition of the constant (x independent) breakdown in the direction of Φ_1, $\langle\Phi_1(x)\rangle\equiv\phi$. This is a good gauge since v is just specified to be $\alpha e\phi$ in the previously considered good gauge $F=(1/\sqrt{\alpha})(\partial\cdot A-v\Phi_2)$. Shifting the field $\Phi_1\to\Phi_1+\phi$, we can evaluate the effective potential $V^{R_\xi}(\phi)$. Identification of two different quantities has been made: the shifted magnitude ϕ and ϕ in F, $\phi=v/\alpha e$. Note that

$$V^{R_\xi}(\phi)=V(\phi,v)\big|_{v=\alpha e\phi}\,.\tag{3.12}$$

$V(\phi,v)$ on the right-hand side of (3.12) is the effective potential calculated with the gauge condition $F=(1/\sqrt{\alpha})(\partial\cdot A-v\Phi_2)$. Speaking more rigorously, (3.12) holds only when the same renormalizations are carried out on both sides. Since constants Z_2, λ, and m^2 depend on v, (3.12) should be read as

$$V^{R_\xi}(\phi)=V(\phi,v;Z_2(v),\lambda(v),m^2(v))\big|_{v=\alpha e\phi}\,.\tag{3.12$'$}$$

From (3.12′)

$$\begin{aligned}\frac{\partial V^{R_\xi}(\phi)}{\partial\phi}&=\left(\frac{\partial V}{\partial\phi}+\alpha e\frac{dV}{dv}\right)_{v=\alpha e\phi}\\&\equiv\frac{\partial V}{\partial\phi}\bigg|_{v=\alpha e\phi}+\alpha e\left[(1+\gamma_2^v v)\frac{\partial}{\partial v}-\gamma_\lambda^v\lambda\frac{\partial}{\partial\lambda}-\gamma_m^v m^2\frac{\partial}{\partial m^2}\right]V\bigg|_{v=\alpha e\phi}.\end{aligned}\tag{3.13}$$

[The term $\gamma_2^v v(\partial/\partial v)$ comes from the change of v induced by the change of $Z_2(v)$.] The second term on the right-hand side of (3.13) is proportional to $\partial V/\partial\phi$ as is seen from (2.32). So we can write (3.13) in the form

$$\frac{\partial V^{R_\xi}(\phi)}{\partial\phi}=\frac{\partial V(\phi,v)}{\partial\phi}\bigg|_{v=\alpha e\phi}(1+\hbar\zeta)\,, \tag{3.14}$$

where we have explicitly shown that the second term on the right-hand side is $O(\hbar)$. Since apparently $1+\hbar\zeta\neq 0$ (at least in the sense of perturbation in $\hbar$), the stationarity condition of R_ξ gauge and that of the good gauge $F=(1/\sqrt{\alpha})(\partial\cdot A-v\Phi_2)$ are equivalent. The values of the effective potential at its stationary points are the same for these two gauges. So we can say that the R_ξ gauge is also a good gauge. There is no trouble in calculating the effective potential in the R_ξ gauge.

We discuss here the arguments of Weinberg.[2] He calculated tadpole graphs $T_1^{R_\xi}$ in the R_ξ gauge in the one-loop level and found that they cannot be written as a derivative of any potential. His result, in the case of model (2.19), is

$$T_1^{R_\xi}=-i\frac{\partial V_1^W}{\partial\phi}\bigg|_{\phi=\phi_0}-m^2e^2\phi_0\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2(\xi k^2-e^2\phi_0{}^2)}\,, \tag{3.15}$$

$$V_1^W=-\frac{i}{2(2\pi)^4}\int d^4k\bigg[3\ln(e^2\phi^2-k^2)+\ln\Big(m^2+\frac{\lambda}{2}\phi^2-k^2\Big)+\ln\Big(m^2+\frac{\lambda}{6}\phi^2-k^2\Big)\bigg]\,, \tag{3.16}$$

where ϕ_0 is given by (3.4). From the existence of the nonderivative term, the second term of (3.15), Weinberg concluded that the effective potential approach in the R_ξ gauge works only in the Landau gauge, i.e., $\xi\to\infty$, where the nonderivative term drops out.

But our arguments above show that all the R_ξ gauges can be used without any trouble. First, we note that the perturbative approach using tadpoles is completely equivalent to the effective-potential approach. (A proof of this statement is given in Appendix D.) Tadpole graphs can be given in terms of the effective potential (see Appendix D). Especially at $O(\hbar)$, [from (D2) and (D6)], the one-loop tadpole T_1 is given in terms of the one-loop effective potential V_1:

$$T_1=-i\frac{\partial V_1}{\partial\phi}\bigg|_{\phi=\phi_0}\,. \tag{3.17}$$

For the case of the R_ξ-gauge, the right-hand side is evaluated by setting $v=\alpha e\phi_0$ *after* the ϕ differentiation, i.e.,

$$T_1^{R_\xi}=\frac{\partial V_1(\phi,v)}{\partial\phi}\bigg|_{\substack{\phi=\phi_0\\ v=\alpha e\phi_0}}\,. \tag{3.18}$$

Incidentally, the $O(\hbar)$ terms in Eq. (3.14) lead to

$$\frac{\partial V_1^{R_\xi}}{\partial\phi}\bigg|_{\phi=\phi_0}=\frac{\partial V_1(\phi,v)}{\partial\phi}\bigg|_{\substack{\phi=\phi_0\\ v=\alpha e\phi_0}}+\zeta_0\frac{\partial V_0(\phi,v)}{\partial\phi}\bigg|_{\substack{\phi=\phi_0\\ v=\alpha e\phi_0}}=T_1^{R_\xi}\,, \tag{3.19}$$

where the stationarity of the tree effective potential V_0 at $\phi=\phi_0$ has been used. Thus, in contrast to Weinberg, we can express the tadpoles $T_1^{R_\xi}$ as the derivatives of the potential in two ways, (3.18) and (3.19). $V_1(\phi,v)$ and $V_1^{R_\xi}(\phi)$ in the model (2.19) are (see Appendix A)

$$V_1(\phi,v)=\frac{i}{2(2\pi)^4}\int d^4k[2\ln(ev\phi-k^2)-3\ln(e^2\phi^2-k^2)-\ln(m^2+\tfrac{1}{2}\lambda\phi^2-k^2)-\ln(k^4+a_1k^2+a_2)]\,, \tag{3.20}$$

$$V_1^{R_\xi}(\phi)=\frac{i}{2(2\pi)^4}\int d^4k[\ln(\alpha e\phi^2-k^2)-3\ln(e^2\phi^2-k^2)-\ln(m^2+\tfrac{1}{2}\lambda\phi^2-k^2)-\ln(m^2+\tfrac{1}{6}\lambda\phi^2+\alpha e^2\phi^2-k^2)]\,, \tag{3.21}$$

where a_1 and a_2 are given in (A4). It is trivial to check (3.18) and (3.19) with $T_1^{R_\xi}$ given in (3.15) and (3.16).

Finally, let us consider briefly Lorentz noninvariant gauges such as the Coulomb gauge $\vec{\nabla}\cdot\vec{A}=0$, the axial gauge $A_3=0$, etc. Any gauge is a good gauge if the even-odd quantum number (3.7) is conserved. So the Coulomb and axial gauges are good. Since these gauges are attained by the gauge-fixing term $\frac{1}{2}F^2=(1/2\alpha)(a_{\mu\nu}\partial^\mu A^\nu+b_\mu A^\mu)^2$ with suitable choice of $a_{\mu\nu}$ and b_μ and $\alpha\to 0$, the arguments of gauge dependence in Sec. II are also applicable, and especially the gauge independence of the values of Γ (or V) at its stationary points which holds for these gauges also. It is interesting to consider more generally the gauge which breaks the symmetries of the theory (Lorentz invariance, internal symmetries other than gauge group, and some other discrete or continuous symmetries). How are these symmetry properties preserved in the values of Γ (or V) at its stationary points? Consider any transformation T of such symmetries. Denote all fields and their transforms as Φ and Φ^T. Note that the action $S[\Phi]$ and the path integration measure $[d\Phi]$ are invariant under such transformation; $[d\Phi]=[d\Phi^T]$ and $S[\Phi]=S[\Phi^T]$. The transformed action is

$$\frac{i}{\hbar}\Gamma_F^T\Big|_{\delta\Gamma/\delta\phi=J=0}=\ln\int[d\Phi^T]\Delta_F[\Phi^T]\times\exp\left(\frac{i}{\hbar}\{S[\Phi^T]-\tfrac{1}{2}F^2(\Phi^T)\}\right),$$
$$=\ln\int[d\Phi]\Delta_F[\Phi^T]\times\exp\left(\frac{i}{\hbar}\{S[\Phi]-\tfrac{1}{2}(TF)^2(\Phi)\}\right),$$

where the transformed gauge-fixing term $TF(\Phi)$ is defined to be $F(\Phi^T)$. By the identity [see (2.37)]

$$\Delta_F^{-1}[\Phi^T]=\int[dg]\delta(F(\Phi^{Tg})-C)$$
$$=\int[dg]\delta(TF(\Phi^g)-C)=\Delta_{TF}^{-1}[\Phi],$$

which holds if T commutes with any gauge transformation g, we get $\Gamma_F^T|_{J=0}=\Gamma_{TF}|_{J=0}$. The transformation T induces a gauge transformation $F\to TF$. Thus *gauge invariance* guarantees the invariance of stationary values of Γ (or V) under the transformation T; $\Gamma_F^T|_{J=0}=\Gamma_{TF}|_{J=0}=\Gamma_F|_{J=0}$. In the Coulomb gauge, for instance, the value of Γ (or V) at a stationary point does not depend on the Lorentz frame.

IV. DISCUSSIONS

What we have shown can be summarized as follows.

(a) Calculate the effective action Γ as a *functional* in any gauge and renormalize it in an arbitrary way. Find the stationary points of the resulting Γ and evaluate the values of the average energy density V, (1.3), at these points, which are gauge-invariant quantities. The smallest among these values is the energy of the true ground state.

(b) If the gauges are limited to good ones, we can define the usual effective potential $V(\phi)$ as a *function* of ϕ (an x-independent variable). Among good gauges gauge invariance of stationary values of $V(\phi)$ of course holds.

We believe that these properties are sufficient conditions for Γ or V to have a *raison d'être* in gauge theory. Some other related problems are discussed below.

(1) When we consider finite temperature, the effective potential becomes a function of temperature T, $V(\phi,T)$. Our conclusions are applicable for $V(\phi,T)$ without any change; $V(\phi,T)$ is gauge independent at its stationary points. Physical quantities derived from V without involving the derivative with respect to ϕ, specific heat for example, are gauge invariant.

The conventional definition of the critical temperature T_c is that $[\partial^2V(\phi,T_c)/\partial\phi^2]|_{\phi=0}=0$. This is a gauge-invariant definition because the on-shell condition is satisfied in this case. However, we can think of another gauge-invariant definition of T_c which is more transparent physically. Suppose we have two stationary points, a symmetric one ($\phi^s=0$) and a spontaneously broken one ($\phi^b\neq 0$). We define T_c as $\mathrm{Re}[V(\phi^s,T_c)]-V(\phi^b,T_c)=0$. [At T_c, $\mathrm{Im}V(\phi^s,T_c)$ should be zero simultaneously.] When we examine scalar QED of (2.19) for example, the only sensible small expansion parameters are $\lambda\sim e^2\sim(1/T)^2$, i.e., high temperature.[19] To zeroth order in these parameters $V(\phi,T)$ has only one stationary point at ϕ^s and the new definition does not work in this order, higher-order corrections becoming important. We believe that the two definitions should be identical if full order is taken into account.

(2) Our results can be extended for the more general effective action $\Gamma(\phi,G^{(2)},G^{(3)},\ldots)$, where $G^{(2)},G^{(3)},\ldots$ are two-point, three-point, ... functions,

$$G^{(2)}(x,y)=\langle 0|\Phi(x)\Phi(y)|0\rangle,$$
$$G^{(3)}(x,y,z)=\langle 0|\Phi(x)\Phi(y)\Phi(z)|0\rangle,$$
$$\ldots.$$

The value of Γ is gauge invariant at the points where Γ is stationary with respect to $\phi,G^{(2)},G^{(3)}$, ..., and this gives the energy of the corresponding state.

(3) Some comments are needed about the Coleman-Weinberg model,[6] massless scalar QED. Our results can of course be applied to this model. However, if one wants to check our results perturbatively in $\hbar$, one immediately gets into difficulty: We use the expansion formula (B6) to evaluate V at a stationary point. This expansion turns out to be impossible because $\ln\bar{\phi}\sim O(1/\hbar)$. The cause of this trouble is that a spontaneously broken stationary point appears only in order $\hbar$. Unless we find a sensible gauge-invariant expansion parameter (other than $\hbar$), we cannot apply our results to this model perturbatively.

(4) We have little to say about the unitary gauge. However, if we sufficiently regularize the theory, *the value of unrenormalized* Γ *at a stationary point in the unitary gauge is obviously the same as in the renormalizable gauge*. If the unitary gauge is to be renormalizable at stationary points (we believe this is the case because there is no reason for unitary gauge to be unrenormalizable), then renormalized Γ in the unitary gauge gives the same values at its stationary points as other renormalizable gauges.

Note added in proof. After completing this paper, we became aware of a paper of N. K. Nielsen [Nucl. Phys. B101, 173 (1975)]. He reached the same conclusions in a class of Fermi (Lorentz) gauges by the help of Ward-Takahashi identities similar to ours. We think that the reason for the trouble of the Dolan-Jackiw gauge is not clear in his paper because of the lack of the notion of two gauge categories, good and bad. We have also noticed the paper by B. de Wit [Phys. Rev. D 12, 1843 (1975)], who observed the condition $\langle F\rangle_{J=0}=0$. His assertions seem to be the same as our statements given in footnotes 17 and 18 of the present paper.

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APPENDIX A: ONE-LOOP CALCULATION

The effective potential of the theory (2.19) up to $O(\hbar)$ is

$$V(\phi)=\frac{1}{2}m^2\phi^2+\frac{1}{4!}\lambda\phi^4+\frac{i\hbar}{32\pi^4}\int(d^nk)\left[2\ln(ev\phi-k^2)-(n-1)\ln(e^2\phi^2-k^2)-\ln\left(m^2+\frac{\lambda}{2}\phi^4-k^2\right)-\ln(k^4+a_1k^2+a_2)\right]$$
$$+\frac{\hbar}{2}\delta m^{2(1)}\phi^2+\frac{\hbar}{4!}\delta\lambda^{(1)}\phi^4+\hbar\left(m^2+\frac{\lambda}{6}\phi^2\right)\phi w^{(1)}+\hbar C^{(1)}, \tag{A1}$$

where

$$\delta m^2/m^2=Z_mZ_2-1$$
$$=\hbar(Z_m^{(1)}+Z_2^{(1)})+O(\hbar^2)\equiv\hbar\delta m^{2(1)}/m^2+O(\hbar^2),$$
$$\delta\lambda/\lambda=Z_\lambda Z_2^2-1$$
$$=\hbar(Z_\lambda^{(1)}+2Z_2^{(1)})+O(\hbar^2)\equiv\hbar\delta\lambda^{(1)}/\lambda+O(\hbar^2), \tag{A2}$$
$$w=0+\hbar w^{(1)}+O(\hbar^2),\quad C=0+\hbar C^{(1)}+O(\hbar^2).$$

We use the formula

$$\int(d^nk)\ln(m^2-k^2)=-i\pi^2\mu^4\Gamma\left(-\frac{n}{2}\right)\left(\frac{m^2}{\mu^2}\right)^{n/2} \tag{A3}$$

(μ is an arbitrary unit of mass), and a_1 and a_2 are given by

$$a_1=-(m^2+\tfrac{1}{6}\lambda\phi^2+2e\phi v),\quad a_2=e^2\phi^2[\alpha(m^2+\tfrac{1}{6}\lambda\phi^2)+v^2]. \tag{A4}$$

Counterterms in (A1) are determined such as to make the expression finite:

$$\hbar\delta m^{2(1)}/m^2=\frac{\hbar}{32\pi^2}\Gamma\left(2-\frac{n}{2}\right)\left(\frac{4}{3}\lambda-2\alpha e^2\right)+\hbar f_m\left(\alpha,v,\frac{m^2}{\mu^2},\lambda,e\right),$$
$$\hbar\delta\lambda^{(1)}/\lambda=\frac{\hbar}{32\pi^2}\Gamma\left(2-\frac{n}{2}\right)\left(\frac{36e^4}{\lambda}+\frac{10}{3}\lambda-4\alpha e^2\right)+\hbar f_\lambda\left(\alpha,v,\frac{m^2}{\mu^2},\lambda,e\right),$$
$$\hbar w^{(1)}=\frac{\hbar}{32\pi^2}\Gamma\left(2-\frac{n}{2}\right)2ev+\hbar f_w\left(\alpha,v,\frac{m^2}{\mu^2},\lambda,e\right), \tag{A5}$$
$$\hbar C^{(1)}=\frac{\hbar}{32\pi^2}\Gamma\left(2-\frac{n}{2}\right)m^4,$$

where the finite functions f_m, f_λ, and f_w depend on the various renormalization conditions we choose; for example, $\partial^2V/\partial\phi^2|_{\phi^2=\mu^2}=m^2$, $\partial^4V/\partial\phi^4|_{\phi^2=\mu^2}=\lambda$, or on-shell renormalization, etc. We need not specify these functions for the present purpose. Similarly, wave-function renormalization is given as

$$\hbar Z_2^{(1)}=\frac{\hbar}{32\pi^2}\Gamma\left(2-\frac{n}{2}\right)2e^2(3-\alpha)+\hbar f_Z\left(\alpha,v,\frac{m^2}{\mu^2},\lambda,e\right). \tag{A6}$$

Thus from (A2) and (A6),

$$\hbar Z_m^{(1)}=\frac{\hbar}{32\pi^2}\Gamma\left(2-\frac{n}{2}\right)\left(\frac{4}{3}\lambda-6e^2\right)+\hbar(f_m-f_Z),$$
$$\hbar Z_\lambda^{(1)}=\frac{\hbar}{32\pi^2}\Gamma\left(2-\frac{n}{2}\right)\left(36\frac{e^4}{\lambda}+\frac{10}{3}\lambda-12e^2\right)+\hbar(f_\lambda-2f_Z). \tag{A7}$$

The left-hand sides of (2.31) and (2.32) are evaluated as

$$\left(\frac{\partial}{\partial\alpha}+\gamma_2^\alpha v\frac{\partial}{\partial v}-\gamma_\lambda^\alpha\lambda\frac{\partial}{\partial\lambda}-\gamma_m^\alpha m^2\frac{\partial}{\partial m^2}\right)V$$
$$=-\frac{i\hbar}{32\pi^4}\int(d^nk)\frac{e^2\phi^2(m^2+\frac{1}{6}\lambda\phi^2)}{k^4+a_1k^2+a_2}-\frac{\hbar}{32\pi^2}\Gamma\left(2-\frac{n}{2}\right)e^2\left(m^2\phi^2+\frac{\lambda}{6}\phi^4\right)$$
$$+\hbar\frac{m^2}{2}\phi^2\frac{\partial}{\partial\alpha}f_m+\hbar\frac{\lambda}{4!}\phi^4\frac{\partial}{\partial\alpha}f_\lambda+\hbar\left(m^2+\frac{\lambda}{6}\phi^2\right)\phi\frac{\partial}{\partial\alpha}w^{(1)}-\hbar\gamma_\lambda^{\alpha(1)}\frac{\lambda}{4!}\phi^4-\hbar\gamma_m^{\alpha(1)}\frac{m^2}{2}\phi^2+O(\hbar^2)\,,\qquad\text{(A8)}$$

$$\left[(1+\gamma_2^v v)\frac{\partial}{\partial v}-\gamma_\lambda^v\lambda\frac{\partial}{\partial\lambda}-\gamma_m^v m^2\frac{\partial}{\partial m^2}\right]V$$
$$=-\frac{i\hbar}{32\pi^4}\int(d^nk)\left(\frac{-2e\phi k^2+2e^2\Phi^2v}{k^4+a_1k^2+a_2}+\frac{2e\phi}{k^2-ev\phi}\right)+\hbar\frac{m^2}{2}\phi^2\frac{\partial}{\partial v}f_m+\hbar\frac{\lambda}{4!}\phi^4\frac{\partial}{\partial v}f_\lambda+\hbar\left(m^2+\frac{\lambda}{6}\phi^2\right)\phi\frac{\partial}{\partial v}w^{(1)}$$
$$-\hbar\gamma_\lambda^{v(1)}\frac{\lambda}{4!}\phi^2-\hbar\gamma_m^{v(1)}\frac{m^2}{2}\phi^2+O(\hbar^2)\,,\qquad\text{(A9)}$$

where use has been made of the fact that the γ's start with $O(\hbar)$ terms which are denoted as $\hbar\gamma^{(1)}$. On the other hand, the right-hand sides of (2.31) and (2.32) are evaluated with reference to Fig. 1:

$$\delta_2^\alpha\frac{\partial V}{\partial\phi}+\frac{1}{2\alpha}\frac{\partial V}{\partial\phi}\left\langle\Phi_{2x}\left(\frac{e}{-\Box-ev\Phi_1}\right)_{xy}(\partial\cdot A-v\Phi_2)_y\right\rangle$$
$$=\hbar\gamma_2^{\alpha(1)}\left(m^2\phi^2+\frac{\lambda}{6}\phi^4\right)+\hbar\left(m^2\phi+\frac{\lambda}{6}\phi^3\right)\frac{\partial}{\partial\alpha}w^{(1)}$$
$$+\frac{1}{2\alpha}\left(m^2\phi+\frac{\lambda}{6}\phi^3\right)\int\frac{(d^nk)}{(2\pi)^4}\left(\frac{e}{k^2-ev\phi}\right)\frac{-i\hbar[k^2(\alpha e\phi-v)+v(k^2-\alpha e^2\phi^2)]}{k^4+a_1k^2+a_2}+O(\hbar^2)\,,\qquad\text{(A10)}$$

$$\delta_2^v\frac{\partial V}{\partial\phi}+\frac{\partial V}{\partial\phi}\left\langle\Phi_{2x}\left(\frac{e}{-\Box-ev\Phi_1}\right)_{xy}\Phi_{2y}\right\rangle$$
$$=\hbar\gamma_2^{v(1)}\left(m^2\phi^2+\frac{\lambda}{6}\phi^4\right)+\hbar\left(m^2\phi+\frac{\lambda}{6}\phi^3\right)\frac{\partial}{\partial v}w^{(1)}+\left(m^2\phi+\frac{\lambda}{6}\phi^3\right)\int\frac{(d^nk)}{(2\pi)^4}\left(\frac{e}{k^2-ev\phi}\right)\frac{i\hbar(k^2-\alpha e^3\phi^2)}{k^4+a_1k^2+a_2}+O(\hbar^2)\,,$$
$$\text{(A11)}$$

where the propagators

$$\int d^4x\,e^{ikx}\langle T\Phi_2(x)\Phi_2(0)\rangle^{\text{bare}}=i\hbar\frac{k^2-\alpha e^2\phi^2}{k^4+a_1k^2+a_2}\,,\qquad\text{(A12)}$$

$$\int d^4x\,e^{ikx}\langle T\Phi_2(x)A_\mu(0)\rangle^{\text{bare}}=-\hbar\frac{k_\mu(\alpha e\phi-v)}{k^4+a_1k^2+a_2}\qquad\text{(A13)}$$

are used. With the help of the definitions of γ^α's,

$$\hbar\gamma_\lambda^{\alpha(1)}=\hbar\left(\frac{\partial}{\partial\alpha}f_\lambda-2\frac{\partial}{\partial\alpha}f_Z\right),\quad\hbar\gamma_m^{\alpha(1)}=\hbar\left(\frac{\partial}{\partial\alpha}f_m-\frac{\partial}{\partial\alpha}f_Z\right),\quad\hbar\gamma_2^{\alpha(1)}=-\frac{\hbar}{32\pi^2}\Gamma\left(2-\frac{n}{2}\right)e^2+\frac{\hbar}{2}\frac{\partial}{\partial\alpha}f_Z\,,\qquad\text{(A14)}$$

and of the corresponding expressions of γ^v's, comparison of (A8) [(A9)] and (A10) [(A11)] shows that (2.31) [(2.32)] holds up to $O(\hbar)$. The check up to $O(\hbar^2)$ proceeds similarly, though it is tedious. In this case, the term $-i\hbar\delta^4(0)\ln Z_2$ appears for the first time to ensure gauge invariance.

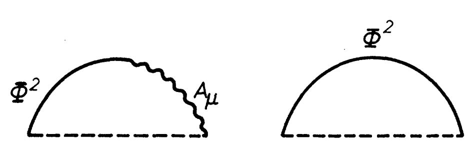

FIG. 1. Graphs that appear in (A10) and (A11).

APPENDIX B: A COMBINATORIAL PROOF OF (2.33) AND (2.34)

In this appendix we prove (2.33), (2.34) up to $O(\hbar^2)$;

$$D_{\alpha,v}V|_{\phi=\tilde\phi}=0\,,\qquad\text{(B1)}$$

with

$$D_{\alpha,v}\equiv\frac{\partial}{\partial(\alpha,v)}+\gamma_2^{\alpha,v}v\frac{\partial}{\partial v}-\gamma_\lambda^{\alpha,v}\lambda\frac{\partial}{\partial\lambda}-\gamma_m^{\alpha,v}m^2\frac{\partial}{\partial m^2}\,,$$
$$\text{(B2)}$$

$$\left.\frac{\partial V}{\partial \phi}\right|_{\phi=\tilde{\phi}}=0\,. \tag{B3}$$

An argument similar to the discussion on (2.18) shows that we can first evaluate V at the stationary point $\phi=\tilde{\phi}$ and then operate $D_{\alpha,v}$. We expand V and $\tilde{\phi}$ in $\hbar$:

$$V(\tilde{\phi})=V_0(\tilde{\phi})+\hbar V_1(\tilde{\phi})+\hbar^2 V_2(\tilde{\phi})+\cdots\,, \tag{B4}$$

$$\tilde{\phi}=\phi_0+\hbar\phi_1+\hbar^2\phi_2+\cdots\,. \tag{B5}$$

Note that $V_0'(\phi_0)=0$ and $\phi_1=-V_1'(\phi_0)/V_0''(\phi_0)$, where V_0', V_1', and V_0'' are derivatives with respect to ϕ evaluated at $\phi=\phi_0$. We get V at a stationary point up to $O(\hbar^2)$:

$$V(\tilde{\phi})=V_0(\phi_0)+\hbar V_1(\phi_0)+\hbar^2\left[V_2(\phi_0)+\tfrac{1}{2}V_1'(\phi_0)\frac{1}{-V_0''(\phi_0)}V_1'(\phi_0)+Z_2^{(1)}\delta^4(0)\right]+O(\hbar^3)\,, \tag{B6}$$

where the term $Z_2^{(1)}\delta^4(0)\hbar^2$ has been explained in (2.40). $-V_0''(\phi_0)$ is the inverse of the bare Φ_1 propagator (multiplied by i) at zero momentum, so we recognize that one-particle *reducible* vacuum graphs are recovered in (B6), which ensures gauge invariance.

Including counterterms we have

$$\phi_0{}^2=\frac{-6m^2}{\lambda}\,, \tag{B7}$$

$$V_0(\phi_0)=\frac{m^2}{2}\phi_0{}^2+\frac{\lambda}{4!}\phi_0{}^4\,, \tag{B8}$$

$$V_1(\phi_0)=V_{\text{count}}^{(1)}+V^{(1\text{-loop})}\,, \tag{B9}$$

$$V_2(\phi_0)=V_{\text{count}}^{(2)}+V_{(1)}^{(1\text{-loop})}+V^{(2\text{-loop})}\,. \tag{B10}$$

Here $V_{\text{count}}^{(1)}$ and $V_{\text{count}}^{(2)}$ are first- and second-order counterterms, which are explicitly given as

$$V_{\text{count}}^{(1)}=\tfrac{1}{2}\delta m^{2(1)}\phi_0{}^2+\frac{1}{4!}\delta\lambda^{(1)}\phi_0{}^4+C^{(1)}\,, \tag{B11}$$

$$V_{\text{count}}^{(2)}=\tfrac{1}{2}\delta m^{2(1)}\phi_0{}^2+\frac{1}{4!}\delta\lambda^{(2)}\phi_0{}^4+C^{(2)}+(\delta m^{2(1)}+\tfrac{1}{6}\delta\lambda^{(1)}\phi_0{}^2)\phi_0 w^{(1)}+\tfrac{1}{2}(m^2+\tfrac{1}{2}\lambda\phi_0{}^2)w^{(1)2}\,. \tag{B12}$$

$V^{(1\text{-loop})}$ and $V^{(2\text{-loop})}$ stand for one-loop and two-loop diagrams without counterterms, and $V_{(1)}^{(1\text{-loop})}$ represents one-loop diagrams containing the first-order counterterms. If we use *symbolic* notations for $V_{\text{count}}^{(1),(2)}$, $V^{(1\text{-loop})}$, $V_{(1)}^{(1\text{-loop})}$, $V^{(2\text{-loop})}$, the Φ_1 propagator $-V_0''(\phi_0)^{-1}$, and the ϕ_0 derivatives $V_{\text{count}}^{(1)\prime}$, $V^{(1\text{-loop})\prime}$, as in Fig. 2, Eq. (B6) is expressed diagramatically as in Fig. 3. Figure 3 manifests the presence of one-particle reducible diagrams. From (A2) we have

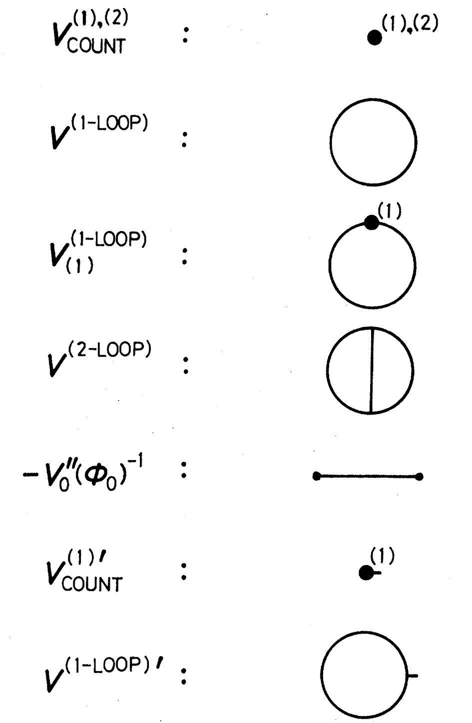

FIG. 2. Symbolic notations for $V_{\text{count}}^{(1),(2)}$, $V^{(1\text{-loop})}$, $V_{(1)}^{(1\text{-loop})}$, $V^{(2\text{-loop})}$, $-V_0''(\phi_0)^{-1}$, $V_{\text{count}}^{(1)\prime}$, and $V^{(1\text{-loop})\prime}$. Symbols for $V^{(1\text{-loop})}$ and $V^{(2\text{-loop})}$ should be read as they stand for all the one-particle irreducible graphs of one loop and two loops.

$$\begin{aligned}
\frac{\delta m^{2(1)}}{m^2}&=Z_m^{(1)}+Z_2^{(1)}\,,\\
\frac{\delta m^{2(2)}}{m^2}&=Z_m^{(2)}+Z_m^{(1)}Z_2^{(1)}+Z_2^{(2)}\,,\\
\frac{\delta\lambda^{(1)}}{\lambda}&=Z_\lambda^{(1)}+Z_2^{(1)}\,,\\
\frac{\delta\lambda^{(2)}}{\lambda}&=Z_\lambda^{(2)}+2Z_\lambda^{(1)}Z_2^{(1)}+Z_2^{(1)2}+2Z_2^{(2)}\,.
\end{aligned} \tag{B13}$$

We show (B1) for the α equation. (The v equation

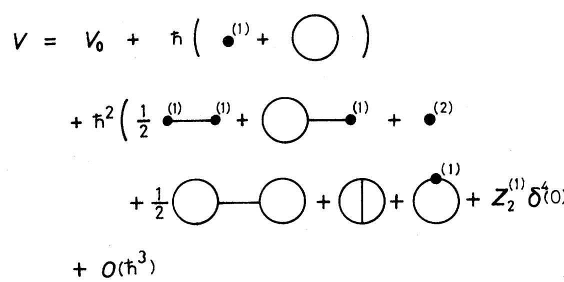

FIG. 3. Diagrammatic representation of Eq. (B6) in terms of symbolic notations in Fig. 2.

can be shown similarly.) The γ^{α}'s are given perturbatively, $\gamma^{\alpha}=\hbar\gamma^{\alpha(1)}+\hbar^2\gamma^{\alpha(2)}+\cdots$, by

$$\gamma_2^{\alpha(1)}=\frac{1}{2}\frac{\partial Z_2^{(1)}}{\partial\alpha},$$

$$\gamma_{\lambda,m}^{\alpha(1)}=\frac{\partial Z_{\lambda,m}^{(1)}}{\partial\alpha},\tag{B14}$$

$$\gamma_{\lambda,m}^{\alpha(2)}=-Z_{\lambda,m}^{(1)}\frac{\partial Z_{\lambda,m}^{(1)}}{\partial\lambda}+\frac{\partial Z_{\lambda,m}^{(2)}}{\partial\alpha}+\frac{1}{2}\frac{\partial Z_2^{(1)}}{\partial\alpha}v\frac{\partial Z_{\lambda,m}^{(1)}}{\partial v}-\frac{\partial Z_{\lambda}^{(1)}}{\partial\alpha}\lambda\frac{\partial Z_{\lambda,m}^{(1)}}{\partial\lambda}-\frac{\partial Z_m^{(1)}}{\partial\alpha}m^2\frac{\partial Z_{\lambda,m}^{(1)}}{\partial m^2}.$$

We here note two facts: First, the graphs in Fig. 3 not involving counterterns are independent of α, v in each order in $\hbar$. (This can be shown by explicit calculations.) Second, because we are treating vacuum graphs, all the Z_2 dependences cancel out. Now we can prove (B1) perturbatively:

$O(\hbar^0)$. This case is trivial, $\partial V_0/\partial\alpha=0$.

$O(\hbar^1)$. The following equations are sufficient:

$$\frac{\partial V_{\text{count}}^{(1)}}{\partial\alpha}=\left(\gamma_\lambda^{\alpha(1)}\lambda\frac{\partial}{\partial\lambda}+\gamma_m^{\alpha(1)}m^2\frac{\partial}{\partial m^2}\right)V_0=\frac{m^2}{2}\phi_0^{\,2}\left(\frac{\partial Z_m^{(1)}}{\partial\alpha}-\frac{1}{2}\frac{\partial Z_\lambda^{(1)}}{\partial\alpha}\right).\tag{B15}$$

$O(\hbar^2)$. We should consider the following terms:

$$\gamma_2^{\alpha(1)}v\frac{\partial V_{\text{count}}^{(1)}}{\partial v}-\gamma_\lambda^{\alpha(1)}\lambda\frac{\partial V_{\text{count}}^{(1)}}{\partial\lambda}-\gamma_m^{\alpha(1)}m^2\frac{\partial V_{\text{count}}^{(1)}}{\partial m^2}-\gamma_\lambda^{\alpha(2)}\lambda\frac{\partial V_0}{\partial\lambda}-\gamma_m^{\alpha(2)}m^2\frac{\partial V_0}{\partial m^2}+\frac{1}{2}\frac{\partial}{\partial\alpha}\left(V_{\text{count}}^{(1)\prime}\frac{1}{-V_0''}V_{\text{count}}^{(1)\prime}\right)+\frac{\partial V_{\text{count}}^{(2)}}{\partial\alpha}\tag{B16}$$

$$-\left(\gamma_\lambda^{\alpha(1)}\lambda\frac{\partial}{\partial\lambda}+\gamma_m^{\alpha(1)}m^2\frac{\partial}{\partial m^2}\right)V^{(1\text{-loop})}+\frac{\partial}{\partial\alpha}\left(V_{(1)}^{(1\text{-loop})}+V^{(1\text{-loop})\prime}\frac{1}{-V_0''}V_{\text{count}}^{(1)\prime}\right)+\frac{\partial}{\partial\alpha}Z_2^{(1)}\delta^4(0).\tag{B17}$$

By using (B11), (B13), and (B14) it is straightforward (though tedious) to show that (B16) and (B17) vanish separately.

APPENDIX C: CHOICE OF GAUGES IN NON-ABELIAN GAUGE THEORY

We discuss the choice of a good gauge for non-Abelian gauge theory, taking a few familiar examples.

(i) O(3) with vector representation. The Lagrangian has vector fields A_μ^i and scalar fields Φ^i $(i=1,2,3)$ and is invariant under following two transformations:

(1) rotation π about the third axis $e^{i\pi I_2}$,

$$(\Phi^1,\Phi^2,A^1,A^2)\to(-\Phi^1,-\Phi^2,-A^1,-A^2)\quad\text{(odd)},$$

$$(\Phi^3,A^3)\to(\Phi^3,A^3)\quad\text{(even)};$$

(2) "G"-parity transformation $Ce^{i\pi I_2}$ ($C=-1$ for Φ, $+1$ for A),

$$(A^1,\Phi^2,A^3)\to(-A^1,-\Phi^2,-A^3)\quad\text{(odd)},$$

$$(\Phi^1,A^2,\Phi^3)\to(\Phi^1,A^2,\Phi^3)\quad\text{(even)}.$$

After the condensation in the Φ^3 direction these quantum numbers are also conserved. Therefore, the effective action Γ has the form

$$\Gamma(\phi^3,(a^3)^2,(\phi^1)^2,(\phi^1a^2),(\phi^2)^2,(\phi^2a^1)),$$

which is stationary at $\phi^1=\phi^2=a^1=a^2=a^3=0$. The most general (Lorentz-invariant) linear good gauge is

$$\frac{1}{2\alpha^{(1)}}(\partial\cdot A^1-v^{(1)}\Phi^2)^2+\frac{1}{2\alpha^{(2)}}(\partial\cdot A^2-v^{(2)}\Phi^1)^2+\frac{1}{2\alpha^{(3)}}(\partial\cdot A^3)^2,$$

with ghost (η) interaction of the form

$$\epsilon_{ijk}\eta^{*i}\partial_\mu\eta^jA_\mu^k,\quad v^1\epsilon_{2ij}\eta^{*1}\eta^i\Phi^j,\quad v^2\epsilon_{1ij}\eta^{*2}\eta^i\Phi^j.$$

Ghost interactions are invariant under transformations (1) and (2) if we assign η^i the same quantum number as A_μ^i ($C=+1$ for ghost interactions). An example of a bad gauge involves $(\partial\cdot A^1-v\Phi^1)^2$, which violates symmetry (2). We expect, therefore, that $a_\mu^3=0$ does not give a stationary point and we should consider Γ involving a_μ^3. Indeed, graphs given in Fig. 4 produce a nonvanishing linear term in a_μ^3 for Γ.

(ii) SU(2) with complex spinor representation. We denote the relevant fields as[20] A_μ^i, and

$$\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}=(Z+i\vec{\tau}\cdot\vec{\Phi})\begin{pmatrix}1\\0\end{pmatrix}.$$

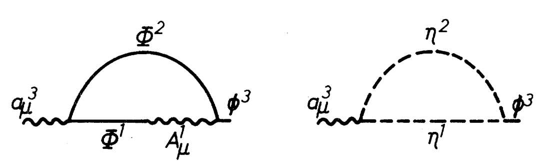

FIG. 4. Graphs that produce linear terms in a_μ^3 for Γ.

The covariant derivative is $D_\mu \xi = (\partial_\mu - ig\frac{1}{2}\vec{\tau}\cdot\vec{A}_\mu)\xi$. We have the following symmetries:

(1) rotation π about the third axis $e^{(i/2)\pi\tau_3}$,

$$(\Phi^1, \Phi^2, A^1, A^2) \to (-\Phi^1, -\Phi^2, -A^1, -A^2) \quad \text{(odd)},$$

$$(\Phi^2, A^2, Z) \to (\Phi^2, A^2, Z) \quad \text{(even)}.$$

The most general good gauge in the case of Z condensation is

$$\sum_{i=1}^{3} \frac{1}{2\alpha^i} (\partial\cdot A^i - v^i \Phi^i)^2 , \tag{C1}$$

with a ghost interaction of the form

$$\epsilon_{ijk}\eta^{*i}\partial_\mu\eta^j A^k_\mu, \quad v^i\epsilon_{ijk}\eta^{*i}\eta^j\Phi^k, \quad v^i\eta^{*i}\eta^i Z . \tag{C2}$$

η^i should transform as A^i_μ under symmetries (1) and (2). Then Γ is stationary at $\phi^i = a^i_\mu = 0$ because Γ has the form $\Gamma(z, (\phi^i)^2, (a^i)^2, a^i\phi^i)$ (not summed in i). If we choose the gauge involving $(\partial\cdot A^1 - v\Phi^2)^2$, for example, which violates symmetry (2) given above, diagrams given in Fig. 5 destroy the stationarity of Γ at $\phi^3 = 0$.

(iii) SU(2) ⊗ U(1) with complex spinor representation. This case is similar to the case (ii). The covariant derivative is now $D_\mu\xi = (\partial_\mu - ig\frac{1}{2}\vec{\tau}\cdot\vec{A}_\mu - ig'B_\mu)\xi$. The B_μ field transforms under symmetries (1) and (2) as $B_\mu \to B_\mu$ (even) and $B_\mu \to -B_\mu$ (odd), respectively. We have for a general good gauge

$$\sum_{i=1,2} \frac{1}{2\alpha_i}(\partial\cdot A^i - v^i\Phi^i)^2 + \frac{1}{2\alpha_A}(\partial\cdot A^3 + a\partial\cdot B - v_A\Phi^3)^2 + \frac{1}{2\alpha_B}(\partial\cdot B + b\partial\cdot A^3 - v_B\Phi^3)^2 ,$$

where $ab \neq 1$.

The above examples suggest that in order to find a general good gauge, first we should look for a *minimal* set of symmetries of a given Lagrangian that guarantees the presupposed condensation. Afterwards, the gauge should be chosen (with corresponding ghost interactions) to be consistent with these symmetries.

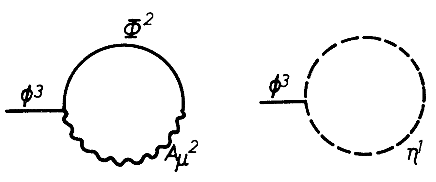

FIG. 5. Graphs that produce linear terms in ϕ^3 for Γ.

APPENDIX D: EQUIVALENCE OF TWO APPROACHES USING THE EFFECTIVE POTENTIAL AND TADPOLES

This equivalence is almost trivial, but, for completeness, here we show an interesting diagrammatical proof.

Let us denote the stationary point $\tilde{\phi}$ as (B5). The tadpole approach[2] determines the quantities $\phi_0, \phi_1, \phi_2, \ldots,$ such that no tadpoles appear in any order of $\hbar$; that is,

$$O(\hbar^0): \quad \partial V_0/\partial\phi|_{\phi=\phi_0} = 0 , \tag{D1}$$

$$O(\hbar^1): \quad -\phi_1 + (\text{all the graphs of one-loop tadpole}) = 0 , \tag{D2}$$

$$O(\hbar^2): \quad -\phi_2 + (\text{all the graphs of two-loop tadpole}) = 0 . \tag{D3}$$

In terms of symbolic notations similar to those in Fig. 2, (D2) and (D3) are represented diagramatically in Fig. 6.

On the other hand, the expectation value $\tilde{\phi}$ is determined by the stationarity of V in the effective potential approach:

$$\partial V/\partial\phi|_{\phi=\tilde{\phi}} = 0 . \tag{D4}$$

By the use of (B4) and (B5), (D4) is rewritten as

$$0 = \partial V_0/\partial\phi|_{\phi=\phi_0} + \hbar(V_0''\phi_1 + V_1') + \hbar^2(V_0''\phi_2 + \tfrac{1}{2}V_0'''\phi_1^2 + V_1''\phi_1 + V_2') + O(\hbar^3) , \tag{D5}$$

where V_0'', V_1', V_0''', etc., are derivatives with respect to ϕ evaluated at $\phi = \phi_0$. Equivalence at $O(\hbar^0)$ is apparent literally, and at $O(\hbar^1)$ and $O(\hbar^2)$,

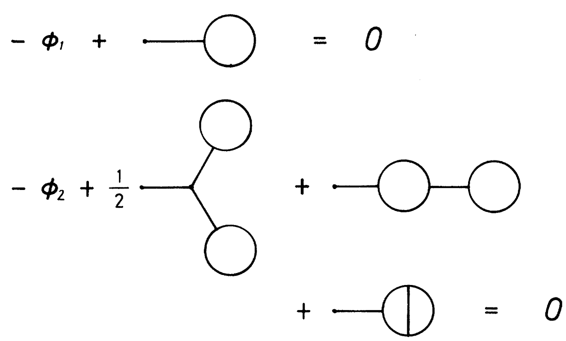

FIG. 6. Diagramatic representations of Eqs. (D2) and (D3) in terms of symbolic notations similar to those in Fig. 2. Note especially that the straight line stands for the Φ_1 propagator $-V_0''(\phi_0)^{-1}$ and that the crossing point of the three straight lines represents the (bare) three-vertex, $d^3V_0/d\phi^3|_{\phi=\phi_0}$.

from (D5),

$$O(\hbar^1):\quad -\phi_1+\frac{V_1'}{-V_0''}=0\,, \tag{D6}$$

$$O(\hbar^2):\quad -\phi_2+\frac{1}{2}\,\frac{1}{-V_0''}\,V_0'''\phi_1^{\,2}+\frac{1}{-V_0''}\,V_1''\phi_1+\frac{1}{-V_0''}\,V_2'=0\,. \tag{D7}$$

The comparison of (D2) [(D3)] and (D6) [(D7)] obviously shows that they are the same equations term by term. [Compare Fig. 6 and (D6) and (D7).] Extension to higher orders is straightforward. (Note that the above comparison should be done between two approaches with the same gauge-fixing condition, of course.)

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[12]If we calculate Z's in terms of renormalized quantities, we should use the formulas

$$Z_3\frac{\partial}{\partial\alpha_0}\ln Z=\frac{\partial}{\partial\alpha}\ln Z-\left[2\gamma_3^\alpha\alpha\frac{\partial}{\partial\alpha}-(\gamma_2^\alpha-\gamma_3^\alpha)v\frac{\partial}{\partial v}+\gamma_\lambda^\alpha\lambda\frac{\partial}{\partial\lambda}+\gamma_m^\alpha m^2\frac{\partial}{\partial m^2}+\gamma_3^\alpha e\frac{\partial}{\partial e}\right]\ln Z\,,$$

$$Z_2^{-1/2}Z_3^{1/2}\frac{\partial}{\partial v_0}\ln Z=\frac{\partial}{\partial v}\ln Z-\left[-(\gamma_2^v-\gamma_3^v)v\frac{\partial}{\partial v}+2\gamma_3^v\alpha\frac{\partial}{\partial\alpha}+\gamma_\lambda^v\lambda\frac{\partial}{\partial\lambda}+\gamma_m^v m^2\frac{\partial}{\partial m^2}+\gamma_3^v e\frac{\partial}{\partial e}\right]\ln Z.$$

Since the second terms of the right-hand sides are one order higher than the first, we can determine γ's perturbatively with these formulas.

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[15]The change of vacuum expectation values due to the change of gauge can be seen by differentiating (2.10) with respect to J and setting $J=0$:

$$\begin{aligned}\Delta\langle\Phi_i\rangle&=\langle\Phi_i\rangle_{F+\Delta F}-\langle\Phi_i\rangle_F\\&=\langle f_i\rangle=(\Lambda_i^\alpha+t_{ij}^\alpha\langle\Phi_j\rangle)\langle[M_F^{-1}(\Phi)]_{\alpha\beta}\Delta F_\beta(\Phi)\rangle\\&\quad+\langle(\Lambda_i^\alpha+t_{ij}^\alpha\Phi_j)[M_F^{-1}(\Phi)]_{\alpha\beta}\Delta F_\beta(\Phi)\rangle_{\text{connected}}\,.\end{aligned}$$

The first term in the last expression expresses the c-number gauge transformation of $\langle\Phi_i\rangle$, which reproduces the usual gauge transformation of classical fields in the limit $\hbar\to 0$. The second term represents the complexity of the q-number nature of the exact gauge transformation.

[16]There are special gauges among the bad gauges. These are c-number gauge transforms of good gauges. The solutions in these gauges are, to full order in $\hbar$, the c-number gauge transforms of the solutions in good gauges because $\Delta F=M_F\Delta u$ in Ref. 15.

[17]We note that

$$\int[d\Phi]\Delta_F[\Phi]\exp\left(\frac{i}{\hbar}\int\{L(x)-\tfrac{1}{2}[F(x)+c(x)]^2\}d^4x\right)\equiv K$$

is independent of parameter field c. Then $\langle F(x)\rangle_{J=0}=\delta K/\delta c(x)|_{c=0}=0$. More generally,

$$\langle F(x_1)F(x_2)\cdots F(x_{2n+1})\rangle_{J=0}=0,$$

$\langle F(x_1)\cdots F(x_{2n})\rangle_{J=0}=(h/i)^n\sum\prod_{i\neq j}\delta(x_i-x_j)$ (the summation extends over all possible pairings of $2n$ points). These relations say that F is essentially free.

[18]Solution (3.9) may seem to satisfy the Euler-Lagrange equation, but note the point: One has to *assume*, to obtain (3.9), *that the surface term can be neglected in required partial integration to get the equation of motion.* Resulting solution (3.9), however, violates this assumption because it does not make F vanish.

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