Construction of a quark spin operator*

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We discuss how, in null-plane quantized quark models, the total angular momentum operator can be decomposed into approximately conserved, commuting orbital angular momentum and quark spin operators. This decomposition is relevant for the construction of a U(6) × U(6) × O(3) ("constituent quark") classification algebra. Such an algebra has been extensively utilized in phenomenological analyses of matrix elements of the vector and axial-vector currents, but has to date been explicitly constructed only in the free-quark model. Extending this construction to theories of interacting quarks, we identify some features of the free-quark result which are not generally applicable. Specifically, we argue that the U(6) × U(6) × O(3) algebra is only approximately boost invariant, and that quark pairs provide correction terms in the calculation of current matrix elements.

I. INTRODUCTION

While it is widely agreed that the spectrum of strongly interacting particles displays an approximate U(6) × U(6) × O(3) symmetry,[1] it is conjectural at best that this symmetry may be relevant for the description of hadronic transitions. The reason for this discrepancy has been the failure to date to construct in any realistic field theory a plausible form for the U(6) × U(6) × O(3) generators. In this paper we will examine theories of interacting quark fields and formulate a procedure for constructing the desired algebra. Aside from demonstrating the existence of the algebra, we show that the U(6) × U(6) × O(3) generators are all *approximately conserved*[2] so that the algebra is in fact a reasonable one to use for the classification of hadronic states.

In physical terms a U(6) × U(6) × O(3) classification algebra implies the existence of objects called constituent quarks[3] in terms of which hadronic states may be simply represented. The constituent quarks carry angular momentum $\frac{1}{2}$ and combine to form states of definite quark spin S and orbital angular momentum L. To construct the quark spin and orbital angular momentum operators we demand that the total angular momentum $\vec{J}$ should be separable in the form[4]

$$\vec{J} = \vec{L} + \vec{S}\,, \tag{1.1}$$

with $\vec{L}$ and $\vec{S}$ forming the generators of two mutually commuting SU(2) algebras.

This problem has, of course, a simple solution in theories quantized on arbitrary spacelike surfaces. We choose, however, to work with theories quantized on null planes,[5,6] a choice which, while complicating the problem immediately at hand, makes our results of direct relevance to a number of physical processes.[7] Deep-inelastic scattering,[6] for example, can be viewed as a probe of the null-plane wave function of the target particle. Current-algebra sum rules[8] have likewise been shown to have a simple structure on the null plane. Of even greater phenomenological interest, perhaps, is the fact that via the PCAC (partial conservation of axial-vector current) hypothesis, pionic transitions can be described[9] in terms of a null-plane integral of the axial-vector current density. Furthermore, since null planes result from the infinite Lorentz boost of spacelike surfaces, it might be expected that the null-plane description is useful for all processes involving high-energy particles.

It is convenient in the null-plane approach to replace the usual spacetime variables x^0 and x^3 by the linear combinations

$$x^{\pm} = (x^0 \pm x^3)/\sqrt{2}\,. \tag{1.2}$$

A null plane is defined as a surface of constant x^+, and null-plane dynamics describe the evolution of a system from one such surface to another. Since we have singled out the direction x^3 in defining our null surfaces, it is obvious that the rotation generators J^1 and J^2 will not leave these surfaces invariant. These operators are thus dynamical quantities in the present approach. It follows that the form of $\vec{L}$ and $\vec{S}$—or, equivalently, the form of the constituent quark fields—will depend in detail on the form of the interaction.

As will become evident in the following section, we cannot simply identify the constituent quarks with the canonical (current quark) fields[10] of our theory. Rather, we seek a transformation[11] which carries the canonical fields into a form appropriate for constituent quarks. The procedure is to take the standard expression for $\vec{J}$ (in terms of the canonical quark fields) and demand that in terms of the constituent quarks it possess an archetypical form compatible with Eq. (1.1). The problem is thus similar to that first discussed by Foldy, Wouthuysen, and Tani,[12] who sought a transformation bringing the equal-time Hamiltonian into a certain archetypical form. Here, of course, the emphasis is on $\vec{J}$ rather than the Hamiltonian, but because of the approximate conservation of $\vec{L}$ and $\vec{S}$ found in our study, this difference is not of very great technical importance. In this regard our work might be viewed as a generalization to the null plane of the Foldy-Wouthuysen-Tani transformation. The idea that this type of transformation might be used to construct a (nonlocal) SU(6) algebra is an old one,[13] but again, these early proposals were made in the context of dynamics on spacelike surfaces.

Although our ultimate interest will be in theories of interacting quarks, we orient ourselves in the next section by discussing an SU(3) triplet of noninteracting quark fields.[14] This will enable us to make a precise specification of the problem we wish to solve and to construct for this simple case an exact solution. The techniques used in obtaining this solution will be enunciated more generally in Sec. III, after which we will apply them (in Sec. IV) to the specific model of an SU(3) triplet of quark fields interacting with an SU(3) singlet scalar field.

The fifth section of the paper is devoted to an analysis of our results. We explain first why the U(6) × U(6) × O(3) algebra is approximately conserved and point to the source which eventually breaks this symmetry. Next, we turn to the phenomenological implications of our results. Since much recent work in this area has been stimulated by the work of Melosh,[11] we emphasize those features of our results which may be contrasted with his work. Specifically, we note that the U(6) × U(6) × O(3) algebra is not boost invariant. This implies that states of different momenta must be classified differently and thus complicates the treatment of unequal-mass transitions. This effect is actually present in the free-quark model, but it has been optimistically ignored in most phenomenological discussions.

A feature of our results which is truly characteristic of interacting theories is the presence of quark-pair terms in the transformation defining the constituent quark. We find that these pair terms are small in exactly the sense in which the U(6) × U(6) × O(3) generators are approximately conserved. Thus we argue that the observed multiplet structure of the hadron spectrum implies that the pair terms are relatively unimportant in hadronic transitions. This is quite an important result, since much of the phenomenology which has flowed from Melosh's work rests exclusively on the fact that pair terms never appear in the free-quark transformation.

II. FREE-QUARK MODEL

We begin our discussion by examining the Lagrangian density

$$\mathcal{L}(x)=\overline{\Psi}(x)(i\partial\!\!\!/-m)\Psi(x) \tag{2.1}$$

from which one can derive the usual Dirac equation

$$(i\partial\!\!\!/-m)\Psi(x)=0\,. \tag{2.2}$$

It is convenient in the null-plane approach to decompose the field $\Psi(x)$ into components

$$\Psi_\pm(x)=\frac{\gamma^\mp\gamma^\pm}{2}\Psi(x)\,. \tag{2.3}$$

Equation (2.2) then provides a constraint between Ψ_+ and Ψ_-,

$$p\Psi_-(x)=\tfrac{1}{2}\gamma^+(i\vec{\gamma}^\perp\cdot\vec{\partial}^\perp+m)\Psi_+(x) \tag{2.4}$$

and an equation of motion

$$\partial_+\Psi_+(x)=\tfrac{1}{2}\gamma^-(\vec{\gamma}^\perp\cdot\vec{\partial}^\perp-im)\Psi_-(x)\,. \tag{2.5}$$

In Eq. (2.4) we have introduced the symbol $p\Psi(x)$ for the derivative $i(\partial/\partial x^-)\Psi(x)$. (A summary of our null-plane notation may be found in Appendix A.) The constraint Eq. (2.4) can be solved by introducing an integral operator p^{-1}, defined by the relation

$$p^{-1}\Psi(x^-)=\frac{-i}{2}\int dy^-\epsilon(x^--y^-)\Psi(y^-)\,. \tag{2.6}$$

We can then rewrite the equation of motion as

$$\partial_+\Psi_+(x)=-i\left(\frac{m^2-\partial_\perp{}^2}{2p}\right)\Psi_+(x)\,. \tag{2.7}$$

Quantization of the theory is accomplished by rewriting the $\Psi_-(x)$ in terms of the $\Psi_+(x)$ and assigning to those fields the canonical anticommutator

$$\{\Psi_+(x),\Psi_+^\dagger(y)\}\delta(x^+-y^+)=\frac{\gamma^-\gamma^+}{2\sqrt{2}}\delta^4(x-y)\,. \tag{2.8}$$

The Lorentz generators for this theory are easily constructed.[15] In terms of a null-plane integral they may be written in the form

$$M^{\mu\nu}=\sqrt{2}\int d^2x^\perp dx^-\Psi_+^\dagger(ix^\mu\partial^\nu-ix^\nu\partial^\mu+\tfrac{1}{2}\sigma^{\mu\nu})\Psi\,. \tag{2.9}$$

Some components of $M^{\mu\nu}$, such as

$$J^3 = M^{12} = \sqrt{2}\int d^2x^\perp dx^- \Psi_+^\dagger(ix^1\partial^2 - ix^2\partial^1 + \tfrac{1}{2}\sigma^3)\Psi_+ \tag{2.10}$$

and, for $i=1,2$,

$$E^i = M^{+i} = \sqrt{2}\int d^2x^\perp dx^- \Psi_+^\dagger(ix^+\partial^i - x^i p)\Psi_+ , \tag{2.11}$$

depend only on Ψ_+ and are quite simple in form. Other components, such as

$$F^i = M^{-i} = i\sqrt{2}\int d^2x^\perp dx^-[\Psi_+^\dagger(x^-\partial^i - x^i\partial_+)\Psi_+ + \tfrac{1}{2}\Psi_+^\dagger\gamma^-\gamma^i\Psi_-] , \tag{2.12}$$

involve the dependent fields Ψ_- as well. When the Ψ_- are expressed in terms of the Ψ_+ [with the aid of Eq. (2.4)] and the x^+ derivative is evaluated [with the aid of Eq. (2.5)], the F^i assume the following rather complicated form[16]:

$$F^i = -\frac{1}{\sqrt{2}}\int d^2x^\perp dx^- \Psi_+^\dagger[-2ix^-\partial^i + \partial^i p^{-1} + x^i(m^2 - \partial_\perp{}^2)p^{-1} + i(m\gamma^i + \sigma^3\epsilon^{ij}\partial^j)p^{-1}]\Psi_+ . \tag{2.13}$$

It is apparent from Eqs. (2.11) and (2.13) that the angular momentum components

$$J^i = \frac{1}{\sqrt{2}}\epsilon^{ij}(F^j - E^j) \tag{2.14}$$

are not particularly simple in the null-plane approach. In contrast with the equal-time approach, there is here no obvious way of separating J^i into spin and orbital angular momentum parts. This should not be very surprising. Rotations around the x^1 and x^2 axes do not leave the null plane (x^+ = constant) invariant. Hence construction of these operators requires an essential knowledge of how the fields evolve in x^+, i.e., of the equations of motion. Indeed, traces of Eqs. (2.4) and (2.5) are evident in the factors of p^{-1} present in our final expression for the F^i, Eq. (2.13).

We seek to separate the operator $\vec{J}$ [Eqs. (2.10) and (2.14)] into commuting operators $\vec{L}$ and $\vec{S}$. To accomplish this task we will construct a unitary transformation V which carries the canonical fields $\Psi_+(x)$ into new fields $q(x)$:

$$q(x) = V\Psi_+(x)V^{-1} . \tag{2.15}$$

In the spirit of the Foldy-Wouthuysen-Tani transformation[12] we demand that $\vec{J}$, in terms of the $q(x)$, should have the simple form

$$\vec{J} = \int d^2x^\perp dx^- q^\dagger\left(\vec{\mathfrak{L}} + \frac{\vec{\omega}}{\sqrt{2}}\right)q , \tag{2.16}$$

where[17]

$$\omega^i = -i\,\mathcal{E}\epsilon^{ij}\gamma^j , \qquad \omega^3 = \sigma^3 , \tag{2.17}$$

and $\vec{\mathfrak{L}}$ contains no Dirac matrices. The symbol $\mathcal{E}$ denotes the operator

$$\mathcal{E} = \frac{p}{|p|} = \frac{p}{(p^2)^{1/2}} . \tag{2.18}$$

We can then identify

$$\vec{S} = \int d^2x^\perp dx^- q^\dagger \frac{\vec{\omega}}{\sqrt{2}} q \tag{2.19}$$

and

$$\vec{L} = \int d^2x^\perp dx^- q^\dagger \vec{\mathfrak{L}}\, q \tag{2.20}$$

as the quark spin and orbital angular momentum operators, respectively.

The unitary nature of V implies that the $q(x)$ obey the same anticommutation relations (2.8) as the fields $\Psi_+(x)$. It follows that the $\vec{S}$ satisfy the commutation relations of the algebra SU(2). It is also clear that $\vec{L}$ and $\vec{S}$ commute and that the $\vec{L}$ consequently form a second SU(2) algebra. One can imbed the $\vec{S}$ in a U(6)×U(6) algebra with generators

$$W_\alpha^{a(\pm)} = \frac{1}{2\sqrt{2}}\int d^2x^\perp dx^- q^\dagger(1\pm\mathcal{E})\omega^a\lambda_\alpha q , \tag{2.21}$$

where $a=0$, 1, 2, or 3 and $\omega^0=2$. The W's all commute with $\vec{L}$ and form with it the U(6)×U(6)×O(3) algebra in terms of which we wish to classify hadronic states.

We have yet, of course, to construct the transformation V. To insure its unitarity, we write it in the form

$$V = e^{iY} , \tag{2.22}$$

where Y is a Hermitian operator on which our attention will now be focused. Since J^3 [Eq. (2.10)] is already of the form (2.16), the transformation V should leave this operator unchanged. Demanding that J^3 and Y commute, we can then concentrate on the J^i [Eq. (2.14)]. The procedure we shall follow in the construction of Y consists of expanding relevant operators in powers of m^{-1}, treating other dimensional quantities ($p^\perp, p^+$, etc.) as m^0. The leading terms in this expansion will be dominant in the formal limit $m\to\infty$. This procedure is reminiscent of the Foldy-Wouthuysen-Tani expansion. Since, however, we work at fixed x^+ rather than fixed x^0, the physical interpretation of the large-m limit is rather different. Indeed each term of our expansion receives contributions from terms of all orders in the corresponding fixed-x^0 expansion (and vice versa). Grouping together terms of J^i with the same number of powers of the quark mass, we write

$$J^i = \int d^2x^\perp dx^- \Psi_+^\dagger (m^2 \mathcal{J}^i_{(0)} + m\mathcal{J}^i_{(1)} + \mathcal{J}^i_{(2)})\Psi_+ , \quad (2.23)$$

where

$$\mathcal{J}^i_{(0)} = -\epsilon^{ij} x^j/2p , \quad (2.24)$$

$$\mathcal{J}^i_{(1)} = -\epsilon^{ij}(i\gamma^j)/2p , \quad (2.25)$$

$$\mathcal{J}^i_{(2)} = -\epsilon^{ij}[-ix^-\partial^j - x^j p + ix^+\partial^j + i\sigma^3\epsilon^{jk}\partial^k/2p - (x^j\partial_\perp{}^2 - \partial^j)/2p] . \quad (2.26)$$

We note that the leading term $\mathcal{J}^i_{(0)}$ commutes with $\vec{\omega}$ and is hence already of a form appropriate for $\mathcal{L}_i$. We conclude that in the infinite-m limit we can identify the fields Ψ_+ and q. The transformation V is then unity and Y in this limit is identically zero.

For finite values of m we will construct Y in terms of an asymptotic expansion

$$Y = \sum_{n=1}^{\infty} m^{-n} Y^{(n)} . \quad (2.27)$$

With the aid of Eqs. (2.15), (2.22), and (2.23), J^i may be written as

$$J^i = \exp\Big(-i\sum_n m^{-n}Y^{(n)}\Big)\int d^2x^\perp dx^- q^\dagger (m^2\mathcal{J}^i_{(0)} + m\mathcal{J}^i_{(1)} + \mathcal{J}^i_{(2)})q \exp\Big(i\sum_n m^{-n}Y^{(n)}\Big)$$

$$= \int d^2x^\perp dx^- \{m^2 q^\dagger \mathcal{J}^i_{(0)} q + m q^\dagger \mathcal{J}^i_{(1)} q - m[iY^{(1)}, q^\dagger \mathcal{J}^i_{(0)} q] + q^\dagger \mathcal{J}^i_{(2)} q - [iY^{(1)}, q^\dagger \mathcal{J}^i_{(1)} q] - [iY^{(2)}, q^\dagger \mathcal{J}^i_{(0)} q] + \tfrac{1}{2}[iY^{(1)}, [iY^{(1)}, q^\dagger \mathcal{J}^i_{(0)} q]] + O(1/m)\} . \quad (2.28)$$

To each order in m we demand that J^i have the form specified by Eq. (2.16). It is clear that given $Y^{(1)}, Y^{(2)}, \ldots, Y^{(n)}$, the term $Y^{(n+1)}$ is specified by a single commutator equation.

To be more specific let us examine the term of order m in Eq. (2.28). Since $\mathcal{J}^i_{(1)}$ [Eq. (2.25)] contains a factor γ^j, it is inadmissable in $\mathcal{L}^i$. This term must be canceled by the $Y^{(1)}$ commutator:

$$\int d^2x^\perp dx^- \{[iY^{(1)}, q^\dagger \mathcal{J}^i_{(0)} q] - q^\dagger \mathcal{J}^i_{(1)} q\} = 0 . \quad (2.29)$$

Using Eqs. (2.24) and (2.25) we arrive at the equation

$$\int d^2x^\perp dx^- \left\{\left[Y^{(1)}, q^\dagger \frac{x^j}{p} q\right] - q^\dagger \frac{\gamma^j}{p} q\right\} = 0 . \quad (2.30)$$

Using the commutation relations (2.8) one readily verifies that a solution of this equation is given by

$$Y^{(1)} = -\sqrt{2}\int d^2x^\perp dx^- q^\dagger \vec{\gamma}^\perp \cdot \vec{\partial}^\perp q . \quad (2.31)$$

Substituting this result back into Eq. (2.28) and turning our attention to terms of order m^0, we obtain a commutator equation for the operator $Y^{(2)}$. Explicitly, we have for J^i the expression

$$J^i = -\epsilon^{ij}\int d^2x^\perp dx^- \left\{q^\dagger [m^2 x^j/2p + i(x^+ - x^-)\partial^j - x^j p + \partial^j/2p - x^j\partial_\perp{}^2/2p] q - \left[iY^{(2)}, q^\dagger \frac{x^j}{2p} q\right] + O(1/m)\right\} . \quad (2.32)$$

To bring this expression into the form (2.16) it is necessary that the $Y^{(2)}$ commutator generate the quark spin term, Eq. (2.19). This provides the equation

$$\int d^2x^\perp dx^- \left\{\left[Y^{(2)}, q^\dagger \frac{x^j}{2p} q\right] + q^\dagger \frac{\varepsilon\gamma^j}{\sqrt{2}} q\right\} = 0 , \quad (2.33)$$

which has a solution

$$Y^{(2)} = 2\int d^2x^\perp dx^- q^\dagger \vec{\gamma}^\perp \cdot \vec{\partial}^\perp |p| q . \quad (2.34)$$

Continuing in this fashion we can construct Y to arbitrarily high orders of the $1/m$ expansion. It is actually possible in this model to sum the series (2.27) and thus arrive at Melosh's result[11]

$$Y = -\sqrt{2}\int d^2x^\perp dx^- q^\dagger \tan^{-1}\left(\frac{\vec{\gamma}^\perp \cdot \vec{\partial}^\perp}{\sqrt{2}|p| + m}\right) q . \quad (2.35)$$

We obtain in this manner the following form for the orbital angular momentum:

$$L^i = -\tfrac{1}{2}\epsilon^{ij} \int d^2x^\perp dx^- q^\dagger \left[i(x^+ - x^-)\partial^j + x^j\left(\frac{m^2-\partial_\perp{}^2}{2p} - p\right)\right] q + \text{H.c.} \tag{2.36}$$

From the explicit form of the null-plane Hamiltonian

$$P^- = \sqrt{2} \int d^2x^\perp dx^- \Psi_+^\dagger(x) \frac{m^2-\partial_\perp{}^2}{2p}\Psi_+(x) = \sqrt{2}\int d^2x^\perp dx^- q^\dagger(x)\frac{m^2-\partial_\perp{}^2}{2p} q(x). \tag{2.37}$$

we see that $q^\dagger \vec{\mathcal{L}} q$ has the simple interpretation of the cross product of a displacement and a momentum density.[18]

A further important feature of $\vec{\mathrm{L}}$ and $\vec{\mathrm{S}}$ is the fact that they are conserved. That

$$[P^-, \vec{\mathrm{S}}] = 0 \tag{2.38}$$

is evident from Eqs. (2.19) and (2.37). Between states of zero transverse momenta, it follows that the commutator of P^- and $\vec{\mathrm{L}}$ is also zero. From the structure of Eq. (2.21) it is clear that all the $W_\alpha^{a(\pm)}$ commute with P^- and that the entire $\mathrm{U}(6)\times\mathrm{U}(6)\times\mathrm{O}(3)$ algebra is conserved.

Returning to the expansion (2.27), we note from Eq. (2.35) that successive powers of m^{-1} are compensated in each order by successive powers of $|p|$. Since this factor is—in the nonrelativistic limit—itself of order m, it is clear that our expansion is not the same as the nonrelativistic expansion of Foldy, Wouthuysen, and Tani. It is clear, in fact, that all orders of our expansion are required to generate just the first term of the corresponding nonrelativistic expansion. Thus our expansion is not useful in studying, say, the static properties of isolated quarks.

Where our expansion will be useful is in the study of tightly bound systems of massive quarks. In the rest frame of such a composite, the total momentum P^+ is just $M/\sqrt{2}$, where M is the composite particle's mass. Its constituents have even smaller momenta, so if $m \gg M$ we might expect the expansion (2.27) to converge.

III. INTERACTING MODELS

We have already noted some similarities of the present work with that of Foldy, Wouthuysen, and Tani. Just as their nonrelativistic (v/c) expansion can be extended to the study of interacting fields, our m^{-1} expansion admits a similar extension. In this section we will discuss some general features of the interacting case. These are illustrated in a specific example which we will solve in the following section.

In any null-plane quantized theory the quark field $\Psi(x)$ may be split into components $\Psi_\pm(x)$ [Eq. (2.3)]. We will regard the Ψ_+ as independent components and quantize the theory using the anticommutation relations (2.8). To begin our construction of V we write out $\vec{\mathrm{J}}$ in terms of the Ψ_+ and other independent fields of the theory. Since J^3 is not a dynamical object, it will always have the form of Eq. (2.10) (plus terms involving fields other than Ψ_+). The separation of L^3 and S^3 is thus trivial to accomplish, and we demand accordingly that V leave J^3 unchanged. The J^i, in contrast, are dynamical quantities: They involve terms which mix the Ψ_+ and other fields and permit no obvious separation of L^i and S^i. Grouping together terms of different orders in m, we write these operators in the form

$$J^i = \int d^2x^\perp dx^- [\Psi_+^\dagger(m^2 \mathcal{J}^i_{(0)} + m\mathcal{J}^i_{(1)} + \mathcal{J}^i_{(2)})\Psi_+ + \mathcal{K}^i]. \tag{3.1}$$

The specific forms of $\mathcal{J}^i_{(1)}$ and $\mathcal{J}^i_{(2)}$ will differ with different interactions. The extra term $\mathcal{K}^i$ is an operator which—like the $\mathcal{J}^i$—does not involve the fields Ψ_+ and $\Psi_+^\dagger$.

We seek a unitary transformation V which upon application of Eq. (2.15) brings J^i into the form

$$J^i = L^i + S^i, \tag{3.2}$$

where S^i is of the form (2.19) and $\vec{\mathrm{L}}$ and $\vec{\mathrm{S}}$ commute. Since $\mathcal{J}^i_{(0)}$ in any interacting theory is exactly the same as in the noninteracting theory [Eq. (2.24)] the transformation V is unity in the limit $m\to\infty$. For finite m we write V in the form (2.22) and develop Y in the asymptotic form (2.27). Transforming the quark fields with Eq. (2.15) and other fields in an analogous way, we rewrite Eq. (3.1) to obtain the following asymptotic expansion for J^i:

$$J^i = \int d^2x^\perp dx^- \{ m^2 q^\dagger \mathcal{J}^i_{(0)} q + m q^\dagger \mathcal{J}^i_{(1)} q - m[iY^{(1)}, q^\dagger \mathcal{J}^i_{(0)} q] + q^\dagger \mathcal{J}^i_{(2)} q - [iY^{(2)}, q^\dagger \mathcal{J}^i_{(0)} q] - [iY^{(1)}, q^\dagger \mathcal{J}^i_{(1)} q] + \tfrac{1}{2}[iY^{(1)}, [iY^{(1)}, q^\dagger \mathcal{J}^i_{(0)} q]] + V^{-1}\mathcal{K}^i V + O(1/m)\}. \tag{3.3}$$

Demanding that to each order in m, J^i should have the form (3.2) we obtain successive commutator equations for $Y^{(1)}$, $Y^{(2)}$, etc. The solutions of these equations—assuming they exist—allow the specification of Y to arbitrarily high orders of the $1/m$ expansion.

In Eq. (3.2) we have specified only the requirement that $\vec{L}$ commute with $\vec{S}$. This guarantees that the L^i generate the appropriate angular momentum algebra [SU(2)] but leaves the exact form of $\vec{L}$—and hence of V—undefined. To avoid this ambiguity we demand that L^i have a quark contribution of the form (2.36) with additional terms representing the angular momentum carried by other fields of the theory. We demand that these terms also have the form characteristic of free-field theories. This insures that to any order of the $1/m$ expansion both $\vec{L}$ and $\vec{S}$ obey (to that order) their proper SU(2) commutation relations. This is not a trivial requirement, as commutators of the $\vec{L}$ mix terms of different orders in m^{-1}.

Although the procedure just outlined is exactly that followed in the free-quark example of the previous section, there is no assurance that in interacting models it will yield any solution for $\vec{L}$ and $\vec{S}$. Furthermore, there is no *a priori* reason that these operators should be approximately conserved. We will show in the following section that solutions do in fact exist; and in Sec. V we will explain why the operators constructed in the $1/m$ expansion turn out to be approximately conserved. Here we note some general features which should be important in physical applications.

First of all, the simple bilinear dependence of Y [Eq. (2.35)] on the quark fields will not be generally true. Rather, we expect the interactions to generate quark-antiquark pairs and produce corresponding 4-quark, 6-quark, etc. terms in Y. These terms will typically enter at increasingly high orders of the $1/m$ expansion, and hence they may in some approximate sense be negligible. The nature of this approximation will be explored In Sec. V.

Secondly, the U(6)×U(6)×O(3) algebra is not boost invariant.[19] This feature is actually present in the free-quark model although its significance seems to have been systematically neglected in previous phenomenological discussions.

Finally, while the U(6)×U(6)×O(3) generators are approximately conserved, they are not conserved exactly. Higher-order terms in the $1/m$ expansion break the symmetry[20]—a welcome feature in light of the fact that SU(6) multiplets are not exactly degenerate and that general theorems[21] forbid nontrivial examples of SU(6)-invariant theories.

IV. A THEORY OF INTERACTING QUARKS

In this section we apply the general procedure of the previous section to a specific theory[6] of interacting quark fields. We study here a triplet of quarks interacting with an SU(3) singlet scalar particle. The Lagrangian density for this theory has the form

$$\mathcal{L}(x)=\bar{\Psi}(x)(i\not{\partial}-m)\Psi(x)+\tfrac{1}{2}[(\partial_\mu\phi)(\partial^\mu\phi)-\mu^2\phi^2] - G\bar{\Psi}(x)\Psi(x)\,\phi(x). \tag{4.1}$$

Separating $\Psi(x)$ into independent and dependent components $\Psi_\pm(x)$ we find a constraint equation

$$p\Psi_-(x)=\tfrac{1}{2}\gamma^+(i\vec{\gamma}^\perp\cdot\vec{\partial}^\perp+m+G\phi)\Psi_+(x) \tag{4.2}$$

and an equation of motion

$$i\partial_+\Psi_+(x)=\tfrac{1}{2}[(m^2-\partial_\perp^2)p^{-1}+iG\phi\vec{\gamma}^\perp\cdot\vec{\partial}^\perp p^{-1} - iG\vec{\gamma}^\perp\cdot\vec{\partial}^\perp p^{-1}\phi+mGp^{-1}\phi + mG\phi p^{-1}+G^2\phi p^{-1}\phi]\Psi_+(x). \tag{4.3}$$

In terms of Ψ_+ and ϕ the null-plane Hamiltonian P^- has the form

$$P^-=\frac{1}{\sqrt{2}}\int d^2x^\perp dx^-\left\{\Psi_+^\dagger[m^2p^{-1}+mG(\phi p^{-1}+p^{-1}\phi)-\vec{\partial}_\perp^2p^{-1}+iG\phi\vec{\gamma}^\perp\cdot\vec{\partial}^\perp p^{-1}-iG\vec{\gamma}^\perp\cdot\vec{\partial}^\perp p^{-1}\phi+G^2\phi p^{-1}\phi]\Psi_+ + \frac{1}{\sqrt{2}}[(\vec{\partial}^\perp\phi)^2+\mu^2\phi^2]\right\}. \tag{4.4}$$

The theory is quantized by assigning to the independent components of the quark fields the canonical anticommutation relations (2.8) and to the scalar field the commutation relation

$$[\phi(x),\partial_-\phi(y)]\,\delta(x^+-y^+)=\frac{i}{2}\,\delta^4(x-y). \tag{4.5}$$

For equal values of x^+ the fields Ψ_+ and ϕ commute with each other.

The Lorentz generators relevant to our discussion are

$$J^3=\int d^2x^\perp dx^-\left[\epsilon^{ij}x^i(\partial_-\phi)(\partial^j\phi)+i\sqrt{2}\,\Psi_+^\dagger\epsilon^{ij}x^i\partial^j\Psi_+ +\Psi_+^\dagger\frac{\sigma^3}{\sqrt{2}}\Psi_+\right], \tag{4.6}$$

$$E^i=\int d^2x^\perp dx^-[x^+(\partial_-\phi)(\partial^i\phi)-x^i(\partial_-\phi)(\partial_-\phi)+\sqrt{2}\,\Psi_+^\dagger(ix^+\partial^i-x^ip)\Psi_+],\tag{4.7}$$

$$F^i=\int d^2x^\perp dx^-\{x^-(\partial_-\phi)(\partial^i\phi)-\tfrac{1}{2}x^i[(\vec{\partial}^\perp\phi)^2+\mu^2\phi^2]+i\sqrt{2}\,[\Psi_+^\dagger(x^-\partial^i-x^i\partial_+)\Psi_++\tfrac{1}{2}\Psi_+^\dagger\gamma^-\gamma^i\Psi_-]\}.\tag{4.8}$$

It is clear from Eq. (4.6) that the separation of L^3 and S^3 is a simple matter if we demand that V leave J^3 invariant. The transverse components of $\vec{J}$ are more complex. They are given in terms of Eqs. (4.7) and (4.8) with the aid of Eq. (2.14). We choose for simplicity to work at $x^+=0$, where E^i assumes the form

$$E^i=-\int d^2x^\perp dx^-[x^i(\partial_-\phi)(\partial_-\phi)+\sqrt{2}\,\Psi_+^\dagger x^ip\Psi_+]\tag{4.9}$$

Using the constraint equation (4.2) and the equation of motion (4.3) we write F^i in the form

$$\begin{aligned}F^i=\int d^2x^\perp dx^-\Big(&x^-(\partial_-\phi)(\partial^i\phi)-\tfrac{1}{2}x^i[(\vec{\partial}^\perp\phi)^2+\mu^2\phi^2]\\&-\frac{1}{\sqrt{2}}\Psi_+^\dagger\{m^2x^ip^{-1}+m\,[Gx^i(\phi p^{-1}+p^{-1}\phi)+i\gamma^ip^{-1}]-2ix^-\partial^i+\partial^ip^{-1}+i\epsilon^{ij}\partial^j\sigma^3p^{-1}\\&+x^i[-\partial_\perp{}^2p^{-1}+iG\phi\vec{\gamma}^\perp\cdot\vec{\partial}^\perp p^{-1}-iG\vec{\gamma}^\perp\cdot\vec{\partial}^\perp p^{-1}\phi+G^2\phi p^{-1}\phi]+iG\gamma^ip^{-1}\phi\}\Psi_+\Big).\end{aligned}\tag{4.10}$$

We thus arrive at an expansion of J^i of the form (3.1) with

$$\mathcal{J}^i_{(0)}=-\frac{\epsilon^{ij}}{2}x^jp^{-1},\tag{4.11}$$

$$\mathcal{J}^i_{(1)}=-\frac{\epsilon^{ij}}{2}(Gx^j\phi p^{-1}+Gx^jp^{-1}\phi+i\gamma^jp^{-1}),\tag{4.12}$$

$$\mathcal{J}^i_{(2)}=-\frac{\epsilon^{ij}}{2}(-2ix^-\partial^j-2x^jp-x^j\partial_\perp{}^2p^{-1}+i\epsilon^{jk}\partial^k\sigma^3p^{-1}+iGx^j\phi\vec{\gamma}^\perp\cdot\vec{\partial}^\perp p^{-1}-iGp^{-1}\vec{\gamma}^\perp\cdot\vec{\partial}^\perp x^j\phi+G^2x^j\phi p^{-1}\phi),\tag{4.13}$$

and

$$\mathcal{K}^i=\frac{\epsilon^{ij}}{\sqrt{2}}\{x^-(\partial_-\phi)(\partial^j\phi)+x^j[(\partial_-\phi)(\partial_-\phi)-\tfrac{1}{2}(\vec{\partial}^\perp\phi)\cdot(\vec{\partial}^\perp\phi)-\tfrac{1}{2}\mu^2\phi^2]\}.\tag{4.14}$$

We seek a transformation V which brings J^i into the canonical form (3.2). For definiteness we demand that L^i have the form

$$\begin{aligned}L^i=-\tfrac{1}{2}\epsilon^{ij}\int d^2x^\perp dx^-\Big\{&q^\dagger\Big[x^j\Big(\frac{m^2-\partial_\perp{}^2}{2p}-p\Big)-ix^-\partial^j\Big]q\\&+\frac{1}{\sqrt{2}}\{x^j[\tfrac{1}{2}(\vec{\partial}^\perp f)\cdot(\vec{\partial}^\perp f)-(\partial_-f)(\partial_-f)+\tfrac{1}{2}\mu^2f^2]-x^-(\partial_-f)(\partial^jf)\}\Big\}\\&+\text{H.c.}\end{aligned}\tag{4.15}$$

Here the quark fields q are defined by Eq. (2.15) and the scalar field f by the analogous expression

$$f(x)=V\phi(x)V^{-1}.\tag{4.16}$$

The L^i form an SU(2) algebra and clearly commute with the $\vec{S}$, Eq. (2.18). Comparing Eqs. (4.15) and (4.11) we see that to the leading order in m, J^i is already of the appropriate form. Thus V is unity in the limit $m\to\infty$, and it is reasonable to develop Y [defined in Eq. (2.22)] in the asymptotic expansion (2.27). Collecting terms of different orders in m, we obtain the expression (3.3) with $\mathcal{J}^i$ and $\mathcal{K}^i$ given by Eqs. (4.11)–(4.14).

Comparison of Eqs. (3.3) and (3.2) gives [with the aid of Eqs. (2.19) and (4.15)] a set of successive commutator equations for $Y^{(1)}$, $Y^{(2)}$, etc. At order m, for example, we note that L^i and S^i have no terms. This means that the terms of order m in Eq. (3.3) must all cancel,

$$\int d^2x^\perp dx^-\{[iY^{(1)},q^\dagger\mathcal{J}^i_{(0)}q]-q^\dagger\mathcal{J}^i_{(1)}q\}=0.\tag{4.17}$$

Introducing the specific forms (4.11) and (4.12) of $\mathcal{J}^i_{(0)}$ and $\mathcal{J}^i_{(1)}$, we obtain the following commutator equation for $Y^{(1)}$:

$$\int d^2x^\perp dx^- [iY^{(1)}, q^\dagger x^j p^{-1} q] = \int d^2x^\perp dx^- q^\dagger [Gx^j(fp^{-1} + p^{-1}f) + i\gamma^j p^{-1}] q. \quad (4.18)$$

The detailed solutions of this and other commutator equations are discussed in Appendix B. Here we merely note the result

$$Y^{(1)} = -\sqrt{2} \int d^2x^\perp dx^- q^\dagger [\vec{\gamma}^\perp \cdot \vec{\partial}^\perp + iG(p^{-1}f)p + iGp(p^{-1}f)] q. \quad (4.19)$$

Equation (4.19) gives us the first-order expression for Y. To obtain the second-order term $Y^{(2)}$ we substitute this result for $Y^{(1)}$ into Eq. (3.3) and examine the resulting terms of order m^0. Comparison with Eq. (3.2) then yields a commutator equation for $Y^{(2)}$. This equation (which is derived in detail in Appendix B) has the form

$$\int d^2x^\perp dx^- [iY^{(2)}, q^\dagger x^j p^{-1} q] = \int d^2x^\perp dx^- (X^j_{\mathrm{FQ}} + X^j_{\mathrm{I}} + X^j_{\mathrm{II}} + X^j_{\mathrm{III}} + X^j_{\mathrm{IV}}) \quad (4.20)$$

where (FQ referring to free quarks)

$$X^j_{\mathrm{FQ}} = -i\sqrt{2}\, q^\dagger \mathcal{E} \gamma^j q, \quad (4.21)$$

$$X^j_{\mathrm{I}} = -\tfrac{1}{2} iGx^j q^\dagger \vec{\gamma}^\perp \cdot [(\vec{\partial}^\perp f)p^{-1} + p^{-1}(\vec{\partial}^\perp f)] q, \quad (4.22)$$

$$X^j_{\mathrm{II}} = iGq^\dagger(\vec{\gamma}^\perp \cdot \vec{\partial}^\perp f x^j p^{-1} - x^j p^{-1} f \vec{\gamma}^\perp \cdot \vec{\partial}^\perp) q, \quad (4.23)$$

$$X^j_{\mathrm{III}} = -\tfrac{1}{2} G^2 x^j q^\dagger [ffp^{-1} + p^{-1}ff + 2(p^{-1}f)(pf)p^{-1} + 2p^{-1}(p^{-1}f)(pf)] q, \quad (4.24)$$

$$X^j_{\mathrm{IV}} = -\frac{G^2}{2\sqrt{2}} x^j [q^\dagger(p^{-1} - \overleftarrow{p}^{-1})q] p^{-2} [q^\dagger(p - \overleftarrow{p})q]. \quad (4.25)$$

The symbols $\overleftarrow{p}$ and $\overleftarrow{p}^{-1}$ in these expressions denote *left*-acting differential and integral operators, respectively.

Each term on the right-hand side of Eq. (4.20) can be handled in sequence, with the result that $Y^{(2)}$ can be written as a sum

$$Y^{(2)} = Y^{(2)}_{\mathrm{FQ}} + Y^{(2)}_{\mathrm{I}} + Y^{(2)}_{\mathrm{II}} + Y^{(2)}_{\mathrm{III}} + Y^{(2)}_{\mathrm{IV}}. \quad (4.26)$$

This solution is contructed in detail in Appendix B. There we derive the explicit expressions

$$Y^{(2)}_{\mathrm{FQ}} = 2 \int d^2x^\perp dx^- q^\dagger \vec{\gamma}^\perp \cdot \vec{\partial}^\perp |p| q, \quad (4.27)$$

$$Y^{(2)}_{\mathrm{I}} = -\frac{1}{\sqrt{2}} G \int d^2x^\perp dx^- q^\dagger \vec{\gamma}^\perp \cdot [(p^{-1}\vec{\partial}^\perp f)p + p(p^{-1}\vec{\partial}^\perp f)] q, \quad (4.28)$$

$$Y^{(2)}_{\mathrm{II}} = -\sqrt{2} G \int d^2x^\perp dx^- \times q^\dagger [(p^{-1}f)\vec{\gamma}^\perp \cdot \vec{\partial}^\perp p + \vec{\gamma}^\perp \cdot \vec{\partial}^\perp p(p^{-1}f)] q, \quad (4.29)$$

$$Y^{(2)}_{\mathrm{III}} = i\sqrt{2} G^2 \int d^2x^\perp dx^- [q^\dagger(p - \overleftarrow{p})q] p^{-1} \times [\tfrac{1}{2} ff + (p^{-1}f)(pf)], \quad (4.30)$$

$$Y^{(2)}_{\mathrm{IV}} = \frac{iG^2}{2} \int d^2x^\perp dx^- [q^\dagger(p - \overleftarrow{p})q] p^{-3} [q^\dagger(p - \overleftarrow{p})q] \times [1 - (p\overleftarrow{p})_1/(p\overleftarrow{p})_2]. \quad (4.31)$$

The notation $(p\overleftarrow{p})_1$ and $(p\overleftarrow{p})_2$ designates operators which act on the first and second pair of quark fields, respectively.

Equations (4.19) and (4.26) specify to order $1/m^2$ a transformation which renders J^i in the form $J^i = L^i + S^i$ [Eq. (3.2)], with L^i and S^i of the free-field form, Eqs. (2.19) and (2.36). Applying this transformation to the null-plane Hamiltonian (4.4) we find that this operator has in terms of q and f the remarkably simple form

$$P^- = -\sqrt{2} \int d^2x^\perp dx^- \left[q^\dagger \frac{m^2 - \partial_\perp^{\,2}}{2p} q - \frac{1}{\sqrt{2}} (\vec{\partial}^\perp f)^2 - \frac{1}{\sqrt{2}} \mu^2 f^2 + O(1/m) \right]. \quad (4.32)$$

It follows that to this order the U(6) × U(6) × O(3) algebra of Eq. (2.21) is *conserved*.

We should point out that the absence of interaction-related terms in Eq. (4.32) is a consequence of demanding that L^i assume its free-field form. If this requirement is relaxed, interaction terms will appear in both L^i and P^-. These terms will not, however, spoil the commutativity of P^- with L^i and S^i—at least not to order m^0. In this context we have extended our calculations to the next order in $1/m$. We find that $\vec{L}$ and $\vec{S}$ still exist and are again conserved. To higher orders we are optimistic that these operators will continue to exist. The expected form of P^- is, however, such that they will no longer be exactly conserved.

One should bear in mind that since we are working on the null plane, our m^{-1} expansion is not the same as the nonrelativistic expansion of Foldy, Wouthuysen, and Tani. Hence the absence from Eq. (4.32) of terms familiar from the Foldy-Wouthuysen-Tani transformation should be no cause for alarm.

V. DISCUSSION

The results of the previous section are remarkable in several respects. In a theory of interacting quarks we have shown that a U(6) × U(6) × O(3) algebra of null-plane charges can be constructed and that the charges forming this algebra are all approximately conserved. The existence of such an algebra may not seem too surprising: An algebra of this type certainly exists in the free-quark model, and it seems natural that quark spin should have meaning even in theories of interacting quarks. That the entire U(6) × U(6) × O(3) algebra should be approximately conserved is, however, a striking fact. It is worthwhile to try to understand why it is so. Mathematically, one notes a parallel between the structure of the null-plane Hamiltonian P^- [Eq. (4.4)] and the Lorentz generators F^i [Eq. (4.10)]. (To the second order in the $1/m$ expansion only that part of J arising from the F's has been important.) Specifically, one can write F^i in the form

$$F^i=\tfrac{1}{2}\int d^2x^\perp dx^-\left[x^-(\partial_-\phi)(\partial^i\phi)-\frac{1}{\sqrt{2}}\Psi_+^\dagger(im\gamma^i p^{-1}-2ix^-\partial^i+\partial^i p^{-1}+i\epsilon^{ij}\partial^j\sigma^3p^{-1})\Psi_+-x^i\mathcal{P}^-(x)\right]+\text{H.c.} \quad (5.1)$$

where $\mathcal{P}^-(x)$ is the Hamiltonian density:

$$P^-=\int d^2x^\perp dx^-\mathcal{P}^-(x). \quad (5.2)$$

All interaction dependence of F^i is thus related to the form of $\mathcal{P}^-(x)$. Physically this relation arises from the fact that angular momentum involves the cross product of a displacement and a momentum density. Thus, in demanding that L^i, when expressed in terms of the q fields, should involve no γ matrices, it is natural that to the leading orders of the $1/m$ expansion, P^- (in terms of the q fields) should also involve no γ matrices. In higher orders of the $1/m$ expansion, contributions to J from the E's are just as important as ones from the F's; any parallel between P^- and J is then destroyed, and exact U(6) × U(6) × O(3) symmetry is lost. This remark applies to the explicit example of Sec. IV as well as to the more general case in which L^i is not restricted to having the form (4.15).

Besides constructing an approximately conserved U(6) × U(6) × O(3) algebra we have been able formally to eliminate all interaction terms in P^- up to order m^{-1}. This is a remarkable result, peculiar to our null-plane m^{-1} expansion. The quasi-free nature of our massive and tightly bound quarks provides a striking parallel to the usual quark-parton model.

The free-quark transformation Eq. (2.35) has formed the basis of a number of phenomenological studies of current matrix elements. Using the transformation (2.22) one can rewrite any current in terms of the constituent quarks q and then use the standard quark-model classification scheme to reduce any matrix element to the product of a reduced matrix element and a Clebsch-Gordan coefficient of the group U(6) × U(6) × O(3). Previous analyses have concentrated on the subgroup $SU(6)_W \times O(2)$.

This subgroup has always been assumed to be boost invariant, an assumption which Eq. (2.35) clearly violates. For transitions between states of equal mass, this effect is of no importance. For unequal-mass transitions, however, there is in general a recoil momentum and the two states are not relatively at rest. Knowing how the $SU(6)_W \times O(2)$ generators transform under boosts we must calculate how the group structure of one state appears in the rest frame of the other. Since Y is not invariant under Lorentz boosts, the group structure will be different in different frames—an effect which previous phenomenological analyses have typically neglected.[22]

This is largely because of the fact that in the free-quark model it is possible to mock up a boost-invariant expression for Y. Melosh,[11] for example, multiplies all factors of p by the operator $M/\sqrt{2}P^+$. This operator acts as the unit operator in the rest frame of any given state. The presence of this operator will, however, complicate the algebraic structure of any operator which does not commute with M. Recognizing this complication amounts to recognizing the recoil effect discussed above. From either viewpoint unequal-mass transitions have a more complicated structure than equal-mass transitions. For processes involving small mass differences this effect will, however, not be too important.

It is clear from Eq. (4.26) that the boost properties of the transformation in the interacting theory are no less complicated than those of the noninteracting theory. The preceding discussion is thus clearly applicable to this model as well. What is new here is the proliferation of terms in the transformation. Actually the algebraic structure of Eq. (4.26) [in terms of $SU(6)_W$] is simply $\underline{1}+\underline{35}$. The presence of pair terms [Eq. (4.31)] means, however, that in higher orders Y will include terms with the exotic structure of $\underline{405}$-and-higher-dimensional $SU(6)_W$ representations.

Particularly interesting in the present approach is the fact that exotic terms enter only in higher orders of the $1/m$ expansion. We have already noted that higher-order terms must be small if approximately degenerate U(6) × U(6) × O(3) multiplets are to be retained. Thus, we can argue that the very existence of such multiplets demands that exotic quark-pair terms be unimportant in the discussion of transitions between the states of these multiplets.

It should be clear that the full $U(6)\times U(6)\times O(3)$ group structure will lead to symmetry predictions beyond those of the $SU(6)_W\times O(2)$ analysis. A preliminary step in this direction has been made by Carlitz and Weyers,[23] who, however, assumed (as in other analyses) the boost invariance of V. These authors interpret $1/m$ as the size of the constituent quark and argue on that basis that the $1/m$ expansion is convergent. Phenomenological arguments, however, are probably still the strongest basis for this viewpoint.

Aside from further phenomenological study, there are several important ways in which the work of this paper might be extended. We believe that the constituent-quark mass m can be spontaneously generated and we are presently investigating models in which this happens. This is crucial, of course, for PCAC applications. We have mentioned that the *a priori* form of $\vec{L}$ is ambiguous and that we have resolved this ambiguity only by the arbitrary definition (4.15). Specifying $\vec{L}$ on more physical grounds, we could remove this arbitrariness and thereby extend the utility of the transformation V. Finally, while we speculate that V can be constructed to arbitrarily high orders of the $1/m$ expansion, we have not gone beyond order $1/m^3$ ourselves. Whether the transformation really exists in higher orders thus remains a challenge to us and our patient readers.

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APPENDIX A

We review here the null-plane quantization rules used throughout the text. A detailed discussion of this procedure is given in Ref. 6. The null-plane components of a four-vector x^μ are

$$x^{\pm}=(x^0\pm x^3)/\sqrt{2} \tag{A1}$$

and

$$\vec{x}^{\perp}=-\vec{x}_{\perp}=(x^1,x^2)\,. \tag{A2}$$

The scalar product is

$$x\cdot y=x^+y^-+x^-y^+-\vec{x}^{\perp}\cdot\vec{y}^{\perp}\,. \tag{A3}$$

In the canonical formulation of null-plane dynamics, the variable x^+ plays the role of "time," and the "Hamiltonian" is correspondingly $P^-=(P^0-P^3)/\sqrt{2}$. The equations of motion involve $\partial_+=\partial/\partial x^+$, while the constraint equations obtained for spin-$\frac{1}{2}$ fields involve only $\partial_-=\partial/\partial x^-$ and $\vec{\partial}^{\perp}=-\partial/\partial\vec{x}^{\perp}$.

In the model of Sec. IV the dynamically independent fields are $\Psi_+(x)=\frac{1}{2}(1+\alpha^3)\Psi(x)$ and $\phi(x)$. These fields have canonical conjugates $\Psi_+^\dagger(x)$ and $\partial_-\phi(x)$, respectively. Quantization of the theory involves imposing the commutation relations

$$\{\Psi_+(x),\Psi_+^\dagger(y)\}\delta(x^+-y^+)=\frac{\gamma^-\gamma^+}{2\sqrt{2}}\delta^4(x-y)\,, \tag{A4}$$

$$[\phi(x),\partial_-\phi(y)]\delta(x^+-y^+)=\frac{i}{2}\delta^4(x-y)\,. \tag{A5}$$

All other equal-x^+ commutators (or anticommutators) vanish.

APPENDIX B

In this appendix we supply the detailed arguments leading to the results (4.19) and (4.26) of Sec. IV. We start by examining Eq. (4.18),

$$\int d^2x^{\perp}dx^-[iY^{(1)},q^\dagger x^j p^{-1}q]$$

$$=\int d^2x^{\perp}dx^-q^\dagger[Gx^j(fp^{-1}+p^{-1}f)+i\gamma^j p^{-1}]q\,, \tag{B1}$$

which is typical of the commutator equations with which we have to deal. If the scalar field f were absent, this equation would coincide with Eq. (2.30) and its solution would be of the free-quark (FQ) form (2.31),

$$Y^{(1)}_{FQ}=-\sqrt{2}\int d^2x^{\perp}dx^-q^\dagger\vec{\gamma}^{\perp}\cdot\vec{\partial}^{\perp}q\,. \tag{B2}$$

Writing

$$Y^{(1)}=Y^{(1)}_{FQ}+Y^{(1)}_{int}\,, \tag{B3}$$

we obtain the following equation for $Y^{(1)}_{int}$:

$$\int d^2x^{\perp}dx^-[iY^{(1)}_{int},q^\dagger x^j p^{-1}q]$$

$$=\int d^2x^{\perp}dx^-q^\dagger Gx^j(fp^{-1}+p^{-1}f)q\,, \tag{B4}$$

To solve this equation we note that an arbitrary operator

$$Z=\sqrt{2}\int d^2x^{\perp}dx^-q^\dagger\Gamma q \tag{B5}$$

(where Γ does not involve q) will obey the following commutation relation:

$$\int d^2x^\perp dx^-[Z, q^\dagger x^j p^{-1} q] = \int d^2x^\perp dx^- q^\dagger x^j (\Gamma p^{-1} - p^{-1}\Gamma) q\,. \quad \text{(B6)}$$

Choosing

$$\Gamma = G[(p^{-1}f)p + p(p^{-1}f)] \quad \text{(B7)}$$

we reproduce the right-hand side of Eq. (B4) and thus obtain the solution

$$Y^{(1)}_{\text{int}} = -iZ = -i\sqrt{2}\,G \int d^2x^\perp dx^- q^\dagger[(p^{-1}f)p + p(p^{-1}f)]q\,. \quad \text{(B8)}$$

To obtain the next-order term in Y we must substitute our result for $Y^{(1)}$ in Eq. (3.3) and study the resulting terms of order m^0. Comparison with Eq. (3.2) gives [with the aid of Eqs. (2.19), (4.15), and (4.17)] the equation

$$\int d^2x^\perp dx^-\{[-iY^{(2)}, q^\dagger \mathcal{J}^i_{(0)}q] - \tfrac{1}{2}[iY^{(1)}, q^\dagger \mathcal{J}^i_{(1)}q] - \tfrac{1}{2}\epsilon^{ij}q^\dagger(i\epsilon^{jk}\partial^k\sigma^3 p^{-1} - i\sqrt{2}\,\mathcal{E}\gamma^j + iGx^j f\vec{\gamma}^\perp\cdot\vec{\partial}^\perp p^{-1} - iG\vec{\gamma}^\perp\cdot\vec{\partial}^\perp x^j p^{-1}f + G^2x^j f p^{-1}f)q\} = 0\,. \quad \text{(B9)}$$

To proceed we must find the explicit form of the commutator

$$C^i = \int d^2x^\perp dx^-[iY^{(1)}, q^\dagger \mathcal{J}^i_{(1)}q]\,. \quad \text{(B10)}$$

Using Eqs. (B3) and (4.12) for $Y^{(1)}$ and $\mathcal{J}^i_{(1)}$ we separate C^i into three pieces:

$$C^i = -\frac{\epsilon^{ij}}{2}(Z^j_{\text{I}} + Z^j_{\text{II}} + Z^j_{\text{III}})\,, \quad \text{(B11)}$$

with

$$\begin{aligned} Z^j_{\text{I}} &= \int d^2x^\perp dx^-[iY^{(1)}_{\text{FQ}}, q^\dagger(Gx^j f p^{-1} + Gx^j p^{-1} f + i\gamma^j p^{-1})q] \\ &= -i\int d^2x^\perp dx^- q^\dagger[\vec{\gamma}^\perp\cdot\vec{\partial}^\perp, Gx^j f p^{-1} + Gx^j p^{-1}f + i\gamma^j p^{-1}]q \\ &= -i\int d^2x^\perp dx^- q^\dagger(G\vec{\gamma}^\perp\cdot\vec{\partial}^\perp f x^j p^{-1} - Gx^j p^{-1} f\vec{\gamma}^\perp\cdot\vec{\partial}^\perp - Gx^j f\vec{\gamma}^\perp\cdot\vec{\partial}^\perp p^{-1} + G\vec{\gamma}^\perp\cdot\vec{\partial}^\perp p^{-1} f x^j - 2\epsilon^{jk}\partial^k\sigma^3 p^{-1})q\,, \end{aligned} \quad \text{(B12)}$$

$$Z^j_{\text{II}} = \int d^2x^\perp dx^-[iY^{(1)}_{\text{int}}, q^\dagger(i\gamma^j p^{-1})q] = i\int d^2x^\perp dx^- G q^\dagger \gamma^j(fp^{-1} + p^{-1}f)q\,, \quad \text{(B13)}$$

$$Z^j_{\text{III}} = \int d^2x^\perp dx^-[iY^{(1)}_{\text{int}}, q^\dagger G x^j(fp^{-1} + p^{-1}f)q]\,. \quad \text{(B14)}$$

Since the scalar field f does not commute with itself [cf. Eq. (4.5)] this last expression will include terms with four quark fields:

$$\begin{aligned} Z^j_{\text{III}} &= G^2\int d^2x^\perp dx^-\left\{q^\dagger[p(p^{-1}f) + (p^{-1}f)p, x^j(fp^{-1} + p^{-1}f)]q + \frac{x^j}{\sqrt{2}}[q^\dagger(p^{-1} - \overleftarrow{p}^{-1})q]p^{-2}[q^\dagger(p - \overleftarrow{p})q]\right\} \\ &= G^2\int d^2x^\perp dx^- x^j\left\{q^\dagger[2fp^{-1}f + ffp^{-1} + p^{-1}ff + 2(p^{-1}f)(pf)p^{-1} + 2p^{-1}(p^{-1}f)(pf)]q + \frac{1}{\sqrt{2}}[q^\dagger(p^{-1} - \overleftarrow{p}^{-1})q]p^{-2}[q^\dagger(p - \overleftarrow{p})q]\right\}. \end{aligned} \quad \text{(B15)}$$

The symbols $\overleftarrow{p}$ and $\overleftarrow{p}^{-1}$ in these expressions denote left-acting operators. Combining Eqs. (B12), (B13), and (B15) and substituting them into Eq. (B9), we arrive at the following commutator equation for $Y^{(2)}$:

$$\int d^2x^\perp dx^-[iY^{(2)}, q^\dagger x^j p^{-1} q] = \int d^2x^\perp dx^-(X^j_{\text{FQ}} + X^j_{\text{I}} + X^j_{\text{II}} + X^j_{\text{III}} + X^j_{\text{IV}})\,, \quad \text{(B16)}$$

with

$$X^j_{\text{FQ}} = -i\sqrt{2}\,q^\dagger\mathcal{E}\gamma^j q\,, \quad \text{(B17)}$$

$$X^j_{\text{I}} = -\frac{iG}{2}x^j q^\dagger\vec{\gamma}^\perp\cdot[(\vec{\partial}^\perp f)p^{-1} + p^{-1}(\vec{\partial}^\perp f)]q\,, \quad \text{(B18)}$$

$$X^j_{\text{II}} = iGq^\dagger(\vec{\gamma}^\perp\cdot\vec{\partial}^\perp f x^j p^{-1} - x^j p^{-1} f\vec{\gamma}^\perp\cdot\vec{\partial}^\perp)q\,, \quad \text{(B19)}$$

$$X^j_{\mathrm{III}} = -\tfrac{1}{2}G^2 x^j q^\dagger[ffp^{-1} + p^{-1}ff + 2(p^{-1}f)(pf)p^{-1} + 2p^{-1}(p^{-1}f)(pf)]q\,, \tag{B20}$$

$$X^j_{\mathrm{IV}} = -\frac{G^2}{2\sqrt{2}} x^j[q^\dagger(p^{-1} - \bar{p}^{-1})q]p^{-2}[q^\dagger(p - \bar{p})q]\,. \tag{B21}$$

To solve Eq. (B16) we write $Y^{(2)}$ as a sum

$$Y^{(2)} = Y^{(2)}_{\mathrm{FQ}} + Y^{(2)}_{\mathrm{I}} + Y^{(2)}_{\mathrm{II}} + Y^{(2)}_{\mathrm{III}} + Y^{(2)}_{\mathrm{IV}} \tag{B22}$$

and construct a series of equations for the corresponding terms in the commutator. For the first term, for example, we require

$$\int d^2x^\perp dx^- [iY^{(2)}_{\mathrm{FQ}}, q^\dagger x^j p^{-1} q] = \int d^2x^\perp dx^- X^j_{\mathrm{FQ}}\,. \tag{B23}$$

This equation is exactly the same as in the free-quark model, Eq. (2.33), and hence has the solution (2.34),

$$Y^{(2)}_{\mathrm{FQ}} = 2\int d^2x^\perp dx^- q^\dagger |p|\, \vec{\gamma}^\perp \cdot \vec{\partial}^\perp q\,. \tag{B24}$$

Proceeding to the next term $Y^{(2)}_{\mathrm{I}}$, we have the equation

$$\int d^2x^\perp dx^- [iY^{(2)}_{\mathrm{I}}, q^\dagger x^j p^{-1} q] = \int d^2x^\perp dx^- X^j_{\mathrm{I}} = \int d^2x^\perp dx^- \tfrac{1}{2}G x^j q^\dagger \vec{\gamma}^\perp \cdot [(\vec{\partial}^\perp f)p^{-1} + p^{-1}(\vec{\partial}^\perp f)]q\,. \tag{B25}$$

The right-hand side of this equation is very similar to Eq. (B6); following the discussion of that term we obtain a solution

$$Y^{(2)}_{\mathrm{I}} = -\frac{1}{\sqrt{2}}G\int d^2x^\perp dx^- q^\dagger \vec{\gamma}^\perp \cdot [(p^{-1}\vec{\partial}^\perp f)p + p(p^{-1}\vec{\partial}^\perp f)]q\,. \tag{B26}$$

The next term of $Y^{(2)}$, $Y^{(2)}_{\mathrm{II}}$, must satisfy the equation

$$\int d^2x^\perp dx^- [iY^{(2)}_{\mathrm{II}}, q^\dagger x^j p^{-1} q] = \int d^2x^\perp dx^- X^j_{\mathrm{II}} = G\int d^2x^\perp dx^- q^\dagger (\vec{\gamma}^\perp \cdot \vec{\partial}^\perp f x^j p^{-1} - p^{-1} x^j f \vec{\gamma}^\perp \cdot \vec{\partial}^\perp) q\,. \tag{B27}$$

This leads to the result

$$Y^{(2)}_{\mathrm{II}} = -\sqrt{2}\,G\int d^2x^\perp dx^- q^\dagger[(p^{-1}f)\vec{\gamma}^\perp \cdot \vec{\partial}^\perp p - \vec{\gamma}^\perp \cdot \vec{\partial}^\perp p(p^{-1}f)]q\,. \tag{B28}$$

The term X^j_{III} is also dealt with in a similar fashion. From the equation

$$\int d^2x^\perp dx^- [iY^{(2)}_{\mathrm{III}}, q^\dagger x^j p^{-1} q] = \int d^2x^\perp dx^- X^j_{\mathrm{III}} = -G^2\int d^2x^\perp dx^- x^j [q^\dagger(p^{-1} - \bar{p}^{-1})q] \times [\tfrac{1}{2}ff + (p^{-1}f)(pf)] \tag{B29}$$

we obtain the result

$$Y^{(2)}_{\mathrm{III}} = i\sqrt{2}\,G^2\int d^2x^\perp dx^- [q^\dagger(p - \bar{p})q] \times p^{-1}[\tfrac{1}{2}ff + (p^{-1}f)(pf)] \tag{B30}$$

Finally, we must deal with the term X^j_{IV}. It contains four quark fields and is thus unlike any of the terms we have previously encountered. The commutator equation involving X^j_{IV} is

$$\int d^2x^\perp dx^- [iY^{(2)}_{\mathrm{IV}}, q^\dagger x^j p^{-1} q] = \int d^2x^\perp dx^- X^j_{\mathrm{IV}}\,. \tag{B31}$$

It is easy to find an expression whose commutator with $q^\dagger x^j p^{-1} q$ gives *inter alia* the term X^j_{IV}. Specifically, we can take

$$A^1 = \frac{i}{2}G^2\int d^2x^\perp dx^- [q^\dagger(p^{-1} - \bar{p}^{-1})q] \times \frac{1}{p^3}[q^\dagger(p - \bar{p})p\bar{p}q]\,, \tag{B32}$$

which satisfies

$$\int d^2x^\perp dx^- [iA^1, q^\dagger x^j p^{-1} q] = \int d^2x^\perp dx^- [X^j_{\mathrm{IV}} - (X^j_{\mathrm{IV}})']\,. \tag{B33}$$

This is of the form (B31) but with a remainder term

$$(X^j_{\mathrm{IV}})' = \frac{G^2}{2\sqrt{2}} x^j [q^\dagger(p^{-1} - \bar{p}^{-1})(p\bar{p})^{-1}q] \times p^{-2}[q^\dagger(p - \bar{p})p\bar{p}q]\,. \tag{B34}$$

This term is quite similar to X^j_{IV}; the term

$$A^2 = \frac{i}{2}G^2\int d^2x^\perp dx^- [q^\dagger(p^{-1} - \bar{p}^{-1})(p\bar{p})^{-1}q] \times \frac{1}{p^3}[q^\dagger(p - \bar{p})(p\bar{p})^2 q] \tag{B35}$$

will commute with $q^\dagger x^j p^{-1} q$ to cancel it. There is, however, another remainder term:

$$\int d^2x^\perp dx^- [iA^2, q^\dagger x^j p^{-1} q] = \int d^2x^\perp dx^- [(X^j_{\mathrm{IV}})' - (X^j_{\mathrm{IV}})'']\,, \tag{B36}$$

with

$$(X^j_{\rm IV})'' = \frac{G^2}{2\sqrt{2}} x^j [q^\dagger (p^{-1} - \bar{p}^{-1})(p\bar{p})^{-2} q] \times p^{-2} [q^\dagger (p - \bar{p})(p\bar{p})^2 q] . \qquad \text{(B37)}$$

Proceeding in this fashion we can construct a solution to Eq. (B31) of the form

$$Y^{(2)}_{\rm IV} = \sum_{n=1}^{\infty} A^n , \qquad \text{(B38)}$$

with a general term A^n of the form

$$A^n = \frac{i}{2} G^2 \int d^2x^\perp dx^- [q^\dagger (p^{-1} - \bar{p}^{-1})(p\bar{p})^{1-n} q] \times p^{-3} [q^\dagger (p - \bar{p})(p\bar{p})^n q] . \qquad \text{(B39)}$$

We can sum the series (B38) to write $Y^{(2)}_{\rm IV}$ in the compact form

$$\begin{aligned} Y^{(2)}_{\rm IV} &= \frac{i}{2} G^2 \sum_{n=1}^{\infty} \int d^2x^\perp dx^- [q^\dagger (p - \bar{p})(p\bar{p})^{-n} q] \times p^{-3} [q^\dagger (p - \bar{p})(p\bar{p})^n q] \\ &= \frac{i}{2} G^2 \int d^2x^\perp dx^- [q^\dagger (p - \bar{p}) q] \times p^{-3} [q^\dagger (p - \bar{p}) q] \times [1 - (p\bar{p})_1 / (p\bar{p})_2] . \end{aligned} \qquad \text{(B40)}$$

The notation $(p\bar{p})_1$ and $(p\bar{p})_2$ designates operators which act on the first and second pair of quark fields, respectively. Combining Eqs. (B24), (B26), (B28), (B30), and (B40), we obtain the complete expression for $Y^{(2)}$, Eq. (B22). This, together with Eq. (B3), specifies the transformation $V = e^{iY}$ to order $1/m^2$ in the $1/m$ expansion.

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[1]For a recent review see J. L. Rosner, Phys. Rep. 11C, 190 (1974).

[2]A search for approximate symmetries of a Dirac particle in an external potential has been made by J. S. Bell and H. Ruegg, Nucl. Phys. B93, 12 (1975). Although we will discuss particles interacting with second-quantized sources and will emphasize the structure of the angular momentum operator over that of the Hamiltonian, our general outlook is quite similar to that of these authors.

[3]M. Gell-Mann, Phys. Lett. 8, 214 (1964); G. Zweig, CERN Reports Nos. TH401 and TH412, 1964 (unpublished).

[4]M. Gell-Mann, Phys. Rev. Lett. 14, 77 (1965).

[5]J. Jersak and J. Stern, Nucl. Phys. B7, 413 (1968); Nuovo Cimento 59, 315 (1969); K. Bardakci and M. B. Halpern, Phys. Rev. 176, 1686 (1968); H. Leutwyler, in *Springer Tracts in Modern Physics*, edited by G. Höhler (Springer, New York, 1969), Vol. 50, p. 29; J. B. Kogut and D. E. Soper, Phys. Rev. D 1, 2901 (1970).

[6]S.-J. Chang, R. G. Root, and T.-M. Yan, Phys. Rev. D 7, 1133 (1973); T.-M. Yan, *ibid.* 7, 1760 (1973).

[7]In all these processes, if we were to construct $\vec{L}$ and $\vec{S}$ on a spacelike surface, we would require their matrix elements between states of infinite momentum. Such a calculation is not a simple one and would in fact be equivalent to the one performed in this paper. Rather than boosting the external states to infinite momenta, we boost the operators, and transform thereby an arbitrary spacelike surface into a null plane.

[8]See R. Jackiw, in *Springer Tracts in Modern Physics*, edited by G. Höhler (Springer, Berlin, 1972), Vol. 62, p. 1, and references therein.

[9]S. Weinberg, Phys. Rev. 177, 2604 (1969).

[10]It is these fields from which the vector and axial-vector currents are constructed [see M. Gell-Mann, Phys. Rev. 125, 1067 (1962)].

[11]This approach was enunciated first by R. Dashen and M. Gell-Mann, Phys. Rev. Lett. 17, 340 (1966). An explicit example of such a transformation has been constructed in the free-quark model by H. J. Melosh, Phys. Rev. D 9, 1095 (1974).

[12]L. L. Foldy and S. A. Wouthuysen, Phys. Rev. 78, 29 (1950); S. Tani, Suryushiron Kenku 1, 15 (1949); Prog. Theor. Phys. 6, 267 (1951).

[13]F. Gürsey, Phys. Lett. 14, 330 (1965); K. T. Mahanthappa and E. C. G. Sudarshan, Phys. Rev. Lett. 14, 458 (1965); Riazuddin and L. K. Pandit, *ibid.* 14, 462 (1965). Other authors who have recently emphasized the Foldy-Wouthuysen-Tani approach include A. Le Yaouanc, L. Oliver, O. Pène, and F. Raynal, Phys. Rev. D 9, 2636 (1974); and J. S. Bell, in *Progress in Particle Physics*, proceedings of the XIII Schladming conference on nuclear physics, edited by P. Urban (Springer, Berlin, 1974) [Acta Phys. Austriaca Suppl. 13, (1974)], p. 395.

[14]This problem has been solved previously by Melosh, Ref. 11. We present in Sec. II a procedure by which the Melosh solution can be constructed and which we can generalize easily to the case of interacting quark fields.

[15]Our null-plane notation is summarized in Appendix A. For an excellent discussion of null-plane quantization see Ref. 6.

[16]ϵ^{ij} denotes the two-dimensional antisymmetric tensor with $\epsilon^{12} = 1$. Repeated indices in Eq. (2.13) and following equations are assumed to be summed over transverse components.

[17]The factor ε is necessary if $\vec{S}$ is to have even charge conjugation. (See Melosh, Ref. 11.) Note the interesting property $[x^-, \varepsilon] = 0$.

[18]One notes that having constructed $\vec{L}$ and $\vec{S}$ we could construct alternate forms for these operators by means of an arbitrary unitary transformation. Even if we demand that $\vec{S}$ has the form (2.19) there is considerable arbitrariness in the form of $\vec{L}$. In our discussion of

interacting models we shall eliminate this arbitrariness by demanding that $\vec{L}$ always have a form analogous to Eq. (2.36).

[19]This point has also been emphasized by J. S. Bell, Ref. 13.

[20]The idea that an exact SU(6) limit is reached when $m \to \infty$ has been suggested by R. Carlitz, D. Heckathorn, J. Kaur, and W.-K. Tung, Phys. Rev. D 11, 1234 (1975), and M. Ida, Kyoto Report No. RIFP-213, 1975 (unpublished).

[21]S. Coleman, Phys. Rev. 138, B1262 (1965); T. F. Jordan, *ibid.* 139, B148 (1965); 140, B766 (1965); S. Weinberg, *ibid.* 139, B597 (1965); L. Michel and B. Sakita, Ann. Inst. Henri Poincaré 2, 167 (1965); S.-J. Chang, R. F. Dashen, and L. O'Raifeartaigh, Phys. Rev. 182, 1819 (1969).

[22]Strictly speaking, there is in the free-quark model no logical inconsistency in neglecting this effect. This is because in the free-quark model an operator like the axial charge never changes the mass of a state. Consequently, one never would deal with unequal-mass transitions and the boost properties of Y would never be relevant. In many real applications one does, however, discuss unequal-mass transitions; for these applications it is essential that the boost properties of Y be recognized and properly treated.

[23]R. Carlitz and J. Weyers, Phys. Lett. 56B, 154 (1975).