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Infinite sequence of conserved currents in the sine-Gordon theory*

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We investigate the infinite sequence of conserved currents in the sine-Gordon theory. We find that they arise from symmetries of the system, that they impose the soliton property on the classical solutions, and that the equivalent theory, the massive Thirring model, reflects their existence.

I. INTRODUCTION

The sine-Gordon theory has several remarkable features which reflect an underlying formal property, the existence of infinitely many symmetries of the system. Classically, the sine-Gordon theory is characterized by an infinite sequence of conserved currents[1-3] and by exact solutions which describe the scattering of solitons.[1] Quantum mechanically, it is equivalent to the massive Thirring model,[4,5] a theory of fermions.

Solitons are localized traveling waves which scatter without change in momentum or shape. In semiclassical treatments[6,7] the soliton is interpreted as a new particle. It is a coherent bound state of the fundamental-boson field. It has been conjectured that the quantum soliton of the sine-Gordon theory is the fundamental fermion of the massive Thirring model.[4]

Previously, the conserved currents were obtained without reference to symmetry considerations. We show in Sec. II that they are determined by Noether's theorem from nonlinear field transformations which leave the action invariant. In Sec. III we show that the soliton property is a consequence of these conserved currents (i.e., that current conservation requires the set of final momenta in any scattering process to be identical to the inital set). If the soliton-fermion identification is valid, we would expect classical scattering amplitudes (the tree approximation) in the massive Thirring model to exhibit the soliton property. We verify this for the $3 \to 3$ amplitude.

In Appendix A we discuss a set of conserved currents which do not arise from symmetries of the system.[1] Appendix B contains the massive-Thirring-model calculation.

II. SYMMETRIES OF THE LAGRANGIAN

The conserved currents are conveniently expressed in light-cone coordinates, defined by

$$x^+ = \frac{t+x}{\sqrt{2}} = x_- ,$$
$$x^- = \frac{t-x}{\sqrt{2}} = x_+ , \tag{2.1}$$
$$g^{++} = g^{--} = 0 , \quad g^{+-} = g^{-+} = 1 .$$

We use the following notation for derivatives of the scalar field:

$$\phi_\nu \equiv \frac{\partial \phi}{\partial x_\nu} \equiv \partial_\nu \phi . \tag{2.2}$$

With these conventions, the sine-Gordon Lagrangian is given by

$$\mathcal{L} = \phi_+ \phi_- + \frac{\alpha_0}{\beta^2}(\cos\beta\phi - 1) , \tag{2.3}$$

where α_0 has the dimensions of mass squared, and β is a dimensionless parameter. The canonical formalism is developed in the usual way, provided that

$$\{\phi(x), \phi(y)\}_{x^+ = y^+} = \tfrac{1}{4}\epsilon(x^- - y^-) \tag{2.4}$$

is the fundamental Poisson bracket. Schwinger's action principle determines the Euler-Lagrange equation:

$$\phi_{+-} = -\frac{\alpha_0}{2\beta}\sin\beta\phi . \tag{2.5}$$

We consider now the infinite sequence of conserved currents derived by Kruskal and Wiley through the generating function[2]:

$$\psi = \phi + \frac{1}{\beta}\sin^{-1}\epsilon\beta\psi_- . \tag{2.6}$$

When ϕ satisfies (2.5), ψ satisfies

$$\partial_+\left(\frac{1 - [1 - \beta^2\epsilon^2(\psi_-)^2]^{1/2}}{\epsilon^2}\right) - \partial_-\left(\frac{\alpha_0}{2}(\cos\beta\psi - 1)\right) = 0 . \tag{2.7}$$

For ϵ an infinitesimal parameter, (2.6) determines ψ as a power series in terms of ϕ. This result is substituted into (2.7) to yield the infinite sequence of conserved currents as coefficients of even powers of ϵ. A dual sequence is obtained by everywhere substituting ∂_+ and ∂_- in Eqs. (2.6) and (2.7). We label the two sequences by the following notation:

$$\partial_+ \tilde{J}^+_{2n,a} + \partial_- \tilde{J}^-_{2n,a} = 0 \,, \quad \partial_+ \tilde{J}^+_{2n,b} + \partial_- \tilde{J}^-_{2n,b} = 0 \,, \tag{2.8}$$

where the index, $2n$, corresponds to the power of ϵ in the expansion of (2.7) and its dual. The lowest-order expression is the energy-momentum tensor:

$$\tilde{J}^+_{0,a} = \theta^{++} = (\phi_-)^2 \,, \quad \tilde{J}^-_{0,a} = \tilde{J}^+_{0,b} = \theta^{+-} = -\frac{\alpha_0}{\beta^2}(\cos\beta\phi - 1) \,, \quad \tilde{J}^-_{0,b} = \theta^{--} = (\phi_+)^2 \,. \tag{2.9}$$

The next currents in the sequence are

$$\tilde{J}^+_{2,a} = -\left[2\phi_-\phi_{---} + (\phi_{--})^2 + \frac{\beta^4}{4}(\phi_-)^4\right] \,, \quad \tilde{J}^-_{2,a} = -\frac{\alpha_0}{2}\left[(\phi_-)^2 \cos\beta\phi + \frac{2}{\beta}\phi_{--}\sin\beta\phi\right] \,, \tag{2.10a}$$

$$\tilde{J}^+_{2,b} = -\frac{\alpha_0}{2}\left[(\phi_+)^2 \cos\beta\phi + \frac{2}{\beta}\phi_{++}\sin\beta\phi\right] \,, \quad \tilde{J}^-_{2,b} = -\left[2\phi_+\phi_{+++} + (\phi_{++})^2 + \frac{\beta^4}{4}(\phi_+)^4\right] \,. \tag{2.10b}$$

To establish the correspondence with symmetries,[8] it is sufficient to show that the constant of the motion

$$Q = \int dx^- J^+ \tag{2.11}$$

generates an infinitesimal transformation of the field

$$\delta\phi = -i\{Q, \phi(x)\} \,, \tag{2.12}$$

which changes the Lagrangian (without application of the Euler-Lagrange equation) by a total divergence:

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\phi_\mu}\delta\phi_\mu = \partial_\mu\Lambda^\mu \,. \tag{2.13}$$

The action, $I = \int d^2x\mathcal{L}$, is then left invariant. The conserved current is recovered in the following way. Using the Euler-Lagrange equation, $\delta\mathcal{L}$ has the alternate expression

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\phi_\mu}\delta\phi\right) .$$

J^μ, defined by

$$J^\mu = \frac{\partial\mathcal{L}}{\partial\phi_\mu}\delta\phi - \Lambda^\mu \,, \tag{2.15}$$

is conserved by virtue of (2.13) and (2.14).

Applying this procedure to the current $\tilde{J}_{2,a}$, we find, using Eq. (2.4), that $\tilde{Q}_{2,a}$, generates the transformation

$$\delta\phi_{2,a} = -\left[\phi_{---} + \frac{\beta^2}{2}(\phi_-)^2\right] . \tag{2.16}$$

The induced change in the Lagrangian is indeed a total divergence which has the following (nonunique) form:

$$\Lambda^+_{2,a} = -\left[\phi_-\phi_{---} + (\phi_{--})^2 + \frac{\beta^4}{4}(\phi_-)^4\right] \,, \quad \Lambda^-_{2,a} = -\left[\phi_+\phi_{---} - 2\phi_{+-}\phi_{--} + \frac{\beta^2}{4}\phi_+(\phi_-)^3 - \frac{\alpha_0}{\beta}\phi_{--}\sin\beta\phi + \frac{\alpha_0}{2}(\phi_-)^2\cos\beta\phi\right] . \tag{2.17}$$

It is thus seen that (2.16) is a symmetry operation. The conserved current determined by (2.15),

$$J^+_{2,a} = (\phi_{--})^2 - \frac{\beta^2}{4}(\phi_-)^4 \,, \quad J^-_{2,a} = \frac{\alpha_0}{2}(\phi_-)^2\cos\beta\phi \,, \tag{2.18}$$

differs from the original expression in (2.10) by total divergences and applications of the Euler-Lagrange equation. It can be shown in a similar way that the dual current

$$J^+_{2,b} = \frac{\alpha_0}{2}(\phi_+)^2\cos\beta\phi \,, \quad J^-_{2,b} = (\phi_{++})^2 - \frac{\beta^4}{4}(\phi_+)^4 \tag{2.19}$$

is generated by the symmetry operation

$$\delta\phi_{2,b} = -\left[\phi_{+++} + \frac{\beta^2}{2}(\phi_+)^3\right] . \tag{2.20}$$

The $O(\epsilon^4)$ currents

$$J^+_{4,a} = (\phi_{---})^2 + \frac{5\beta^2}{2}(\phi_-)^2(\phi_{--})^2 + \frac{5\beta^2}{3}(\phi_-)^2\phi_{---} + \frac{\beta^4}{8}(\phi_-)^6 \,,$$
$$J^-_{4,a} = -\frac{5\beta^2}{3}(\phi_-)^3\phi_{+--} - \frac{3\alpha_0\beta^2}{8}(\phi_-)^4\cos\beta\phi + \frac{3\alpha_0\beta}{2}(\phi_-)^2\phi_{--}\sin\beta\phi + \frac{\alpha_0}{2}(\phi_{--})^2\cos\beta\phi \,, \tag{2.21}$$

and the dual, $J^\mu_{4,b}$ [obtained by everywhere interchanging the symbols + and − in (2.21)], are generated, respectively, by

$$\delta\phi_{4,a}=\tfrac{1}{2}\phi_{-----}+\frac{5\beta^2}{4}[\phi_-(\phi_{--})^2+(\phi_-)^2\phi_{---}]+\frac{3\beta^4}{16}(\phi_-)^5\,, \tag{2.22a}$$

$$\delta\phi_{4,b}=\tfrac{1}{2}\phi_{+++++}+\frac{5\beta^2}{4}[\phi_+(\phi_{++})^2+(\phi_+)^2\phi_{+++}]+\frac{3\beta^4}{16}(\phi_+)^5\,. \tag{2.22b}$$

The interaction term in the Lagrangian apparently allows an infinite sequence of highly nonlinear symmetry operations. The interpretation of the corresponding currents is given in the next section.

III. REPRESENTATIONS OF THE CONSERVED QUANTITIES

To gain insight into the nature of the conserved quantities, we present first a representation for the case $\beta\to 0$. In this limit, the sine-Gordon theory describes a free field which has the following light-cone quantization:

$$\Phi(x)=\frac{1}{(4\pi)^{1/2}}\int_0^\infty \frac{dk^+}{\sqrt{k^+}}[e^{-ikx}a^\dagger(k^+)+e^{ikx}a(k^+)]\,. \tag{3.1}$$

In the above expression, $k^-=\alpha_0/2k^+$ and the creation and annihilation operators satisfy the commutation relations

$$\begin{aligned}&[a(k^+),a^\dagger(k^{+\prime})]=\delta(k^+-k^{+\prime})\,,\\&[a(k^+),a(k^{+\prime})]=[a^\dagger(k^+),a^\dagger(k^{+\prime})]=0\,.\end{aligned} \tag{3.2}$$

From (3.1) and (3.2) we obtain for the constants of the motion operators in normal-ordered form

$$Q_{0,a}=\int dx^-:(\Phi_-)^2:=\int_0^\infty dk^+(k^+)a^\dagger(k^+)a(k^+)\,, \tag{3.3a}$$

$$Q_{0,b}=\int dx^-:(\Phi\Phi_{+-}):=\int_0^\infty dk^+(k^-)a^\dagger(k^+)a(k^+)\,, \tag{3.3b}$$

$$Q_{2,a}=\int dx^-:(\Phi_{--})^2:=\int_0^\infty dk^+(k^+)^3a^\dagger(k^+)a(k^+)\,, \tag{3.3c}$$

$$Q_{2,b}=\int dx^-:(\Phi_+\Phi_{++-}):=\int_0^\infty dk^+(k^-)^3a^\dagger(k^+)a(k^+)\,. \tag{3.3d}$$

The normal-ordering process amounts to a subtraction of infinite constants which does not destroy current conservation. It is clear that the succeeding operators are given by

$$Q_{2n,a}=\int_0^\infty dk^+(k^+)^{2n+1}a^\dagger(k^+)a(k^+)\,, \tag{3.4a}$$

$$Q_{2n,b}=\int_0^\infty dk^+(k^-)^{2n+1}a^\dagger(k^+)a(k^+)\,. \tag{3.4b}$$

We mention that although the quantities

$$(k^+)^{2n}a^\dagger(k^+)a(k^+)\,,$$
$$(k^-)^{2n}a^\dagger(k^+)a(k^+)$$

are also constants of the motion, they do not correspond to the local current densities derived from symmetries of the system. The conservation laws (3.4) impose an infinite number of independent conditions on the momenta of the individual particles undergoing scattering processes. To satisfy them, quantum scattering amplitudes must conserve the set of light-cone momenta.[9] (Implicit in this constraint is conservation of particle number and type.) Of course, the constraint is trivial for a free-field theory.[10] We shall show, however, that it carries over to the soliton solutions of the classical interaction theory.

The n-soliton solutions, $\phi_n(x)$, are parameterized by n momenta, $k_i^-=64\alpha_0/\beta^4k_i^+$, and n localizations, x_i^-. They are defined up to a constant, $C_n=2\pi m/\beta$ (where m is an integer), and are generated recursively from the vacuum solution, $\phi_0(x)=0$, by the Bäcklund transformation[1]:

$$\phi_{n,+}=\phi_{n-1,+}+\frac{\beta k_n^-}{4}\sin\beta\left(\frac{\phi_n+\phi_{n-1}}{2}\right)\,, \tag{3.5a}$$

$$\phi_{n,-}=-\phi_{n-1,-}-\frac{\beta k_n^+}{4}\sin\beta\left(\frac{\phi_n-\phi_{n-1}}{2}\right)\,. \tag{3.5b}$$

Here, k_n^- and k_n^+ can be either positive or negative. By cross differentiation, both ϕ_n and ϕ_{n-1} must satisfy (2.5). There are two solutions for ϕ_1, the solition, ϕ_S, and the antisoliton, ϕ_A,

$$\begin{aligned}\phi_S&=\frac{4}{\beta}\tan^{-1}\exp\left\{+\frac{\beta^2}{8}[\,|k_n^-|x^+-|k_n^+|(x^--X^-)]\right\}\,,\\ \phi_A&=\frac{4}{\beta}\tan^{-1}\exp\left\{-\frac{\beta^2}{8}[\,|k_n^-|x^+-|k_n^+|(x^--X^-)]\right\}\,,\end{aligned} \tag{3.6}$$

which are distinguished by the boundary condition

$$\begin{aligned}&\phi_S(x^+,+\infty)-\phi_S(x^+,-\infty)=-\frac{2\pi}{\beta}\,,\\&\phi_A(x^+,+\infty)-\phi_A(x^+,-\infty)=+\frac{2\pi}{\beta}\,.\end{aligned} \tag{3.7}$$

The general solutions, ϕ_n, have the boundary condition

$$\phi_n(x^+,+\infty)-\phi_n(x^+,-\infty)=\frac{-2\pi N}{\beta}\,, \tag{3.8}$$

where N is the number of solitons minus antisolitons contained in ϕ_n.

Let us now consider the light-cone Hamiltonian for this solution:

$$Q_{0a}[\phi_n]=\int dx^-(\phi_{n,-})^2\,. \tag{3.9}$$

From (3.5) we obtain the recursive relation

$$(\phi_{n,-})^2 = (\phi_{n-1,-})^2 + \frac{k_n^+}{2}\partial_-\left[\cos\beta\left(\frac{\phi_n - \phi_{n-1}}{2}\right)\right]. \tag{3.10}$$

By repeating this relation and using (3.7) with an appropriate choice of constants, C_i, we obtain for $Q_{0,a}[\phi_n]$

$$Q_{0,a}[\phi_n] = \sum_{i=1}^{n} k_i^+ . \tag{3.11a}$$

In a similar way, we can show that

$$Q_{0,b}[\phi_n] = \sum_{i=1}^{n} k_i^- , \tag{3.11b}$$

$$Q_{2,a}[\phi_n] = \frac{\beta^4}{3\times 2^7}\sum_{i=1}^{n}(k_i^+)^3 , \tag{3.11c}$$

$$Q_{2,b}[\phi_n] = \frac{\beta^4}{3\times 2^7}\sum_{i=1}^{n}(k_i^-)^3 . \tag{3.11d}$$

The generalization to higher-order currents (apart from numerical factors) is obvious. We see then that the constants of the motion for the classical soliton solutions have the same representation as in the free-field theory. In this case , the constraint imposed on ϕ_n is nontrivial; the infinite sequence of conserved currents requires the soliton property.

IV. THE MASSIVE THIRRING MODEL

To see whether this property holds in the massive Thirring model, it is necessary to consider the n-point scattering amplitudes, where n is at least six. For lower amplitudes, the energy-momentum conservation is sufficient to fix the final momenta. The Lagrangian for this theory is

$$\mathcal{L} = \bar\psi(i\not\partial - m)\psi - \tfrac{1}{2}g(\bar\psi\gamma^\mu\psi)(\bar\psi\gamma_\mu\psi) . \tag{3.12}$$

We compute the $3\to 3$ scattering amplitude in the tree approximation:

$$S_{3,3} = -(2\pi)^2 g^2 \delta(k_1 + k_2 + k_3 - k_4 - k_5 - k_6) \times \left(m^6 \Big/ \prod_{i=1}^{6} E_i\right)^{1/2} M_{3,3} . \tag{3.13}$$

The invariant amplitude, $M_{3,3}$, is determined by the graph in Fig. 1:

$$M_{3,3} = \sum_{\{\sigma\}\{\delta\}} \mathrm{sgn}\sigma\, \mathrm{sgn}\delta\, \bar u_{\delta(6)}\gamma_\mu u_{\sigma(3)} \bar u_{\delta(5)}\gamma^\mu \times \frac{\not k_{\sigma\delta} + m}{k_{\sigma\delta}^2 - m^2 + i\epsilon}\gamma^\nu u_{\sigma(2)}\bar u_{\delta(4)}\gamma_\nu u_{\sigma(1)} , \tag{3.14}$$

where σ and δ are permutation indices, $k_{\sigma\delta} = k_{\sigma(1)} + k_{\sigma(2)} - k_{\delta(4)}$, and the $u_i \equiv u(k_i)$ are the spinor solutions to the Dirac equation. We express the denominators in (3.14) as

$$\frac{1}{k^2 - m^2 + i\epsilon} = \mathrm{P}\left(\frac{1}{k^2 - m^2}\right) - i\pi\delta(k^2 - m^2) , \tag{3.15}$$

where $\mathrm{P}(x)$ denotes the principal part of x. In Appendix B we show that the principal parts of (3.14) cancel. $M_{3,3}$ is thus a sum over the δ functions which constrain the set of final momenta to be identical to the set of initial momenta.

CONCLUSIONS

We have shown that an infinite sequence of nonlinear symmetry operations is a fundamental property of the sine-Gordon theory. These symmetries determine the conserved currents, which in turn determine the soliton property of the classical solutions. We have shown, furthermore, that the interpretation of the fermion in the massive Thirring model as a quantum soliton is consistent on the classical level. Whether or not conservation of the set of momenta persists quantum mechanically in the massive Thirring model can be seen by looking at higher orders of perturbation theory where ordering problems and anomalies can destroy formal current conservation.

Note added. While writing this manuscript, I learned that Berg, Karowski, and Thun have also computed tree amplitudes in the massive Thirring model. They give an elegant proof of our result and show, in addition, that the $2\to 4$ amplitude vanishes.[11]

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APPENDIX A

In Ref. 1, a different sequence of conserved currents is generated by a method similar to that de-

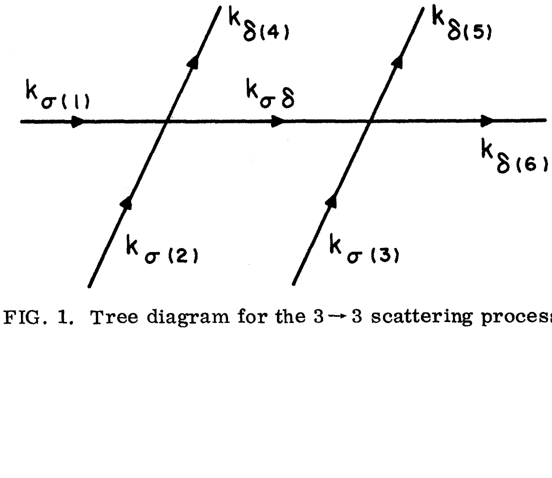

FIG. 1. Tree diagram for the $3\to 3$ scattering process.

scribed in Sec. II. The generating function, ψ, is defined by half of the Bäcklund transformation:

$$\psi_- = -\phi_- - \frac{2^4\alpha_0}{\beta^3 k^-}\sin\beta\left(\frac{\psi-\phi}{2}\right), \tag{A1}$$

where k^- is considered an infinitesimal parameter. ψ is determined in terms of ϕ and substituted into the conservation law

$$\partial_+(\psi_-)^2 + \partial_-\left[-\frac{\alpha_0}{\beta^2}(\cos\beta\psi - 1)\right] = 0, \tag{A2}$$

to yield an infinite sequence of conserved currents as coefficients of $(k^-)^n$. As in Sec. II, the zeroth-order expression is conservation of $\theta^{+\mu}$. The higher-order laws can be put into a simple form by using the identities obtained from (A1):

$$(\psi_-)^2 = (q_-)^2 + \partial_-\left[\frac{2^5\alpha_0}{\beta^4 k^-}\cos\beta\left(\frac{\psi-\phi}{2}\right)\right], \tag{A3a}$$

$$\frac{\alpha_0}{\beta^2}\cos\beta\psi = \frac{\alpha_0}{\beta^2}\cos\beta\phi + \partial_+\left[\frac{2^5\alpha_0}{\beta^4 k^-}\cos\beta\left(\frac{\psi-\phi}{2}\right)\right]. \tag{A3b}$$

Substituting (A3) into (A2) and subtracting conservation of $\theta^{+\mu}$, we obtain

$$\partial_+\left[\partial_-\cos\beta\left(\frac{\psi-\phi}{2}\right)\right] - \partial_-\left[\partial_+\cos\beta\left(\frac{\psi-\phi}{2}\right)\right] = 0. \tag{A4}$$

There is also a sequence of dual currents generated by the other half of the Bäcklund transformation:

$$\partial_+\left[\partial_-\cos\beta\left(\frac{\psi+\phi}{2}\right)\right] - \partial_-\left[\partial_+\cos\beta\left(\frac{\psi+\phi}{2}\right)\right] = 0.$$

We see explicitly that all higher-order currents are trivially conserved, with vanishing constants of the motion and, consequently, no corresponding symmetries of the Lagrangian.

APPENDIX B

We choose the following conventions:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad g^{01} = g^{10} = 0, \quad g^{00} = -g^{11} = 1. \tag{B1}$$

The positive-energy spinor solution to the Dirac equation is

$$u(k) = \frac{1}{[4(1+E)]^{1/2}}(1+\not{k})\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{B2}$$

where we have set $m=1$.

In computing the principal part of $M_{3,3}$, we choose the frame of reference $\vec{k}_1 = \vec{k}_2$ and the special case $\vec{k}_3 = 0$. Under these conditions, we obtain

$$\mathrm{P}(M_{3,3}) = \prod_{i=1}^{6} \frac{8}{[4(1+E_i)]^{1/2}} \frac{k_1(3E_1^2 + 6E_1 - 1)}{E_1} B_{3,3}. \tag{B3}$$

We show that

$$B_{3,3} = \sum_{i,j,k=4}^{6} \frac{\epsilon_{ijk}(E_i+1)[k_j(E_k+1) - k_k(E_j+1)]}{E_1 - E_i} = 0, \tag{B4}$$

by choosing the notation

$$b_1 + E_1 - E_4, \quad b_2 = E_1 - E_5, \quad b_3 = E_1 - E_6.$$

Energy-momentum conservation requires

$$2\prod_{i=1}^{3} b_i\left(3 + \sum_{j=1}^{3} b_j\right) + \sum_{i\neq j=1}^{3}(b_i^2 b_j + b_i b_j) = 0. \tag{B5}$$

In the b_i notation

$$B_{3,3} = \frac{2(b_1+b_3+2)}{k_5\prod_{i=1}^{3} b_i}\{\tfrac{1}{2}(b_1+b_3)(b_1+b_2+2)[(b_2+b_3+2)(b_3-b_1)b_2 + (b_1+b_3+2)(b_3-b_2)b_1]$$
$$+ \tfrac{1}{2}[b_2(b_3-b_1) - b_1(b_1+b_3+2)][b_2+b_3+2)(b_1-b_2)b_3 + (b_1+b_2+2)(b_3-b_2)b_1]\}. \tag{B6}$$

With a little algebra the expression in curly brackets in (B6) reduces to the left-hand side of (B5). We have verified numerically that $B_{3,3} = 0$ for $\vec{k}_3 \neq 0$.

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